Indian J. pure appl. Math., 15(7) : 731–748, July 1984

ON THE EXTENSION OF A VECTOR-SUBMEASURE

P. K. PAUL

Department of Mathematics, Uluberia College, Uluberia, Howrah, West Bengal

AND

S. K. KUNDU

Department of Mathematics, Jadavpur University, Calcutta 700032

(*Received* 12 *April* 1983; *after revision* 11 *October* 1983)

In this paper we have discussed a method of extending an order bounded, exhausting and continuous from above submeasure (vector-valued) defined on a ring of sets $\mathscr{R}$ to a σ-ring $\mathscr{T}$ in such a manner that $\mathscr{R}$ is dense in $\mathscr{T}$ with respect to a topology which the extended submeasure induces.

1. INTRODUCTION

Drewnowski (1972a) has shown that a submeasure (real-valued) defined on a ring of subsets of a given set generates a *FN*-topology. Also, he discussed (Drewnowski 1972b) the method of extending an exhausting and order continuous submeasure (real-valued) defined on a ring of subsets of a given set uniquely to that defined on a σ-ring containing the given ring of subsets preserving the same properties.

In this paper we have generalized some of these results of Drewnowski to submeasures with values in an L-normed Banach lattice T. In doing so, we have sometimes departed from Drewnowski's definition and approach. In fact, our methods of proof rest on order—structure of T and its relationship with the norm associated with T; and as such they differ from those of Drewnowski substantially.

For details of a vector lattice we may refer to Birkhoff (1967) and Schaefer (1974).

2. PRELIMINARIES AND KNOWN RESULTS

A Banach lattice T is defined to be a real Banach space which is also a vector lattice such that $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for $x, y \in T$, where $\|\cdot\|$ is the norm of the vector space T and for any $x \in T$, $|x| = (x \vee 0) + (-x \vee 0)$, 0 being

the additive identity of T. T is called an L-normed Banach lattice or an AL-space if $\| x+y \| = \| x \| + \| y \|$ for all $x, y \geqslant 0$.

An order interval $[x, y]$, where $x, y \in T$ is the set of all $z \in T$ satisfying $x \leqslant z \leqslant y$; a subset $C \subset T$ contained in some order interval of T, is called order bounded.

T is said to be order complete if every non-void order bounded set has a least upper bound and a greatest lower bound in T.

A function $f: X \to T$ is said to be order bounded if its range is order bounded.

We use the following terminology and notations throughout the paper.

Sets are denoted by $A, B, \ldots$; ϕ means the empty set. Classes (Families) of sets will be written by $\mathscr{A}, \mathscr{B}, \ldots$ and sets of classes by $\Gamma, \Omega, \ldots$. Notation for set operations is that commonly used, in particular $A \Delta B$ means $(A \setminus B) \cup (B \setminus A)$ (symmetric difference).

For the classes $\mathscr{A}, \mathscr{C}$ we define

$$\mathscr{A} \mathring{\cup} \mathscr{B} = \{ A \cup B : A \in \mathscr{A}, B \in \mathscr{C} \}; \mathscr{A} \mathring{\cap} \mathscr{B}, \mathscr{A} \setminus \mathring{\mathscr{B}}, \mathscr{A} \mathring{\Delta} \mathscr{B}$$

are defined similarly. If $\mathscr{A} = \{A\}$, we write $A \mathring{\cup} \mathscr{B}$ etc.

Unless otherwise mentioned, T will represent an AL-space. V will represent the positive cone (the set of all positive elements ($\geqslant 0$)) of T. We write $\bar{V} = V \cup \{\lambda\}$, where λ is such that for any $x \in T$, $x < \lambda$.

We assume that $x = \lambda \Leftrightarrow \| x \| = \infty$.

If (M, τ) be a topological space and if $A \subset M$, we denote the closure of A in (M, τ) by $\bar{A}$ or $\overline{A^{\tau}}$.

If $f: X \to Y$, $A \subset X$, $f \mid A$ is the restriction of f to A.

$E_n \uparrow$ means that $E_1 \subset E_2 \subset \ldots$

$E_n \downarrow$ means that $E_1 \supset E_2 \supset \ldots$

$E_n \uparrow E$ means that $E_n \uparrow$ and $\lim\limits_{n} E_n = E$.

$E_n \downarrow E$ means that $E_n \downarrow$ and $\lim\limits_{n} E_n = E$.

η will always represent a submeasure (Def. 2.4).

$\Gamma(\eta)$ will denote the topology generated by η (Th. 3.1).

$\mathbf{N}$ will denote the set of positive integers.

Let $\mathscr{R}$ be a collection of subsets of a set $X \neq \phi$ which forms a ring under the operation of 'addition' (Δ) and 'multiplication' ($\cap$). Unless otherwise mentioned, $\mathscr{R}$ will represent such a ring of subsets of X.

The following theorems are known.

Theorem 2A (Schaefer 1974, Ch. II, § 5.10)—In any Banach lattice T, the following assertions are equivalent :

(a) T is order complete and each continuous linear form on T is order continuous.

(b) Each directed downward ($\geqslant$) family in T with infimum 0, norm converges to 0.

Theorem 2B (Schaefer 1974, Ch. II, § 8.3, § 2.4, § 1.7)—Every AL-space T is order complete and the continuity, order continuity and order boundedness of linear forms on T are equivalent.

Observations : (i) If $\{f_\alpha\} \subset V$ be a directed downward ($\geqslant$) family with $\inf_\alpha f_\alpha = f$ in V, then $\inf_\alpha \| f_\alpha \| = \| f \|$.

PROOF : $\{f_\alpha - f\} \subset V$ is a directed downward ($\geqslant$) family with infimum 0. It follows from Theorems 2A and 2B that $\lim_\alpha \| f_\alpha - f \| = 0$.

This proves that $\lim_\alpha \| f_\alpha \| = \| f \|$.

It readily follows that $\inf_\alpha \| f_i \| = \| f \|$

(ii) If $\{f\} \subset V$ be a directed upward ($\leqslant$) family with $\sup_\alpha f_\alpha = f$ in V, then $\sup_\alpha \| f_\alpha \| = \| f \|$.

The proof is an easy consequence of observation (i).

Lemma 2.1—If ν be a V-valued non-decreasing set function, defined on a class $\mathscr{B}$ of subsets of a set X, which is closed with respect to finite intersection and if $E \subset X$, then $\inf \{\nu(A) : E \subset A \in \mathscr{B}\} = a, (a \in V)$ implies $\inf \{\| \nu(A) \| : E \subset A \in \mathscr{B}\} = \| a \|$.

PROOF : It is easy to see that $P = \{\nu(A) : E \subset A \in \mathscr{B}\}$ is a directed subset ($\geqslant$) of V such that $\inf P = a$ exists in V. It now follows from observation (i) that $\inf \{\| \nu(A) \| : E \subset A \in \mathscr{B}\} = \| a \|$.

Lemma 2.2—If ν is a V-valued non-decreasing set function, defined on a class $\mathscr{B}$ of subsets of a set X, which is closed with respect to finite union and if $E \subset X$, then $\sup \{\nu(A) : E \supset A \in \mathscr{B}\} = a, (a \in V)$ implies $\sup \{\| \nu(A) \| : E \supset A \in \mathscr{B}\} = \| a \|$. Proof is similar to that of Lemma 2.1.

We require the following definitions :

Definition 2.1. (Drewnowski 1972a)—The ordered pair $(\mathscr{R}, \Gamma)$, where $\mathscr{R}$ is a ring, Γ a topology on $\mathscr{R}$ is called a topological ring of sets (TRS) if the ring operations 'Δ' and '$\cap$' are continuous with respect to the product topology.

We shall refer to Γ, as an r-topology.

Definition 2.2 (Drewnowski 1972a)— An r-topology Γ in a ring $\mathscr{R}$ is said to be a Frechet—Nikodym topology (in short, *FN*-topology) if, for each neighborhood $\mathscr{U}$ of ϕ, there is a neighborhood $\mathscr{V}$ of ϕ such that $\mathscr{V} \overset{\circ}{\cap} \mathscr{R} \subset \mathscr{U}$.

A ring equipped with *FN*-topology will be called a *FN*-ring.

Definition 2.3 (Drewnowski 1972a)—A base Ω at ϕ in $(\mathscr{R}, \Gamma)$ is called a normal base of neighborhoods of ϕ if every $\mathscr{U} \in \Omega$ is a normal subclass of $\mathscr{R}$, i.e. $B \in \mathscr{U}$ provided $B \in \mathscr{R}$ and $B \subset A$ for some $A \in \mathscr{U}$.

Definition 2.4—A non-decreasing set function $\eta : \mathscr{R} \to \bar{V}$ is called a submeasure if (i) $\eta(\phi) = 0$;

and (ii) $\| \eta(A \cup B) \| \leqslant \| \eta(A) \| + \| \eta(B) \|$, whenever $A, B \in \mathscr{R}$ and $A \cap B = \phi$.

We note that if η is a submeasure, then $\| \eta(\bigcup_{i=1}^{n} A_i) \| \leqslant \sum_{i=1}^{n} \| \eta(A_i) \|$ for every finite sequence of arbitrary sets $A_1, A_2, \ldots A_n \in \mathscr{R}$.

Definition 2.5—A $\bar{V}$-valued set function μ defined on $\mathscr{R}$ is said to be exhausting (briefly exh.) if $\lim_n \| \mu(E_n) \| = 0$ for each sequence $\{E_n\}$ of pairwise disjoint sets from $\mathscr{R}$.

Definition 2.6—A $\bar{V}$-valued set function μ defined on $\mathscr{R}$ is said to be continuous from above (briefly c. f. a.) if $\lim_n \| \mu(E_n) \| = 0$ for each sequence $\{E_n\} \subset \mathscr{R}$ such that $E_n \downarrow \phi$.

Definition 2.7—A non-empty class $\mathscr{S}$ of sets is called a σ-ring if

(a) $(E, F \in \mathscr{S}) \Rightarrow E \setminus F \in \mathscr{S}$

and

(b) $(E_i \in \mathscr{S}, i = 1, 2, \ldots) \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathscr{S}$.

Definition 2.8—If $\mathscr{R}$ and $\mathscr{S}$ and are two σ-rings such that $\mathscr{R} \subset \mathscr{S}$, then $\mathscr{S}$ is called the null completion of $\mathscr{R}$ if and only if for every $E \in \mathscr{S}$, there exist $F, H \in \mathscr{R}$ such that $F \subset E \subset H$ and $\eta(H \setminus F) = 0$.

We now state the following known theorems which we shall require in the sequel.

Theorem 2C (Drewnowski 1972a)—If $(\mathscr{R}, \Gamma)$ is a TRS and $\mathscr{F}$ is a subring of the ring $\mathscr{R}$, then $\mathscr{F}^{(\Gamma)}$ is a subring of $\mathscr{R}$.

Theorem 2D (Drewnowski 1972a)—If $(\mathscr{R}, \Gamma)$ is a TRS and Ω is a base of (the filter of all) neighborhoods of ϕ in $\mathscr{R}$, then for each $A \in \mathscr{R}$, $A \mathbin{\mathring{\Delta}} \Omega = \{A \mathbin{\mathring{\Delta}} \mathscr{U}, \mathscr{U} \in \Omega\}$ is a base of (the filter of all) neighborhoods of A in $\mathscr{R}$.

Theorem 2E (Drewnowski 1972a)—In order that a filter (or filter base) Ω of neighbourhoods of ϕ determines an r-topology in $\mathscr{R}$, it is necessary and sufficient that the following conditions be satisfied:

(a) For each $\mathscr{U} \in \Omega$ there exists $\mathscr{V} \in \Omega$ such that $\mathscr{V} \mathbin{\mathring{\Delta}} \mathscr{V} \subset \mathscr{U}$.

(b) For each $\mathscr{U} \in \Omega$ there exists $\mathscr{V} \in \Omega$ such that $\mathscr{V} \mathbin{\mathring{\Omega}} \mathscr{V} \subset \mathscr{U}$.

(c) For each $A \in \mathscr{R}$ and $\mathscr{U} \in \Omega$ there exists $\mathscr{V} \in \Omega$ such that $A \mathbin{\mathring{\cap}} \mathscr{V} \subset \mathscr{U}$.

Theorem 2F (Vladimirov 1969)—Let Ω be filter base of neighborhoods of ϕ in $\mathscr{R}$ and suppose that

(a) Each class $\mathscr{U} \in \Omega$ is normal in $\mathscr{R}$.

(b) For each $\mathscr{U} \in \Omega$ there is $\mathscr{V} \in \Omega$ such that $\mathscr{V} \mathbin{\mathring{\cup}} \mathscr{V} \subset \mathscr{U}$.

Then Ω is a normal base of neighbourhoods of ϕ for some *FN*-topology in $\mathscr{R}$.
From now onwards we take the submeasure η to be *AL*-space (T)-valued.

3. Generation of a *FN*-Topology by a Submeasure and some Properties of a c. f. a. Submeasure and also of an Exhausting Submeasure

Theorem 3.1—If η is an order bounded submeasure on $\mathscr{R}$, then the class $\mathscr{F}$ of all $\mathscr{U}_\epsilon$ $(0 < \epsilon)$, where $\mathscr{U}_\epsilon = \{A \in \mathscr{R}: \| \eta(A) \| \leqslant \epsilon\}$ constitutes a normal base of neighborhoods at ϕ for a *FN*-topology.

[This topology is called the *FN*-topology generated (or determined) by η on $\mathscr{R}$ and is denoted by $\Gamma(\eta)$].

Proof: It is easy to see that $\mathscr{F}$ is a filter base. Moreover, $\mathscr{F}$ satisfies conditions (a), (b), (c) of Theorem 2E and also conditions (a), (b) of Theorem 2F. Hence the theorem.

Theorem 3.2—Let η be a c. f. a. submeasure on $\mathscr{R}$. Suppose $\{E_n\} \subset \mathscr{R}$ and is such that $E_n \uparrow E$ (or $E_n \downarrow E$) in $\mathscr{R}$. Then $\| \eta(E) \| = \| \eta(\lim_n E_n) \| = \lim_n \| \eta(E_n) \|$.

Proof: Suppose $E_n \uparrow E$. Then $(E \setminus E_n) \downarrow \phi$.

Consequently, $\lim_n \| \eta (E \setminus E_n) \| = 0$, by definition.

Since $\| \eta (E) \| \leqslant \| \eta (E \setminus E_n) \| + \| \eta (E_n) \|$, for every n, we have $\| \eta (E) \| \leqslant \lim_n \| \eta (E_n) \|$.

Again, since $E_n \subset E$, $\lim_n \| \eta (E_n) \| \leqslant \| \eta (E) \|$.

It follows that $\| \eta (E) \| = \| \eta (\lim_n E_n) \| = \lim_n \| \eta (E_n) \|$.

The other case ($E_n \downarrow E$) may be similarly disposed of.

Theorem 3.3—Every non-decreasing sequence $\{E_n\} \subset \mathscr{R}$ is Cauchy with respect to an exh. submeasure, i.e., $\lim_{n,m} \| \eta (E_n \Delta E_m) \| = 0$.

PROOF: Let $\{E_n\} \subset \mathscr{R}$ be a non-decreasing sequence which is not cauchy. Then there is $\epsilon > 0$ and a positive integer $N(\epsilon)$ such that for all

$m, n \geqslant N(\epsilon)$, $\| \eta (E_n \Delta E_n) \| > \epsilon$.

We can find a strictly increasing sequence

$\{n_k\} \subset \mathbb{N}$ such that $\| \eta (E_{n_{k+1}} \Delta E_{n_k}) \| > \epsilon$, $k = 1, 2, \ldots$.

We set $P_{n_k} = E_{n_{k+1}} \Delta E_{n_k} = E_{n_{k+1}} \setminus E_{n_k}$.

Clearly $P_{n_k} \cap P_{n_{k+1}} = \phi$, $k = 1, 2, \ldots$.

Now $\{ P_{n_k} \} \subset \mathscr{R}$ is a disjoint sequence

such that $\| \eta (P_{n_k}) \| > \epsilon$, $k = 1, 2, \ldots$.

This contradicts the fact that η is exh.

This proves the theorem.

Theorem 3.4 —Let η be a c. f. a. submeasure on $\mathscr{R}$. Let $\{E_n\} \subset \mathscr{R}$ be disjoint with $\bigcup_{i=1}^{\infty} E_i = E$ in $\mathscr{R}$. Then $\| \eta (E) \| \leqslant \sum_{i=1}^{\infty} \| \eta (E_i) \|$.

PROOF: Let $F_n = \bigcup_{i=1}^{n} E_i$ and $G_n = E \setminus F_n$; then $F_n, G_n \in \mathscr{R}$.

Also, $F_n \uparrow$ and $G_n \downarrow \phi$.

Now, $\| \eta (E) \| = \| \eta (F_n \cup G_n) \| \leqslant \| \eta (F_n) \| + \| \eta (G_n) \|$

$$\leqslant \sum_{i=1}^{n} \| \eta (E_i) \| + \| \eta (G_n) \|.$$

It follows that $\| \eta (E) \| \leqslant \lim_n \sum_{i=1}^{n} \| \eta (E_i) \|$, since η is c. f. a.

$$= \sum_{i=1}^{\infty} \| \eta (E_i) \|.$$

Theorem 3.5—If $\mathscr{R}$ is a σ-ring and η be a c.f.a. submeasure on $\mathscr{R}$, then η is exh.

Also, if $\{E_n\} \subset \mathscr{R}$ be disjoint and be such that $\bigcup_{i=1}^{\infty} E_i = E \in \mathscr{R}$, then

$$\| \eta (E) \| \leqslant \sum_{i=1}^{\infty} \| \eta (E_i) \|.$$

PROOF : Let $\{E_n\} \subset \mathscr{R}$ be disjoint.

Let $F_n = \bigcup_{k=n}^{\infty} E_k$. Then $F_n \downarrow \phi$ so that $\lim_n \| \eta (F_n) \| = 0$. Now, since $\eta (E_n) \leqslant \eta (F_n)$ for all n, it follows that $\lim_n \| \eta (E_n) \| = 0$. Thus η is exh.

Second part follows directly from Theorem 3.4.

4. EXTENSION OF A SUBMEASURE

Theorem 4.1—Let $\mathscr{S}$ be a σ-ring generated by a ring $\mathscr{R}$ and let η be an order bounded and c. f. a. submeasure on $\mathscr{S}$. Then $\mathscr{R}$ is dense in $(\mathscr{S}, \Gamma (\eta))$.

PROOF : Clearly $\overline{\mathscr{R}} = \overline{\mathscr{R}}^{\Gamma(\eta)}$ is a subring of $\mathscr{S}$ by Theorem 2C.

Let $\{A_n\} \subset \overline{\mathscr{R}}$ be a disjoint sequence with $A = \bigcup_{n=1}^{\infty} A_n$.

Clearly $C_n = \bigcup_{k=0}^{n} A_k \in \overline{\mathscr{R}}$ for every n.

Let $B_n = A \Delta C_n = A \Delta (\bigcup_{k=1}^{n} A_k) = A \setminus (\bigcup_{k=1}^{n} A_k) = \bigcup_{k=n+1}^{\infty} A_k$.

Then $B \downarrow \phi$.

Let $\epsilon > 0$ and $\mathscr{V} = \left\{ P \in \mathscr{S} : \| \eta (P) \| \leqslant \frac{\epsilon}{2} \right\}$ be a neighborhood of ϕ in $\mathscr{S}$.

Then the neighbourhood $C_n \Delta \mathscr{V}$ of C_n must contain an element, say, $P_n = C_n \Delta W_n \in \mathscr{R}, W_n \in \mathscr{V}$.

Then

$$\| \eta (C_n \Delta P_n) \| \leqslant \frac{\epsilon}{2} \text{ for every } n. \qquad \ldots(4.1.1)$$

Now, $\lim_n \| \eta (A \Delta C_n) \| = \lim_n \| \eta (B_n) \| = 0$, since η is c. f. a. and $B_n \downarrow \phi$.

Therefore we can find a positive integer N such that $\| (\eta A \Delta C_n) \| < \frac{\epsilon}{2}$ for all $n \geqslant N$.

Thus $\| \eta (A \Delta C_N) \| < \frac{\epsilon}{2}$. $\qquad \ldots(4.1.2)$

Now, $\| \eta (A \Delta P_N) \| \leqslant \| \eta (A \Delta C_N) \| + \| \eta (C_N \Delta P_N) \|$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \text{by (4.1.1) and (4.1.2).}$$

Again, since $A \Delta P_N \in \mathscr{S}$, it follows that $A \Delta P_N \in \mathscr{U}_\epsilon$, where $\mathscr{U}_\epsilon = \{P \in \mathscr{S} : \| \eta (P) \| \leqslant \epsilon\}$ is a neighbourhood of ϕ in $\mathscr{S}$. Then $P_n = A \Delta (A \Delta P_n)$ is in $A \Delta \mathscr{U}_\epsilon$, Therefore each neighborhood of A contains an element of $\mathscr{R}$ (Theorem 2D). Hence $A \in \mathscr{R}$. Accordingly, $\mathscr{R}$ is a σ-ring and $\overline{\mathscr{P}} = \mathscr{R}$.

This completes the proof.

Theorem 4.2—Let $\mathscr{R}$ be a ring of subsets of a set X and let η be an order bounded, exh. and c. f. a. submeasure on $\mathscr{R}$. Let $\mathscr{R}_\sigma$ be the class of all sets of the form $A_1 \cup A_2 \cup \ldots$, where $A_n \in \mathscr{R}$, $n = 1, 2, \ldots$.

If for each $E \in \mathscr{R}_\sigma$,

$$\hat{\eta}(E) = \sup \{\eta (F) : F \subset E, F \in \mathscr{R}\},$$

then the following properties hold :

(i) $\hat{\eta} \mid \mathscr{R} = \eta$, $\hat{\eta}$ is non-decreasing, $\hat{\eta} (\phi) = 0$.

(ii) If $E_n \in \mathscr{R}_\sigma$ and $F = \bigcup_{n=1}^{\infty} E_n$, then $\| \hat{\eta} (F) \| \leqslant \sum_{n=1}^{\infty} \| \hat{\eta} (E_n) \|$.

(iii) If $E_n \in \mathscr{R}$, $E_n \uparrow E$, then $\lim_n \| \hat{\eta} (E \setminus E_n) \| = 0$.

(iv) $\hat{\eta}$ is exh. on $\mathscr{R}_\sigma$.

(v) $\hat{\eta}$ is c. f. a. on $\mathscr{R}_\sigma$.

PROOF : It is easy to see that $\mathscr{R}_\sigma$ is closed with respect to countable unions and finite intersections. Also, $E \setminus F \in \mathscr{R}_\sigma$ if $E \in \mathscr{R}_\sigma$ and $F \in \mathscr{R}$.

Let $\{P_n\} \subset \mathscr{R}$ such that $\bigcup_{n=1}^{\infty} P_n = E$. Then there exists a sequence $\{F_n\} \subset \mathscr{R}$,

$F_n \uparrow$, where $F_n = \bigcup_{i=1}^{n} P_i$ and $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} P_n = E.$ ...(a)

By Lemma 2.2 it follows that $\| \hat{\eta}(E) \| = \sup \{ \| \eta(F) \| : F \subset E, F \in \mathscr{R} \}$. It is obviously true that $\| \hat{\eta}(E) \| = \sup \{ \| \eta(F_n) \| : F_n \subset E, F_n \uparrow E, F_n \in \mathscr{R} \}$.

It readily follows that

$$\| \hat{\eta}(E) \| = \lim_n \| \eta(F_n) \|. \qquad \ldots(4.2.1)$$

We now proceed to prove properties (i)–(v).

(i) It is obvious.

(ii) Let $E_n \in \mathscr{R}_\sigma$ and let $E_k^n \in \mathscr{R}$ $(k, n \in \mathbf{N})$

be such that $E_k^n \uparrow_k E_n$ (as in (a)).

We write $F_m = E_m^1 \cup E_m^2 \cup \ldots \cup E_m^m$.

Then $F_m \in \mathscr{R}$. Clearly $F_m \uparrow F = \bigcup_{n=1}^{\infty} E_n$ and $\| \hat{\eta}(F) \| = \lim_m \| \eta(F_m) \|$.

Now, $\| \eta(F_m) \| = \| \eta(E_m^1 \cup E_m^2 \cup \ldots \cup E_m^m) \|$

$$\leqslant \| \eta(E_m^1) \| + \| \eta(E_m^2) \| + \ldots + \| \eta(E_m^m) \|$$

$$\leqslant \| \hat{\eta}(E_1) \| + \| \hat{\eta}(E_2) \| + \ldots + \| \hat{\eta}(E_m) \|$$

$$= \sum_{k=1}^{m} \| \hat{\eta}(E_k) \|, \quad \text{for } m = 1, 2, \ldots,$$

It follows that $\lim_m \| \eta(F_m) \| \leqslant \lim_m \sum_{k=1}^{m} \| \hat{\eta}(E_k) \|$

i.e., $\| \hat{\eta}(F) \| \leqslant \sum_{k=1}^{\infty} \| \hat{\eta}(E_k) \|$.

(iii) Since $\{E_n\} \subset \mathscr{R}$, $E_n \uparrow E$ and η is exh., it follows from Theorem 3.3 that $\{E_n\}$ is Cauchy w.r.t. η,

i.e., $\lim_{n,m} \| \eta (E_m \Delta E_n) \| = 0.$

Letting $m > n$, $E_m \Delta E_n = E_m \setminus E_n$.

Thus $\lim_{n,m} \| \hat{\eta} (E_m \setminus E_n) \| = 0.$

Since $(E_m \setminus E_n) \uparrow_m (E \setminus E_n)$, we get

$$\| \hat{\eta} (E \setminus E_n) \| = \lim_m \| \eta (E_m \setminus E_n) \|, \quad \text{for every } n,$$

and so, $\lim_n \| \hat{\eta} (E \setminus E_n) \| = 0.$

(iv) Let $E_n \in \mathscr{R}_\sigma$ such that $\{E_n\}$ is disjoint.

We have $\| \hat{\eta} (E_n) \| = \sup \{ \| \eta (P) \| : P \subset E_n, P \in \mathscr{R} \}$.

We choose $\epsilon > 0$ arbitrarily.

Then there exists some $B_n \in \mathscr{R}$ such that $B_n \subset E_n$

and

$$\| \hat{\eta} (E_n) \| < \| \eta (B_n) \| + \frac{\epsilon}{2^n} \quad n = 1, 2, \ldots.$$

Now $\{B_n\}$ is disjoint, since $\{E_n\}$ is so. Also, since η is exh., $\lim_n \| \eta (B_n) \| = 0.$

Consequently, $\lim_n \| \hat{\eta} (E_n) \| = 0.$

Thus $\hat{\eta}$ is exh. on $\mathscr{R}_\sigma$.

(v) Suppose $\hat{\eta}$ is not c.f.a. on $\mathscr{R}_\sigma$.

Let $\{P_n\} \subset \mathscr{R}_\sigma$ be such that $P_n \downarrow \phi$ but $\lim_n \| \hat{\eta} (P_n) \| \neq 0$. Then there is a real number p (> 0) such that

$$\| \hat{\eta} (P_n) \| > 2p, \quad n = 1, 2, \ldots.$$

Let $\{ P_n^i \} \subset \mathscr{R}$ be such that $P_n^i \uparrow_i P_n$, $n = 1, 2, \ldots.$

Then by (iii), $\lim_i \| \hat{\eta} (P_n \setminus P_n^i) \| = 0.$

We now find a positive integer N_n such that $\| \hat{\eta} (P_n \setminus P_n^{N_n}) \| < \frac{p}{2^n}$, $i > N_n$.

We write $P_n^{N_n} = Q_n$. Then $Q_n \subset P_n$, $Q_n \in \mathscr{R}$

and $\| \hat{\eta} (P_n \setminus Q_n) \| < \frac{p}{2^n}$, $n = 1, 2, \ldots$

Let $S_n = Q_1 \cap Q_2 \cap \ldots \cap Q_n$. Then $S_n \in \mathscr{R}$.

Also, $S_n \subset Q_n \subset P_n$.

It is easy to see that

$$\| \hat{\eta} (P_n \setminus S_n) \| \leqslant \sum_{i=1}^{n} \| \hat{\eta} (P_n \setminus Q_i) \|$$

$$\leqslant \sum_{i=1}^{n} \| \hat{\eta} (P_i \setminus Q_i) \|$$

$$< \sum_{i=1}^{n} \frac{p}{2^i}$$

$$< p, \ n = 1, 2, \ldots .$$

We now get $2p < \| \hat{\eta} (P_n) \| \leqslant \| \hat{\eta} (S_n) \| + \| \hat{\eta} (P_n \setminus S_n) \|$, $n = 1, 2, \ldots$.

Therefore $\| \hat{\eta} (S_n) \| > p$, $n = 1, 2, \ldots$.

Since $\hat{\eta} \mid \mathscr{R} = \eta$, $\| \eta (S_n) \| > p$, $n = 1, 2, \ldots$.

This is absurd since $S_n \downarrow \phi$ and η is c. f. a. on $\mathscr{R}$.

Hence $\hat{\eta}$ is c. f. a. on $\mathscr{R}_\sigma$. This completes the proof.

Theorem 4.3—Let $\mathscr{R}$ be a ring of subsets of a set X. If η be an order bounded exh. and c. f. a. submeasure on $\mathscr{R}$, then there exists a σ-ring $\mathscr{R}_0$ of subsets of X such that $\mathscr{R} \subset \mathscr{R}_0$ and η can be extended to a c. f. a. submeasure $\tilde{\eta}$ on $\mathscr{R}_0$ such that

(a) $\mathscr{R}_0 = \overline{\mathscr{R}}^{\Gamma(\tilde{\eta})}$;

(b) If $F \subset E \in \mathscr{R}_0$ and $\tilde{\eta} (E) = 0$, then $F \in \mathscr{R}_0$ and $\tilde{\eta} (F) = 0$;

(c) If a c. f. a. submeasure γ on $\mathscr{R}_0$ be such that

$\gamma \mid \mathscr{R} = \eta$, then $\| \gamma (E) \| = \| \tilde{\eta} (E) \|$ for every $E \in \mathscr{R}_0$;

(d) The σ-ring $\mathscr{R}_0$ is the null completion of the σ-ring $\mathscr{S}$ generated by $\mathscr{R}$.

PROOF : (a) We take $\mathscr{R}_\sigma$ and $\overset{\wedge}{\eta}$ as in Theorem 4.2.

Next, let $\widetilde{\mathscr{R}} = \{E : E \subset X$ and $E \subset F$ for some $F \in \mathscr{R}_\sigma\}$

Clearly $\mathscr{R}_\sigma \subset \widetilde{\mathscr{R}}$ and $\widetilde{\mathscr{R}}$ is a σ-ring.

We define for every $E \in \widetilde{\mathscr{R}}$.

$$\widetilde{\eta}(E) = \inf \{\overset{\wedge}{\eta}(F) : E \subset F \in \mathscr{R}_\sigma\}. \qquad \ldots(4.3.1)$$

Obviously $\widetilde{\eta} \mid \mathscr{R}_\sigma = \overset{\wedge}{\eta}$, $\widetilde{\eta}$ is non-decreasing and $\widetilde{\eta}(\phi) = 0$.

We first prove four properties of $\widetilde{\eta}$.

(i) If $E_n \in \widetilde{\mathscr{R}}$, $n = 1, 2, \ldots$, then $\| \widetilde{\eta} (\bigcup_{n=1}^{\infty} E_n) \| \leqslant \sum_{n=1}^{\infty} \| \widetilde{\eta} (E_n) \|$.

(ii) If $\mathscr{R}_0 = \overline{\mathscr{R}_\sigma}^{\Gamma(\widetilde{\eta})}$, then $E \in \mathscr{R}_0$ if and only if there is a sequence $\{E_n\} \subset \mathscr{R}_\sigma$, such that $\lim_n \| \widetilde{\eta} (E \Delta E_n) \| = 0$.

(iii) If $E \in \mathscr{R}_0$, then there exists a sequence $\{H_n\} \subset \mathscr{R}_\sigma$, $H_n \downarrow$, auch that $E \subset H_n$, $n = 1, 2, \ldots$, and $\lim_n \| \widetilde{\eta} (H_n \setminus E) \| = 0$.

(iv) The submeasure $\widetilde{\eta}$ is c. f. a. on $\mathscr{R}_0$.

(i) : If $\sum_{n=1}^{\infty} \| \widetilde{\eta} (E_n) \| = \infty$, then trivially

$$\| \widetilde{\eta} (\bigcup_{n=1}^{\infty} E_n) \| \leqslant \sum_{n=1}^{\infty} \| \widetilde{\eta} (E_n) \|.$$

We may, therefore, suppose that $\sum_{n=1}^{\infty} \| \widetilde{\eta} (E_n) \| < \infty$.

We choose $\epsilon > 0$ arbitrarily; it follows from the definition of $\widetilde{\eta}$ and Lemma 2.1 that for each $n \in \mathbb{N}$, there exists a set $P_n \in \mathscr{R}_\sigma$ such that $E_n \subset P_n$ and

$$\| \overset{\wedge}{\eta} (P_n) < \| \| + \widetilde{\eta} (E_n) \| \frac{\epsilon}{2^n} . \qquad \ldots(4.3.2)$$

Accordingly

$$\| \tilde{\eta} \left(\bigcup_{n=1}^{\infty} E_n \right) \| \leqslant \| \tilde{\eta} \left(\bigcup_{n=1}^{\infty} P_n \right) \|$$

$$= \| \hat{\eta} \left(\bigcup_{n=1}^{\infty} P_n \right) \|$$

$$\leqslant \sum_{n=1}^{\infty} \| \hat{\eta} (P_n) \|, \qquad \text{by Th. 4.2 (ii)}$$

$$< \sum_{n=1}^{\infty} \| \tilde{\eta} (E_n) \| + \epsilon, \qquad \text{by (4.3.1)}$$

Since ϵ is arbitrary, it follows that

$$\| \tilde{\eta} \left(\bigcup_{n=1}^{\infty} E_n \right) \| \leqslant \sum_{n=1}^{\infty} \| \tilde{\eta} (E_n \|. \qquad \ldots(4.3.3)$$

Thus $\tilde{\eta}$ is a submeasure satisfying (4.3.3).

(ii) : Let $E \in \mathscr{R}_0$. Suppose $\mathscr{V} = \{F : F \in \tilde{\mathscr{R}}, \| \tilde{\eta} (F) \| \leqslant \epsilon\}$, $\epsilon > 0$, is an arbitrary neighborhood of ϕ in $\tilde{\mathscr{R}}$. Then the neighborhood $E \Delta \mathscr{V}$ of E contains an element $E \Delta W = P$ (say) of $\mathscr{R}_\sigma$, where $W \in \mathscr{V}$. Then $\| \eta (\tilde{W}) \| \leqslant \epsilon$, i.e., $\| \tilde{\eta} (E \Delta P) \| \leqslant \epsilon$.

It now follows that given a sequence $\left\{ \frac{\epsilon}{2^n} \right\}$, there exists a sequence $\{E_n\} \subset \mathscr{R}_\sigma$ such that $\| \tilde{\eta} (E \Delta E_n) \| \leqslant \frac{\epsilon}{2^n}$, $n = 1, 2, \ldots$.

Consequently, $\lim_n \| \tilde{\eta} (E \Delta E_n) \| = 0$.

Conversely, Let $E \in \tilde{\mathscr{R}}$ and $\lim_n \| \tilde{\eta} (E \Delta E_n) \| = 0$, where $\{E_n\} \subset \mathscr{R}_\sigma$. We can easily show as in Theorem 4.1 that $E \in \overline{\mathscr{R}_\sigma}^{\Gamma(\tilde{\eta})}$.

We now use property (ii) to show that $\mathscr{R}_0 = \overline{\tilde{\mathscr{R}}}^{\Gamma(\tilde{\eta})}$. Let $\epsilon > 0$ be chosen arbitrarily. Let $E \in \mathscr{R}_0$. Then by (ii) there exists a sequence $\{E_n\} \subset \mathscr{R}_\sigma$ such that $\lim_n \| \tilde{\eta} (E \Delta E_n) \| = 0$.

Accordingly we can find a positive integer N such that $\| \tilde{\eta} (E \Delta E_N) \| < \frac{\epsilon}{2}$.

Let $E_k^n \in \mathscr{R}$ $(k, n \in \mathbf{N})$ and $E_k^n \uparrow_k E_n$.

Then by Theorem 4.2 (iii),

$$\lim_k \| \overset{\wedge}{\eta} (E_n \Delta E_k^n) \| = \lim_k \| \overset{\wedge}{\eta} (E_n \setminus E_k^n) \| = 0, \quad n = 1, 2, \ldots.$$

Since $\overset{\sim}{\eta} \mid \mathscr{R}_\sigma = \overset{\wedge}{\eta}$, it follows that $\lim_k \| \overset{\sim}{\eta} (E_n \Delta E_k^n) \| = 0, \quad n = 1, 2, \ldots.$

As before we can show that $E \in \overline{\mathscr{R}}^{\Gamma(\overset{\sim}{\eta})}$. Consequently, $\mathscr{R}_0 \subset \overline{\mathscr{R}}^{\Gamma(\overset{\sim}{\eta})}$. Also, since $\mathscr{R} \subset \mathscr{R}_\sigma$, $\overline{\mathscr{R}}^{\Gamma(\overset{\sim}{\eta})} \subset \overline{\mathscr{R}_\sigma}^{\Gamma(\overset{\sim}{\eta})} = \mathscr{R}_0$.

Hence $\mathscr{R}_0 = \overline{\mathscr{R}}^{\Gamma(\overset{\sim}{\eta})}$.

It now follows from Theorem 2C that $\mathscr{R}_0$ is a ring.

(iii) : Since $E \in \mathscr{R}_0 = \overline{\mathscr{R}}^{\Gamma(\overset{\sim}{\eta})}$, as in (ii), there exists a sequence $\{E_n\} \subset \mathscr{R}$ such that $\lim_n \| \overset{\sim}{\eta} (E \Delta E_n) \| = 0$.

Let $\epsilon > 0$ be arbitrarily chosen; it follows from the definition of $\overset{\sim}{\eta}$ and Lemma 2.1 that for each $n \in \mathbf{N}$, there exists a set $G_n \in \mathscr{R}_\sigma$ such that $E \Delta E_n \subset G_n$ and $\| \overset{\wedge}{\eta} (G_n) \| < \| \overset{\sim}{\eta} (E \Delta E_n) \| + \frac{\epsilon}{2^n}$.

Since

$$\overset{\sim}{\eta} \mid \mathscr{R}_\sigma = \overset{\wedge}{\eta}, \| \overset{\sim}{\eta} (G_n) \| < \| \overset{\sim}{\eta} (E \Delta E_n) \| + \frac{\epsilon}{2^n}. \qquad \ldots(4.3.4)$$

we write $H_n = (E_1 \cup G_1) \cap \ldots \cap (E_n \cup G_n)$.

Clearly $\{H_n\} \subset \mathscr{R}_\sigma$, $H_n \downarrow$.

Again $E = (E \setminus E_n) \cup (E \cap E_n) \subset (E \setminus E_n) \cup E_n \subset E_n \cup G_n$ for each n.

Thus $E \subset H_n$ for each n.

Now $H_n \setminus E \subset (E_n \cup G_n) \setminus E \subset G_n$. Since $\overset{\sim}{\eta}$ is non-decreasing, it readily follows from (4.3.4) that $\lim_n \| \overset{\sim}{\eta} (H_n \setminus E) \| = 0$.

(iv) : Let $\epsilon > 0$ be chosen arbitrarily.

Let $\{E_n\} \subset \mathscr{R}_0$ and $E_n \downarrow \phi$.

Then according to (iii) for each $n \in \mathbf{N}$ there exists a sequence $\{P_n^i\} \subset \mathscr{R}_\sigma$,

$P_n^i \downarrow$ such that $E_n \subset P_n^i$, $i=1, 2, \ldots$, and $\lim_i \| \tilde{\eta} (P_n^i \setminus E) \| = 0$.

We can now obtain a non-increasing sequence $\{H_n\} \subset \mathscr{R}_\sigma$, $E_n \subset H_n$ such that

$$\| \tilde{\eta} (H_n \setminus E_n) \| < \frac{\epsilon}{2^n}, \quad n = 1, 2, \ldots \qquad \ldots(4.3.5)$$

Let $H = \bigcap_{n=1}^{\infty} H_n$; then $H = H \setminus \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (H \setminus E_n) \subset \bigcup_{n=1}^{\infty} (H_n \setminus E_n)$

which gives

$$\| \tilde{\eta} (H) \| \leqslant \| \tilde{\eta} \{ \bigcup_{n=1}^{\infty} (H_n \setminus E_n) \} \|$$

$$\leqslant \sum_{n=1}^{\infty} \| \tilde{\eta} (H_n \setminus E_n) \|, \quad \text{by (i)}$$

$$< \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon, \quad \text{by (4.3.5).}$$

Since ϵ is arbitrary, it follows that

$$\| \tilde{\eta} (H) \| = 0. \qquad \ldots(4.3.6)$$

Next, suppose $\{H_n^i\} \subset \mathscr{R}$ and $H_n^i \uparrow_i H_n$.

Then by Theorem 4.2 (iii), $\lim_i \| \hat{\eta} (H_n \setminus H_n^i) \| = 0$.

Proceeding exactly as in Theorem 4.2 (v) we can find a sequence $\{D_n\} \subset \mathscr{R}$, $D_n \downarrow$, $D_n \subset H_n$ such that $\| \hat{\eta} (H_n \setminus D_n) \| < \epsilon$, $n = 1, 2, \ldots$.

Since $\tilde{\eta} \mid \mathscr{R}_\sigma = \hat{\eta}$, it follows that

$$\| \tilde{\eta} (H_n \setminus D_n) \| < \epsilon, \quad n = 1, 2, \ldots . \qquad \ldots(4.3.7)$$

Using the definition of $\tilde{\eta}$ and (4.3.6), we can show that there exists $F \in \mathscr{R}_\sigma$ such that

$$H \subset F \text{ and } \| \tilde{\eta} (F) \| < \epsilon. \qquad \ldots(4.3.8)$$

We choose $\{F_n\} \subset \mathscr{R}$ such that $F_n \uparrow F$.

We set $G_n = D_n \setminus F_n$; $G_n \in \mathscr{R}$, $n = 1, 2, \ldots$, $G_n \downarrow \phi$ and $H_n \subset (H_n \setminus D_n) \cup G_n \cup F_n$.

It follow that

$$\| \tilde{\eta} (H_n) \| \leqslant \| \tilde{\eta} \{(H_n \setminus D_n) \cup G_n \cup F_n\} \|$$

(*equation continued on p.* 746)

$$\leqslant \| \tilde{\eta}\,(H_n \setminus D_n) \| + \| \tilde{\eta}\,(G_n) \| + \| \tilde{\eta}\,(F_n) \|$$

$$\leqslant \| \tilde{\eta}\,(H_n \setminus D_n) \| + \| \eta\,(G_n) \| + \| \tilde{\eta}\,(F) \|$$

$$< 2\,\epsilon + \| \eta\,(G_n) \|, \quad \text{by (4.3.7) and (4.3.8).}$$

Since ϵ is arbitrary, it follows that $\| \tilde{\eta}\,(H_n) \| \leqslant \| \eta\,(G_n) \|$.

Since η is given to be c. f. a. on $\mathscr{R}$, $\| \eta\,(G_n) \| \to 0$ as $n \to \infty$ and hence $\lim_n \| \tilde{\eta}\,(H_n) \| = 0$

It readily follows that $\lim_n \| \tilde{\eta}\,(E_n) \| = 0$.

Thus $\tilde{\eta}$ is c. f. a. on $\mathscr{R}_0$.

We now prove that $\mathscr{R}_0$ is a σ-ring

Let $\{E_n\} \subset \mathscr{R}_0$ and $E = \bigcup_{n=1}^{\infty} E_n$.

From (iii) it follows that, given $\epsilon > 0$, there exists a set $Q_n \in \mathscr{R}_\sigma$ such that $E_n \subset Q_n$ and $\| \tilde{\eta}\,(Q_n \setminus E_n) \| < \frac{\epsilon}{2^n}$, $n = 1, 2, \ldots$. Let $Q = \bigcup_{n=1}^{\infty} Q_n$. Then $Q \in \mathscr{R}_\sigma$, $E \subset Q$ and in view of (i), $\| \tilde{\eta}\,(Q \,\Delta\, E) \| < \epsilon$. So, $E \in \mathscr{R}_0$.

Thus we have already proved that $\mathscr{R}_0$ is a σ-ring containing $\mathscr{R}$, $\tilde{\eta}$ is a c.f.a. submeasure on $\mathscr{R}_0$ and is an extension of η and the condition (a) is satisfied.

(b) : Let $E \in \mathscr{R}_0$, $\tilde{\eta}\,(E) = 0$. Then $\| \tilde{\eta}\,(E) \| = 0$.

Snce $\mathscr{R}_0 \subset \tilde{\mathscr{R}}$, $E \in \tilde{\mathscr{R}}$. Accordingly $E \subset Z$ for some $Z \in \mathscr{R}_\sigma$. Now $F \subset E$ gives $F \subset Z \in \mathscr{R}_\sigma$. Thus $F \in \tilde{\mathscr{R}}$ and $\| \tilde{\eta}\,(F) \| \leqslant \| \tilde{\eta}\,(E) \| = 0$. Consequently, $\| \tilde{\eta}\,(F) \| = 0$. So, $\tilde{\eta}\,(F) = 0$.

Next, we shall prove that $F \in \mathscr{R}_0$.

Let $\epsilon > 0$ be chosen arbitrarily. Since $E \in \mathscr{R}_0$, from (a) it follow that there exists $B \in \mathscr{R}$ such that

$$\| \tilde{\eta}\,(E \,\Delta\, B) \leqslant \epsilon. \qquad \ldots(4.3.9)$$

Since $\| \tilde{\eta}(E) \| = \| \tilde{\eta}(F) \| = 0$ and $\tilde{\eta}$ is non-decreasing, it can be shown that

$$\| \tilde{\eta}(E \cup B) \| = \| \tilde{\eta}(E \Delta B) \| = \| \tilde{\eta}(B) \| \qquad \ldots(4.3.10)$$

and

$$\| \tilde{\eta}(F \cup B) \| = \| \tilde{\eta}(F \Delta B) \| = \| \tilde{\eta}(B) \|. \qquad \ldots(4.3.11)$$

Using (4.3.9), (4.3.10), (4.3.11) we now obtain

$$\| \tilde{\eta}(F \Delta B) \| \leqslant t, \quad \text{where } B \in \mathscr{R}. \qquad \ldots(4.3.12)$$

Consequently, $F \in \mathscr{R}_0$.

(c) : Let a c. f. a. submeasure γ on $\mathscr{R}_0$ be such that $\gamma \mid \mathscr{R} = \eta$.

Let $F \in \mathscr{R}_\sigma$. Then there exists $\{F_n\} \subset \mathscr{R}$ such that $F_n \uparrow F$. From the definition of $\hat{\eta}$ it follows that $\hat{\eta}(F) \leqslant \gamma(F)$. Using (4.2.1) and Theorem 3.2 we can show that $\hat{\eta}(F) = \gamma(F)$. Thus $\gamma \mid \mathscr{R}_\sigma = \hat{\eta}$.

Let $E \in \mathscr{R}_0$; then as in (a) (iii), there exists $\{H_n\} \subset \mathscr{R}_\sigma$, $E \subset H_n$, $H_n \downarrow$ such that

$$\lim_n \| \tilde{\eta}(H_n \setminus E) \| = 0. \qquad \ldots(4.3.13)$$

This gives

$$\| \tilde{\eta}(E) \| = \lim_n \| \hat{\eta}(H_n) \| = \lim_n \| \gamma(H_n) \|. \qquad \ldots(4.3.14)$$

Let $\epsilon > 0$ be chosen arbitrarily. As $H_n \setminus E \in \tilde{\mathscr{R}}$, it follows from the definition of $\tilde{\eta}$ that for each $n \in \mathbf{N}$, there exists $G_n \in \mathscr{R}_\sigma$ such that $H_n \setminus E \subset G_n$ and $\| \hat{\eta}(G_n) \| < \| \tilde{\eta}(H_n \setminus E) \| + \frac{\epsilon}{2^n}$.

Consequently, $\lim_n \| \hat{\eta}(G_n) \| = 0$, by (4.3.13).

Since $\| \gamma(H_n \setminus E) \| \leqslant \| \gamma(G_n) \| = \| \hat{\eta}(G_n) \|$, $\| \gamma(H_n \setminus E) \| \to 0$ as $n \to \infty$.
It readily follows that $\| \gamma(E) \| = \lim_n \| \gamma(H_n) \|$.

From (4.3.14) we have $\| \gamma(E) \| = \| \tilde{\eta}(E) \|$ for every $E \in \mathscr{R}_0$.

(d) Let $E \in \mathscr{R}_0$. Then by (a) (iii) there exists a sequence $\{H_n\} \subset \mathscr{R}_\sigma$, $H_n \downarrow$ such that $E \subset H_n$, $n = 1, 2, \ldots$, and $\lim_n \| \tilde{\eta}(H_n \setminus E) \| = 0$.

Let $H = \bigcap_{n=1}^{\infty} H_n$; then $E \subset H \in \mathscr{S}$ and

so, $\| \tilde{\eta}(H \setminus E) \leqslant \tilde{\eta}(H_n \setminus E), \quad n = 1, 2, \ldots,$

Hence $\| \tilde{\eta}(H \setminus E) \| \leqslant 0.$

Again $H \setminus E \in \mathscr{R}_0$. By (a) (iii) there exists a sequence $\{P_n\} \subset \mathscr{R}_0$, $P_n \downarrow$, $H \setminus E \subset P_n$ for every n such that $\lim_n \| \overset{\wedge}{\eta}(P_n \setminus (H \setminus E)) \| = 0.$

So, $\lim_n \| \tilde{\eta}(P_n) \| = \| \tilde{\eta}(H \setminus E) \| = 0.$

Now, $H \setminus E \subset \bigcap_{n=1}^{\infty} P_n = P \in \mathscr{S}.$

Also, $\tilde{\eta}(P) = \tilde{\eta}\left(\bigcap_{n=1}^{\infty} P_n\right) \leqslant \tilde{\eta}(P_n)$ for every n.

It readily follows that $\| \tilde{\eta}(P) \| = 0.$

Now $H = (H \setminus E) \cup E \subset P \cup E.$

We thus get $H \setminus P = E \setminus P \subset E \subset H$

and $H \setminus P, P \in \mathscr{S}$ and $\| \tilde{\eta}(H \setminus (H \setminus P)) \| = \| \tilde{\eta}(H \cap P) \| = 0.$

Therefore, $H \setminus P \subset E \subset H$ and $H \setminus P, P \in \mathscr{S}$ and $\tilde{\eta}(H \setminus (H \setminus P)) = \tilde{\eta}(H \cap P) = 0.$

Thus $\mathscr{R}_0$ satisfies (d).

This completes the proof.

ACKNOWLEDGEMENT

The authors express their sincere thanks and gratefulness to the referee for his valuable suggestions to improve the paper.

REFERENCES

Birkhoff, G. (1967). Lattice Theory, 3rd edition. Amer. Math. Soc. Coll. Publication.

Drewnowski, L. (1972a). Topological Rings of Sets, Continuous Set Functions, Integration I. *Bull. Acad. Polon. Sci. Sér. Sci. Math., Astr. Phys.*, **20**, 269–76.

——— (1972b). Topological rings of sets, continuous set function, integration II. *Bull. Acad. Polon. Sci, Ser. Sér. Math., Astr. Phys.*, **20**, pp. 277–86.

Schaefer, H. H. (1974). Banach lattices and Positive Operators, Springer-Verlag, Berlin.

Vladimirov, D. A. (1969). Boolean algebra (in Russian). Moskva.