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ON COMMON FIXED POINTS OF HEMICONTRACTIVE MAPPINGS

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A generalisation of Ishikawa iteration scheme is studied which gives common fixed points of a family of hemicontractive mappings.

Ishikawa (1974) has developed an iteration scheme which converges to a fixed point of a lipschitzian pseudocontractive map in a Hilbert space. Hicks and Kubicek (1977) have studied the convergence of an iteration scheme to fixed points of demicontractive maps. The purpose of this paper is to extend and generalise the result of Ishikawa by simultaneously taking a more generalised iteration scheme involving a family of maps and secondly by taking less restrictive hemicontractive maps.

Before we state our result let us recall the following definitions.

Let E be a subset of a Hilbert space H and T be a self map on E. Let $F(T)$ denote the fixed point set of T. T is said to be demicontractive if there exists a constant λ, $0 \leqslant \lambda < 1$ such that for each $x \in E$ and $q \in F(T)$

$$\| Tx-q \|^2 \leqslant \| x-q \|^2 + \lambda \| x-Tx \|^2.$$

T is said to be hemicontractive if T is demicontractive with $\lambda = 1$, i.e., for each $x \in E$ and $q \in F(T)$,

$$\| Tx-q \|^2 \leqslant \| x-q \|^2 + \| x-Tx \|^2.$$

T is said to be pseudocontractive if for each $x, y \in E$,

$$\| Tx-Ty \|^2 \leqslant \| x-y \|^2 + \| (I-T)x - (I-T)y \|^2$$

where I denotes identity map.

Clearly a pseudocontractive map or a demicontractive map is hemicontractive. We give an example of a map which is hemicontractive but not demicontractive.

$T : [-1, 1] \to [-1, 1]$ defined by $Tx = 0$, $-1 \leqslant x \leqslant 0$;

$Tx = x - 1/[1 - \log x]$, $0 < x \leqslant 1$.

Now we prove the following :

Theorem—Let $\{T_j, j = 1, 2, \ldots, k\}$, $k \geqslant 2$ be a family of hemicontractive maps defined on a convex, compact subset E of a Hilbert space H and have at least one common fixed point in E. Let the sequence $\{x_n\}$ in E be defined by

$$x_1 \in E$$

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\, T_k\, u_{k-1}\ (n) \qquad \ldots(1)$$

where

$$u_0\ (n) = x_n,\ u_j\ (n) = (1-\beta_n)\, x_n + \beta_n\, T_j\, u_{j-1}\ (n) \qquad \ldots(2)$$

for $j = 1, 2, \ldots, k$ and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1]$ such that

(i) $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$, for $n = 1, 2, 3, \ldots$

(ii) $\lim_{n \to \infty} \beta_n = 0$

(iii) $\sum_{n=1}^{\infty} \alpha_n\, \beta_n^{k-1} = \infty$ for each $k \geqslant 2$.

Let the family of maps $\{T_j\}$ satisfy

$$\| T_i\, x - T_j\, y \| \leqslant M \| x - y \| \qquad \ldots(3)$$

for all $x, y \in E$ and all pairs (i, j), M being a positive constant. Then the sequence $\{x_n\}$ converges to a common fixed point of the family of maps $\{T_j\}$.

PROOF : Ishikawa (1974) has shown that for any x, y, z in a Hilbert space and a real number s,

$$\| sx + (1-s)\, y - z \|^2 = s \| x - z \|^2 + (1-s) \| y - z \|^2 - s(1-s) \| x - y \|^2. \qquad \ldots(4)$$

By assumption if p denotes a common fixed point of the family $\{T_j\}$, then, since each T_j is hemicontractive, we have, for $j = 1, 2, \ldots k$,

$$\| T_j\, x - p \|^2 \leqslant \| x - p \|^2 + \| x - T_j\, x \|^2. \qquad \ldots(5)$$

Using (4) we have from (1)

$$\| x_{n+1}-p \|^2 = (1-\alpha_n) \| x_n-p \|^2 + \alpha_n \| T_k u_{k-1}(n) - p \|^2 - \alpha_n (1-\alpha_n) \| x_n - T_k u_{k-1}(n) \|^2. \quad \ldots(6)$$

Since each T_j is a self map on E and E is convex, $u_j(n)$ and $T_j u_{j-1}(n)$ are in E for each j and n. We have the following systems of relations for $j = 1, 2, \ldots, k-1$. Using (5) we have

$$\| T_{k-j+1} u_{k-j}(n) - p \|^2 \leqslant \| u_{k-j}(n)-p \|^2 + \| u_{k-j}(n)-T_{k-j+1} u_{k-j}(n) \|^2. \quad \ldots(7)$$

Using (2) and (4) we have

$$\| u_{k-j}(n)-p \|^2 = \| (1-\beta_n) x_n + \beta_n T_{k-j} u_{k-j-1}(n)-p \|^2 = (1-\beta_n) \| x_n-p \|^2 + \beta_n \| T_{k-j} u_{k-j-1}(n)-p \|^2 - \beta_n (1-\beta_n) \| x_n-T_{k-j} u_{k-j-1}(n) \|^2 \quad \ldots(8)$$

and

$$\| u_{k-j}(n)-T_{k-j+1} u_{k-j}(n) \|^2 = (1-\beta_n) \| x_n-T_{k-j+1} u_{k-j}(n) \|^2 + \beta_n \| T_{k-j} u_{k-j-1}(n)-T_{k-j+1} u_{k-j}(n) \|^2 - \beta_n (1-\beta_n) \| x_n-T_{k-j} u_{k-j-1}(n) \|^2. \quad \ldots(9)$$

Now multiplying β_n^{j-1} in relations (7), (8) and (9) and adding each one of them over j, we respectively obtain (10), (11) and (12) :

$$\sum_{j=1}^{k-1} \beta_n^{j-1} \| T_{k-j+1} u_{k-j}(n)-p \|^2 \leqslant \sum_{j=1}^{k-1} \beta_n^{j-1} \| u_{k-j}(n)-p \|^2 + \sum_{j=1}^{k-1} \beta_n^{j-1} \| u_{k-j}(n)-T_{k-j+1} u_{k-j}(n) \|^2 \quad \ldots(10)$$

$$\sum_{j=1}^{k-1} \beta_n^{j-1} \| u_{k-j}(n)-p \|^2 = (1-\beta_n) \| x_n-p \|^2 \sum_{j=1}^{k-1} \beta_n^{j-1} + \beta_n \sum_{j=1}^{k-1} \beta_n^{j-1} \| T_{k-j} u_{k-j-1}(n)-p \|^2 - \beta_n (1-\beta_n) \sum_{j=1}^{k-1} \beta_n^{j-1} \| x_n-T_{k-j} u_{k-j-1}(n) \|^2 \quad \ldots(11)$$

$$\sum_{j=1}^{k-1} \beta_n^{j-1} \| u_{k-j}(n) - T_{k-j+1} u_{k-j}(n) \|^2$$

$$= (1-\beta_n) \sum_{j=1}^{k-1} \beta_n^{j-1} \| x_n - T_{k-j+1} u_{k-j}(n) \|^2$$

$$+ \beta_n \sum_{j=1}^{k-1} \beta_n^{j-1} \| T_{k-j} u_{k-j-1}(n) - T_{k-j+1} u_{k-j}(n) \|^2$$

$$- \beta_n (1-\beta_n) \sum_{j=1}^{k-1} \beta_n^{j-1} \| x_n - T_{k-j} u_{k-j-1}(n) \|^2. \qquad \ldots(12)$$

Since T_1 is hemicontractive,

$$\| T_1 x_n - p \|^2 \leqslant \| x_n - p \|^2 + \| x_n - T_1 x_n \|^2. \qquad \ldots(13)$$

Now calculating according to (6) + α_n (10) + α_n (11) + α_n (12) + $\alpha_n \beta_n^{k-1}$ (13) and eliminating common terms on both sides we have

$$\| x_{n+1} - p \|^2 \leqslant \| x_n - p \|^2$$

$$+ \alpha_n \beta_n \sum_{j=1}^{k-1} \beta_n^{j-1} \| T_{k-j} u_{k-j-1}(n) - T_{k-j+1} u_{k-j}(n) \|^2$$

$$- [\alpha_n (\beta_n - \alpha_n) \| x_n - T_k u_{k-1}(n) \|^2$$

$$- \alpha_n \beta_n (1-\beta_n) \sum_{j=1}^{k-2} \beta_n^{j-1} \| x_n - T_{k-j} u_{k-j-1}(n) \|^2$$

$$+ 2\alpha_n \beta_n (1-\beta_n) \sum_{j=1}^{k-1} \beta_n^{j-1} \| x_n - T_{k-j} u_{k-j-1}(n) \|^2]$$

$$+ \alpha_n \beta_n^{k-1} \| x_n - T_1 x_n \|^2$$

$$\leqslant \| x_n - p \|^2$$

$$+ \alpha_n \beta_n \sum_{j=1}^{k-1} \beta_n^{j-1} \| T_{k-j} u_{k-j-1}(n) - T_{k-j+1} u_{k-j}(n) \|^2$$

$$- \alpha_n \beta_n (1-\beta_n) \sum_{j=1}^{k-2} \beta_n^{j-1} \| x_n - T_{k-j} u_{k-j-1}(n) \|^2$$

$$- \alpha_n \beta_n^{k-1} (1-2\beta_n) \| x_n - T_1 x_n \|^2, \quad \text{(Since } \beta_n \geqslant \alpha_n)$$

$$\leqslant \| x_n - p \|^2$$

$$+ \alpha_n \beta_n \sum_{j=1}^{k-1} \beta_n^{j-1} \| T_{k-j} u_{k-j-1}(n) - T_{k-j+1} u_{k-j}(n) \|^2$$

$$- \alpha_n \beta_n^{k-1} (1-2\beta_n) \| x_n - T_1 x_n \|^2. \qquad \ldots(14)$$

Now using (2) and (3) we have for each $j = 1, 2, \ldots, k-1$

$$\| T_{k-j+1} u_{k-j}(n) - T_{k-j} u_{k-j-1}(n) \| \leqslant M \| u_{k-j}(n) - u_{k-j-1}(n) \|$$

$$= M\beta_n \| T_{k-j} u_{k-j-1}(n) - T_{k-j-1} u_{k-j-2}(n) \|.$$

A repeated application of the above inequality gives

$$\| T_{k-j+1} u_{k-j}(n) - T_{k-j} u_{k-j-1}(n) \|^2 \leqslant (M\beta_n)^{2k-2j} \| T_1 x_n - x_n \|^2$$

for $j = 1, 2, \ldots, k-1$. Hence from (14) we get

$$\| x_{n+1} - p \|^2 \leqslant \| x_n - p \|^2 + \alpha_n \beta_n \sum_{j=1}^{k-1} \beta_n^{j-1} (M\beta_n)^{2k-2j} \| T_1 x_n - x_n \|^2$$

$$- \alpha_n \beta_n^{k-1} (1 - 2\beta_n) \| x_n - T_1 x_n \|^2$$

$$= \| x_n - p \|^2 - \alpha_n \beta_n^{k-1} [1 - 2\beta_n - M^2 \beta_n^2 ((M^2 \beta_n)^{k-1}$$

$$- 1)/(M^2 \beta_n - 1)] \| T_1 x_n - x_n \|^2. \qquad \ldots(15)$$

Writing

$$\theta_n = 1 - 2\beta_n - M^2 \beta_n^2 ((M^2 \beta_n)^{k-1} - 1)/(M^2 \beta_n - 1)$$

(15) reduces to

$$\alpha_n \beta_n^{k-1} \theta_n \| T_1 x_n - x_n \|^2 \leqslant \| x_n - p \|^2 - \| x_{n+1} - p \|^2. \qquad \ldots(16)$$

Since $\beta_n \to 0$, $\theta_n \to 1$ as $n \to \infty$. Hence there exists a number N such that $\theta_n > \frac{1}{2}$ for all $n \geqslant N$. Choose $s > r \geqslant N$. Now from (16) summing up for $n = r$ to $n = s$ we obtain

$$\frac{1}{2} \sum_{n=r}^{s} \alpha_n \beta_n^{k-1} \| T_1 x_n - x_n \|^2 \leqslant \sum_{n=r}^{s} \alpha_n \beta_n^{k-1} \theta_n \| T_1 x_n - x_n \|^2$$

$$\leqslant \| x_r - p \|^2 - \| x_{s+1} - p \|^2.$$

Since the right hand side of the above inequality is bounded and $\sum_{n=1}^{\infty} \alpha_n \beta_n^{k-1} = \infty$ for each k, we have $\liminf_{n \to \infty} \| T_1 x_n - x_n \|^2 = 0$. By compactness of E there exists a subsequence $\{x_{n_t}\}$ such that $x_{n_t} \to q$ as $t \to \infty$ and $\| T_1 q - q \| = 0$. Now q is a common fixed point of the family $\{T_j\}$; for,

$$\| T_j q - q \| \leqslant \| T_j q - T_j x_{n_t} \| + \| T_j x_{n_t} - T_1 q \|$$

$$\leqslant M \| q - x_{n_t} \| + M \| x_{n_t} - q \| \text{ (using (3))}$$

$$= 2M \| x_{n_t} - q \| \to 0 \text{ as } t \to \infty.$$

Hence $\| T_j q - q \| = 0$ giving $T_j q = q$ for each j. Since q is a common fixed point of $\{T_j\}$ we can show from (16) that $\{ \| x_n - q \| \}$ is an ultimately monotonically decreasing sequence and hence convergent. This along with the fact that $x_{n_t} \to q$ as $t \to \infty$ implies that $x_n \to q$ as $n \to \infty$.

This completes the proof of the theorem.

Remarks: In case $k = 2$ and $T_1 = T_2$ this result includes the theorem of Ishikawa (1974) as a corollary. This theorem further shows that the Ishikawa iteration can be extended to lipschitzian hemicontractive mappings. It is obviously clear that the lipschitzian demicontractive maps have been taken care of.

References

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