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A CHARACTERIZATION OF MEDIAN GRAPHS

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A new characterization of median graphs is given by using specially limited convex sets of graphs.

Let $G = (V, E)$ be a finite connected, undirected graph without loops and multiple lines. The set V is the set of points of G and E its set of lines. We shall follow here the terminology used in Harary's book (1969). The length of an ab-path is the number of its lines. The shortest distance between points a and b is denoted by $d(a, b)$. A point t is called a median for points a, b and c in G, if the following equations hold

$$d(a, b) = d(a, t) + d(t, b); \; d(a, c) = d(a, t) + d(t, c);$$

$$d(b, c) = d(b, t) + d(t, c).$$

A graph G is called a median graph if there is exactly one median t for every three points a, b, c, in G. A comprehensive report on median graphs and related algebras is given in the recent paper of Bandelt and Hedlikova (1983). The purpose of this paper is to give a new characterization for median graphs.

A nonempty set $A \subset V$ is called convex set if A contains all the points on any shortest path joining two points of A. If A and B are two convex sets of G and $A \cap B$ is nonempty, the set $A \cap B$ is also a convex set of G. Thus we can define the least convex containing a set $C \neq \phi$ of V as the intersection of all convex sets containing C; thisl east convex is denoted by $[C]$. When A is a convex set, then clearly $[A] = A$. The least convex set containing $[A]$ and $[C]$ is denoted by $[A] \vee [C]$. We shall use the brief expression $[a, b]$ instead of the complete one $[\{a, b\}]$. Every pointset inducing a complete subgraph of G is a convex of G; thus every single point as well as the endpoints of a line are convexies of G. The set $SP(a, b)$, where $a, b \in V$, is defined as follows : $SP(a, b) = \{w \mid w$ is on a shortest ab-path$\}$. A nonempty set $A \subset V$ is called 2-closed, if for every three points $a, b, c \in V$ with

$a \neq b$, $a, b \in A$ and $(a, c), (b, c) \in E$ also the point c belongs to A. A graph G is called modularly annihilated if for every three points $a, b, m \in V$ with $[b, m] \subset [a, m]$, the relations $[x, m] \subset [b, m]$ and $[a, m] \cap [y, m] \subset [b, m]$ imply $[a, m] \cap ([x, m] \vee [y, m]) \subset [b, m]$, where $x, y \in V$. The name modularly annihilated is motivated by Mandelker's (1970) result on modular lattices.

Theorem—A finite and connected graph $G = (V, E)$ is a median graph if and only if the conditions (i)–(iii) below hold:

(i) G is modularly annihilated;

(ii) $[a, b] = SP(a, b)$ for every two points $a, b \in V$;

(iii) every convex set A of G is 2-closed.

PROOF: Assume first that the conditions (i)–(iii) hold for G; we shall show that G is a median graph. Condition (iii) implies that G cannot contain triangles. Indeed, if a, b, c constitute a triangle, then a and b constitute a convex set as the endpoints of a line, and because every convex set is 2-closed, $c \in [a, b] = \{a, b\}$, which is absurd. Let a, b and c be three points on a cycle C_k such that a shortest path between those points constitute the cycle. We show next that the cycle cannot be of length $k \geqslant 5$; we give the proof for $k = 5$ but the proof for all cases $k > 5$ is similar and hence omitted. Let $d(a, c) = \min\{d(a, c), d(a, b), d(b, c)\}$, and thus $d(a, b) = d(b, c) = 2$, because the cycle is of length 5 and the shortest paths between points a, b, c constitute the cycle. Let e be the nearest point to a on the shortest ab-path on the cycle. Then $[e, b] \subset [a, b]$ and $[a, b] \cap [c, b] \subset [e, b]$. However, $[e, b] \vee [c, b]$ contains all the points of the cycle and thus $[a, b] \cap ([c, b] \vee [e, b] \not\subset [e, b]$, which contradicts (i).

Next we show that all three points $a, b, c \in V$ have at least one median t. If there is no median t for $a, b, c \in V$, then the points on a shortest ab-path, ac-path and bc-path contain a cycle with points a', b', c' (see Fig. 1 and note that a may coincide with a', b with b' and c with c', respectively). This cycle cannot be of

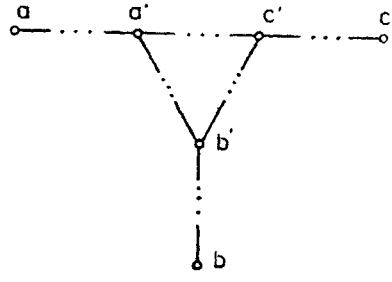

FIG. 1.

length 4, because then a shortest path between a and b, a and c, b and c can be selected such that a median exists and then the median is some among the points a', b' and c'. Thus the cycle is a triangle or C_k with $k \geqslant 5$; all these alternatives are impossible according to the proof above. Thus there is at least one median for all $a, b, c \in V$.

As a final step we show that the median is unique. Assume that $a, b, c \in V$ are three points having at least two medians t_1 and t_2. These two points cannot be on a single shortest ab-path because these points are also on a shortest ac-path as well as on a shortest bc-path. Thus t_1 and t_2 are on different shortest ab-paths, and according to the symmetry, also on different shortest ac-paths and bc-paths (see Fig. 2) According to (ii), $[a', b'] = SP(a', b')$, $[a', c'] = SP(a', c')$ and $[b', c'] = SP(b', c')$. This is possible only if $SP(t_1, t_2) \subset SP(a', b') \cap SP(a', c') \cap SP(b', c')$, which means that t_1 and t_2 are adjacent. If one of the cycles A_1, A_2, A_3, where A_1 contains the points a', t_1, t_2, A_2 the points b', t_1, t_2 and A_3 the points c', t_1, t_2, is of length 4, then one of the given shortest paths between points a', b', c' in the figure is no longer shortest; a contradiction. Thus the cycles A_1, A_2, A_3 are of length 3 or $k \geqslant 5$, which are contradictions too. Hence any three points a, b, c in G have only one median, and the first part of the theorem follows.

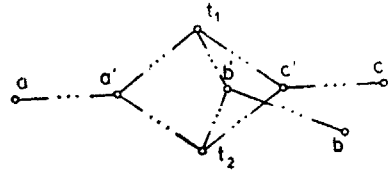

FIG. 2.

Assume conversely that G is a median graph. The points on a triangle do not have a median, and thus G does not contain triangles. If A is a convex set of G with points a and b ($a \neq b$), and if $(a, c), (b, c) \in E$, then $c \in A$, because $(a, b) \notin E$ (G does not contain triangles) and then some shortest ab-path goes over c. Hence (iii) holds for G. (i) holds because the convex sets of G constitute a median algebra (Bandelt *et al.* 1983, Theorem 2.7) and as a median algebra these sets satisfy the modularity condition of (i) (Bandelt *et al.* 1983, Chapter 3). So it remains to show that $[a, b] = SP(a, b)$ for any two points $a, b \in V$.

If $[a, b] \neq SP(a, b)$, there are at least two different shortest ab-paths, one containing the point t_1 and another the point t_2 (see Fig. 2). Moreover, there is a shortest t_1 t_2-path containing a point c' and this path is shorter (or equal) than

t_1 t_2-paths over points a' and b', respectively. Thus $c' \in [a', b']$ and $c' \notin SP(a', b')$. This construction leads to one of three absurd situations : 1) The cycle containing points t_1, b', t_2, c' is of length $k \geqslant 5$ and the points t_1, t_2, b' are on this cycle such that a shortest t_1 t_2-path over c' is shorter than the t_1 t_2-path over b'. Then t_1, t_2 and b' have no median. 2) Replace b' in (1) by a'. 3) Now we can assume that the t_1 t_2-paths over a' b' c', respectively, are of the same length two and then the points a', b' and c' have two medians. Hence also (ii) holds for G, and the theorem follows.

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