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SUMMABILITY CHARACTERIZATION FOR *B*-CONVEX AND SUPER REFLEXIVE SPACES*

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Summability characterizations for *B*-convex and super reflexive spaces are given.

A Banach space X is said to be B-convex, if there is a positive number $\epsilon < 1$ such that, for some positive integer n and for every n-sequence of vectors $\{x_i\}$, $1 \leqslant i \leqslant n$ in the unit ball U of X,

$$\inf \{\| \sum_{i=1}^{n} \epsilon_i x_i \| : \epsilon_i = \pm 1\} \leqslant n(1 - \epsilon).$$

A Banach space Y is said to be finitely representable in a Banach space X if for every finite dimensional subspace Z of Y and $0 < \epsilon < 1$, there is an isomorphism T of Z onto a subspace of X such that $\|T\| \, \|T^{-1}\| < 1 + \epsilon$.

A Banach space X is said to be super reflexive if no non-reflexive space is finitely representable in X.

B-convex and super reflexive spaces were extensively studied by James (1964, 1972, 1974). Literature on these topics can be found in Van Dulst (1978).

Suppose $T = (c_{mn})$, $m \geqslant 1$, $n \geqslant 1$, is an infinite real matrix. T is said to be sign ordered if $c_{mp} > 0$, $c_{mn} < 0$ imply $p < n$. That is, each positive entry in a row precedes every negative entry in that row. T is lower triangular if $c_{mn} = 0$ for $n > m$ for each $m \geqslant 1$. Discussion on matrices of the above type can be found in Zygmund (1959).

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If T is a lower triangular matrix and $\{x_n^m\}$, $m \geqslant 1, n \geqslant 1$, is a double sequence from the Banach space X, arranged in a matrix form, with x_n^m in the nth row and mth column, then the T-means of the double sequence is the sequence $\{t_m\}$, where $t_m = \sum_{n=1}^{m} c_{mn} x_n^m$.

Lemma 1—If X is a normed linear space and x_i, $1 \leqslant i \leqslant n$ are n vectors in the unit ball of X and if for some $\epsilon > 0$,

$$\|\sum_{i=1}^{n} x_i\| \geqslant n(1-\epsilon), \text{ then } \|\sum_{i=1}^{n} a_i x_i\| \geqslant 1-n\epsilon \text{ if } a_i \geqslant 0 \text{ and } \sum_{i=1}^{n} a_i = 1.$$

PROOF :

$$\|\sum_{i=1}^{n} a_i x_i\| \geqslant \|\sum_{i=1}^{n} x_i\| - \sum_{i=1}^{n} (1-a_i)\|x_i\|$$

$$\geqslant n(1-\epsilon) - \sum_{i=1}^{n} (1-a_i) = 1-n\epsilon.$$

The following characterization of super reflexive spaces can be found in James and Schaffer (1972).

Theorem 2—A Banach space X is super reflexive if and only if for some $\epsilon > 0$ and for some $n \geqslant 1$,

$$\inf_{1 \leqslant k \leqslant n} \|\sum_{i=1}^{k} x_i - \sum_{k+1}^{n} x_i\| \leqslant n(1-\epsilon)$$

for every n-sequence $\{x_i\}$ of vectors in the unit ball of X.

Lemma 3 – Suppose X is a B-convex space. Then there exist a positive number $\epsilon < 1$ and an integer $M \geqslant 2$ such that for every positive integer k, for every M^k vectors y_i $(1 \leqslant i \leqslant M^k)$, in the unit ball U of X and for some choice of ϵ_i $(\epsilon_i = \pm 1)$,

$$\|\sum_{i=1}^{M^k} \epsilon_i y_i\| \leqslant M^k (1-\epsilon)^k.$$

PROOF : Since X is a B-convex space, there exist an integer $M \geqslant 2$ and a positive number $\epsilon < 1$ such that for every M vectors y_i $(1 \leqslant i \leqslant M)$, in U and for some choice of ϵ_i $(\epsilon_i = \pm 1)$,

$$\sum_{i=1}^{M} \epsilon_i y_i \leqslant M(1-\epsilon).$$

Now the result follows, by induction.

A result in super reflexive spaces, some what similar to the above is the following.

Lemma 4—Suppose X is a super reflexive space. Then there exist a positive number $\delta < 1$ and an integer $M \geqslant 2$ such that for every positive integer $k \geqslant M$, for every k vectors y_i $(1 \leqslant i \leqslant k)$, in the unit ball U of X and for some j $(1 \leqslant j \leqslant k)$, depending on the sequence,

$$\|\sum_{i=1}^{j} y_i - \sum_{j+1}^{k} y_i\| \leqslant k\delta.$$

PROOF: From Theorem 2, there exist a positive $\epsilon < 1$ and $N \geqslant 2$ such that

$$\inf_{1 \leqslant k \leqslant N} \|\sum_{i=1}^{K} x_i - \sum_{k+1}^{N} x_i\| \leqslant N(1-\epsilon) \qquad \ldots(1)$$

for every N vectors x_i $(1 \leqslant i \leqslant N)$, in the unit ball.

Now, let $\delta = 1-\epsilon^2$. Then there exists a positive integer L such that $(q+1)(1-\epsilon) < q\delta$ for all $q \geqslant L$.

Write $M = LN + 1$. If $n \geqslant M$, then for some $q \geqslant L$, we have $Nq < n \leqslant N(q+1)$. Now, let $\{y_i\}$, $1 \leqslant i \leqslant n$, be any n-sequence in the unit ball. Write

$$z_i = \begin{cases} y_i \text{ if } 1 \leqslant i \leqslant n \\ 0 \text{ if } n < i \leqslant N(q+1). \end{cases}$$

Define

$$x_i = (q+1)^{-1} \left(\sum_{k=1}^{q+1} z_{(i-1)(q+1)+k}\right) \quad \text{for } 1 \leqslant i \leqslant N-1$$

and

$$x_N = (q+1)^{-1} \sum_{k=(N-1)(q+1)+1}^{N(q+1)} z_k.$$

Then $x_1, \ldots, x_N$ are in the unit ball. Hence, from (1), for some k,

$$\|\sum_{i=1}^{k} x_i - \sum_{k+1}^{N} x_i\| \leqslant N(1-\epsilon)$$

so that, for some j,

$$\|\sum_{i=1}^{j} y_i - \sum_{j+1}^{n} y_i\| \leqslant N(q+1)(1-\epsilon) < Nq\,\delta < n\,\delta.$$

Theorem 5—A Banach space X is B-convex (resp. super reflexive) if and only if for every double sequence $\{x_n^m\}$ in the unit ball U of X, there is a lower triangular (resp. lower triangular and sign ordered) matrix $T = (c_{mn})$ such that

$$\sum_{n=1}^{\infty} |\, c_{mn} \,| = 1 \quad \text{for all } m \qquad \ldots(2)$$

$$\limsup \|t_m\| < 1 \text{ where } t_m = \sum_{n=1}^{m} c_{mn}\, x_n^m . \qquad \ldots(3)$$

PROOF : We first consider the B-convex case. Suppose the condition is satisfied and X is not B-convex. Suppose $0 < \epsilon < 1$. Then, for every positive integer m, there is an m-sequence $\{x_i^m\}$, $1 \leqslant i \leqslant m$, in U such that

$$\|\sum_{i=1}^{m} \epsilon_i\, x_i^m\| > m(1 - \epsilon m^{-2}) \qquad \ldots(4)$$

for all ϵ_i ($\epsilon_i = \pm 1$).

Now, corresponding to the double sequence $\{x_n^m\}$ (with $x_n^m = 0$ if $n > m$), there exists a lower triangular matrix $T = (c_{mn})$ satisfying (2) and (3). But, by Lemma 1, applied to the sequence (sgn c_{mn}) x_n^m, we get, from (4)

$$\|t_m\| = \|\sum_{n=1}^{m} c_{mn}\, x_n^m\| \geqslant 1 - \epsilon m^{-1}.$$

Consequently, $\lim \|t_m\| = 1$, a contradiction.

Conversely suppose that X is B-convex.

Choose ϵ and M as in Lemma 3. Let p be such that, for $k \geqslant p$,

$$M(1-\epsilon)^k < 1 - \epsilon. \qquad \ldots(5)$$

Suppose $\{x_n^m\}$ is any double sequence in U. Define

$$c_{mn} = \begin{cases} 0 & \text{if } n > m \\ m^{-1} & \text{if } n \leqslant m \text{ and } m \leqslant M^p. \end{cases}$$

Suppose $M^r < m \leqslant M^{r+1}$ where $r \geqslant p$.

Write

$$y_i = \begin{cases} x_i^m & \text{if } 1 \leqslant i \leqslant m \\ 0 & \text{if } m+1 \leqslant i \leqslant M^{r+1}. \end{cases}$$

Then, by Lemma 3, for some choice of ϵ_i^m $(\epsilon_i^m = \pm 1)$,

$$\|\sum_{i+1}^{M^{r+1}} \epsilon_i^m y_i\| \leqslant M^{r+1} (1-\epsilon)^{r+1} < m(1-\epsilon) \quad \text{[by (5)]}. \qquad \ldots(6)$$

Now, define $c_{mn} = \epsilon_n^m \; m^{-1}$ if $1 \leqslant n \leqslant m$ with ϵ_n^m as above. If we write $T = (c_{mn})$, where c_{mn} are as defined above, then clearly (2) is satisfied and for $m > M^p$, we have

$$\|t_m\| = \|\sum_{l-1}^{m} c_{ml} x_i^m\| = \|\sum_{i-1}^{m} \epsilon_s^m m^{-1} y_i\| < 1 - \epsilon \quad \text{[by (6)]} \qquad \ldots(7)$$

so that (3) is also satisfied.

The proof for the super reflexive case is essentially similar, but here, making use of Lemma 4, we get, instead of (7), for large m, $\|t_m\| \leqslant \delta\ (< 1)$.

Incidentally, we also proved the following characterization.

Theorem 6—A Banach space X is B-convex (resp. super reflexive) if and only if for every double sequence $\{x_n^m\}$ in the unit ball, there is a lower triangular (resp. lower triangular and sign ordered) matrix $T = (c_{mn})$ such that

$$\sum_{n-1}^{\infty} | c_{mn} | = 1 \quad \text{for all } m$$

and

$$\liminf \|t_m\| < 1. \qquad \ldots(8)$$

Question : A closer look at the proof of Theorem 5 shows that condition (8) in Theorem 6 can be replaced by

$$\liminf \|t_m\| = 0 \qquad \ldots(9)$$

for the B-convex case. [In fact, from (6), we have,

$$\|t_{M^{r+1}}\| \leqslant (1-\epsilon)^{r+1} \quad \text{for all } r \geqslant p].$$

Is this true for the super reflexive case ?

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