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CONVOLUTION CONDITIONS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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In a recent paper Silverman *et al.* (1978) obtained characterizations for convex, starlike and spiral-like functions, in terms of convolutions. For each of these classes $\mathcal{F}$, they determined a function g, depending on $\mathcal{F}$ such that $1/z\,(f * g) \neq 0$ is both necessary and sufficient for f to be in $\mathcal{F}$. In this paper we find similar convolution conditions for a function f to belong to certain subclasses of analytic functions related to the Janowski class $P(A,B)$.

1. INTRODUCTION

The convolution or Hadamard product of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. The Polya-Schoenberg (1958) conjecture and its proof (Ruscheweyh and Sheil-Small 1973) motivated further study of properties of convolutions and led to several generalizations of the conjecture (Ruscheweyh 1977, Suffridge 1976). The work done on Hadamard products increased resulting in a wide range of applications to extremal problems in univalent functions (Ruscheweyh 1975, Ruscheweyh and Singh 1976).

In a recent paper Silverman *et al.* (1978) obtained characterizations for convex, starlike and spiral-like functions, in terms of convolutions. For each of these classes $\mathcal{F}$, they determined a function g, depending on $\mathcal{F}$, such that $\frac{1}{z}(f * g) \neq 0$ is both necessary and sufficient for f to be in $\mathcal{F}$. In this paper we obtain simialr convolution conditions for the classes $K(A, B)$, $S^*(A, B)$, $K_\lambda(A, B)$ and $S^*_\lambda(A, B)$, as described below.

Let $E = \{z : |z| < 1\}$ and $H = \{w, w$ analytic in E, $w(0) = 0, |w(z)| < 1, z \in E\}$. Then $P(A,B) = \{p : p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, -1 \leqslant B < A \leqslant 1, w \in H\}$.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E. We say that :

(i) $f \in K(A, B)$ if $\left(1 + \frac{zf''}{f'}\right) \in P(A, B)$

(ii) $f \in S^*(A, B)$ if $\frac{zf'}{f} \in P(A, B)$

(iii) $f \in K_\lambda(A, B)$ if

$$\frac{\frac{(zf')'}{f'} - i \sin \lambda}{\cos \lambda} \in P(A, B)$$

(iv) $f \in S^*_\lambda(A, B)$ if $\frac{\frac{zf'}{f} - i \sin \lambda}{\cos \lambda} \in P(A, B)$, where λ is real and satisfies $|\lambda| < \pi/2$.

2. Convolution Conditions

In the sequel f is analytic in E and satisfies $f(0) = f'(0) - 1 = 0$.

Theorem 1—A function f belongs to $K(A, B)$ in $|z| < R \leqslant 1$ if and only if

$$\frac{1}{z}\left[f * \frac{zx + \frac{z^2x(A+B) + 2z^2}{B-A}}{(1-z)^3}\right] \neq 0$$

$(|z| < R, |x| = 1)$.

Proof : The function f belongs to $K(A, B)$ if and only if $1 + \frac{zf''}{f'} \in P(A, B)$. But the functions $p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, -1 \leqslant B < A \leqslant 1, w \in H$ are subordinate to $\frac{1 + Az}{1 + Bz}$. They map the unit circle $|z| = 1$ onto the boundary of the circle on the line joining $\frac{1 - A}{1 - B}$ and $\frac{1 + A}{1 + B}$ as diameter. When $B = -1$, the image of the unit circle is the line $\operatorname{Re} z = \frac{1 - A}{2}, -1 < A \leqslant 1$. Further $\frac{(zf')'}{f'} = 1$ at $z = 0$ and $(1, 0)$ lies inside the image circle. The functions $\frac{1 + Aw(z)}{1 + Bw(z)}$ are analytic and hence map regions onto regions. Therefore every point in the interior of the unit disc goes over to an interior point of the image disc. Thus $f \in K(A, B)$ is equivalent to $\frac{(zf')'}{f'} \neq \frac{1 + Ax}{1 + Bx}$ $(|z| < R, |x| = 1, Bx \neq -1)$. ...(1)

This simplifies to

$$(1 + Bx)(zf')' - (1 + Ax)f' \neq 0. \qquad \ldots(2)$$

Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have:

$$zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

and

$$(zf'(z))' = 1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} = f' * \frac{1}{(1-z)^2}.$$

Therefore (2) becomes

$$f' * \left(\sum_{n=1}^{\infty} \{(1 + Bx)n - (1 + Ax)\} z^{n-1}\right) \neq 0.$$

Simplifying we get

$$f' * \left(\frac{(1 + Bx) - (1 - z)(1 + Ax)}{(1 - z)^2}\right) \neq 0.$$

Thus (2) is equivalent to

$$\frac{1}{z}\left[zf' * \frac{xz(B - A) + (1 + Ax)z^2}{(1 - z)^2}\right] \neq 0. \qquad \ldots(3)$$

Since $zf' * g = f * zg'$, (3) becomes

$$\frac{1}{z}\left[f * \frac{zx + \dfrac{z^2x(A + B) + 2z^2}{B - A}}{(1 - z)^3}\right] \neq 0 \; (|z| < R, |x| = 1).$$

Remark : When $B = -1$, $A = 1$ we get convolution conditions characterizing convex functions as in Silverman *et al.* (1978) with a suitable modification.

Theorem 2—f belongs to $S^*(A, B)$ in $|z| < R \leqslant 1$ if and only if

$$\frac{1}{z}\left[f * \frac{xz(B - A) + (1 + Ax)z^2}{(1 - z)^2}\right] \neq 0 \; (|z| < R, |x| = 1).$$

Proof : Since $f \in S^*(A, B)$ if and only if $g \in K(A, B)$ where

$$g(z) = \int_0^z \frac{f(\zeta)}{\zeta}\, d\zeta, \text{ we have}$$

$$\frac{1}{z}\left[g * \frac{zx + \dfrac{z^2x(A+B)+2z^2}{(B - A)}}{(1 - z)^3}\right] = \frac{1}{z}\left[f * \frac{xz(B-A) + (1 + Ax)z^2}{(1 - z)^2}\right].$$

The result follows immediately from Theorem 1.

Remark : Similar convolution conditions characterizing starlike functions as in Silverman *et al.* (1978) can be obtained from our result in the case $B = -1$ and $A = +1$.

Theorem 3—For $|z| < R \leqslant 1$, λ real with $|\lambda| < \pi/2$ and $|x| = 1$, we have $f \in K_\lambda(A, B)$ if and only if

$$\frac{1}{z}\left[f * \frac{zx + \dfrac{z^2x(\eta + B) + 2z^2}{(B - \eta)}}{(1-z)^3}\right] \neq 0 \text{ where}$$

$\eta = (A \cos \lambda + i B \sin \lambda)\, e^{-i\lambda}$.

Proof : $f \in K_\lambda(A, B)$ in $|z| < R \leqslant 1$ if and only if

$$\frac{\dfrac{e^{i\lambda}(zf')'}{f'} - i \sin \lambda}{\cos \lambda} \neq \frac{1 + Ax}{1 + Bx} \quad (|z| < R, |x| = 1, Bx \neq -1).$$

This simplifies to

$$(1 + Bx)(zf')' - (\eta x + 1) f' \neq 0 \text{ with } \eta = (A \cos \lambda + iB \sin \lambda)\, e^{-i\lambda} \qquad \ldots(4)$$

(4) can be obtained from (2) by replacing A by η. Now proceeding exactly as in Theorem 1, the result follows.

Theorem 4—For $|z| < R \leqslant 1$, λ real with $|\lambda| < \pi/2$ and $|x| = 1$, we have $f \in S_\lambda^*(A, B)$ if and only if

$$\frac{1}{z}\left[f * \frac{xz(B - \eta) + (1 + \eta x) z^2}{(1-z)^2}\right] \neq 0$$

where $\eta = (A \cos \lambda + i B \sin \lambda)\, e^{-i\lambda}$.

Proof : The result follows from Theorem 3 in the same way as Theorem 2 followed from Theorem 1.

Remark : When $B = -1$ and $A = 1$ we obtain similar convolution conditions characterizing spiral-likeness of functions as in Silverman *et al.* (1978).

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