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OSCILLATORY MHD CHANNEL FLOW AND HEAT TRANSFER

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An approximate analysis of an oscillatory MHD channel flow and heat transfer under transverse magnetic field has been carried out. The transient and steady velocity, the transient and steady magnetic field are shown graphically. The numerical values of the amplitude and phase of the skin-friction and the first and second harmonic of the rate of heat transfer are given in tabular form. It is observed that the amplitude of the skin-friction increases with increasing M (the Hartmann number) and decreases with increasing the frequency ω. Also with increasing ω, the amplitude of the first and the second harmonic of the rate of heat transfer decreases.

1. INTRODUCTION

MHD channel flows are studied since long because of their important applications in MHD generator, MHD pump, MHD flowmeter, etc. Cowling (1957), Sutton and Sherman (1965), Hughes and Young (1966), presented the steady MHD channel flows on the basis of different physical aspects. Chang and Yen (1961) studied the unsteady MHD flows and solved the problem by transform technique. However, another aspect of unsteady MHD flows is the pressure gradient oscillating in the form

$$-\frac{1}{\rho}\frac{\partial p'}{\partial x'} = A(1 + \epsilon e^{i\omega' t'}), \quad \epsilon \ll 1 \qquad \ldots(1)$$

such a study, including the effect of the induced magnetic field, is being performed here. The assumption of steady channel flow is rather unrealistic from the practical point of view. As the MHD flow is under a pressure gradient, it is possible that under some external conditions, the pressure gradient need not remain constant. So one of the physical condition on the pressure gradient is expressed as in (1), where A is a constant.

2. Mathematical Analysis

Consider a fully developed flow of an electrically conducting viscous incompressible fluid between two infinite parallel plates. The origin is taken along the centreline of the channel with x'-axis taken along the direction of the flow and y'-axis is taken normal to x'-axis which is also the direction of the applied uniform magnetic field. The plates are assumed to be non-conducting. Then the fully developed unsteady flow is governed by the following equations :

$$\frac{\partial u'}{\partial t'} = -\frac{1}{\rho}\frac{\partial p'}{\partial x'} + \nu\frac{\partial^2 u'}{\partial y'^2} + \frac{\mu_e H_0}{4\pi\rho}\frac{\partial H_x'}{\partial y'} \qquad \ldots(2)$$

$$\frac{\partial H_x'}{\partial t'} = H_0\frac{\partial u'}{\partial y'} + \eta\frac{\partial H_x'}{\partial y'^2}. \qquad \ldots(3)$$

The energy equation is given by

$$\rho c_P \frac{\partial T'}{\partial t'} = k\frac{\partial^2 T'}{\partial y'^2} + \mu\left(\frac{\partial u'}{\partial y'}\right)^2 + \frac{j_z^2}{\sigma}. \qquad \ldots(4)$$

Here t' is the time, u' the velocity component, ν the kinematic viscosity, ρ the density, μ_e the magnetic permeability, H_0 the applied magnetic field, H_x' the induced magnetic field, $\eta = (4\pi\mu_e\sigma)^{-1}$ the magnetic diffusivity, σ the electrical conductivity, T' the temperature of the fluid, k the thermal conductivity, c_p the specific heat at constant pressure, J_z is the current density. The last twot erms on the right-hand side of eqn. (4) represent the viscous and Joule dissipation terms.

The boundary conditions are

$$u'(\pm L) = 0, \quad H_x'(\pm L) = 0 \qquad \ldots(5)$$

$$T'(+L) = T_2', \quad T'(-L) = T_1'.$$

Here T_1', T_2' are the plate temperatures.

To solve equation (2) and (3), we assume that

$$u' = u_0' + \epsilon\, e^{i\omega' t'} u_1', \; H_x' = H_{x_0}' + \epsilon\, e^{i\omega' t'} H_{x_1}', \; \epsilon < 1. \qquad \ldots(6)$$

Substituting (1), (6) in eqns. (2) and (3), equating the harmonic and non-harmonic parts, neglecting the coefficients of ϵ^2, we get

$$0 = A + \nu\frac{d^2 u_0'}{dy'^2} + \frac{\mu_e H_0}{4\pi\rho}\frac{dH_{x_0}'}{dy'} \qquad \ldots(7)$$

$$i\omega' u_1' = A + \nu \frac{d^2 u_1'}{dy'^2} + \frac{\mu_e H_0}{4\pi\rho} \frac{d H_{x_1}'}{dy'} \qquad \ldots(8)$$

$$0 = H_0 \frac{d u_0'}{dy'} + \eta \frac{d^2 H_{x_0}'}{dy'^2} \qquad \ldots(9)$$

$$i\omega' H_{x_1}' = H_0 \frac{du_1'}{dy'} + \eta \frac{d^2 H_{x_1}'}{dy'^2}. \qquad \ldots(10)$$

The boundary conditions are

$$u_0 (\pm L) = u_1 (\pm L) = H_{x_0}' (\pm L) = H_{x_1}' (\pm L) = 0. \qquad \ldots(11)$$

Introducing the following non-dimensional quantities

$$y = y'/L, h_0 = H_{x_0}'/H_0 R_m, h_1 = H_{x_1}'/H_0 R_m$$

$$R_m = 4\pi\mu_e \sigma L A^*, u_0 = u_0'/A^*, u_1 = u_1'/A^* \qquad \ldots(12)$$

$$A^* = AL^2/\nu, M^2 = \mu_e^2 H_0^2 L^2 \sigma/\mu, \omega = \omega' L^2/\nu, \mathrm{Re} = LA^*/\nu$$

in eqns. (7) to (11), we have

$$\frac{d^2 u_0}{dy^2} + M^2 \frac{dh_0}{dy} = -1 \qquad \ldots(13)$$

$$\frac{d^2 u_1}{dy^2} + M^2 \frac{dh_1}{dy} + 1 = i\omega u_1 \qquad \ldots(14)$$

$$\frac{du_0}{dy} + \frac{d^2 h_0}{dy^2} = 0 \qquad \ldots(15)$$

$$\frac{du_1}{dy} + \frac{d^2 h_1}{dy^2} = i\omega \frac{R_m}{\mathrm{Re}} h_1 \qquad \ldots(16)$$

$$u_0 (\pm 1) = 0, u_1 (\pm 1) = 0, h_0 (\pm 1) = 0, h_1 (\pm 1) = 0 \qquad \ldots(17)$$

Here R_m is the magnetic Reynolds number and M is the Hartmann number.

The solutions of eqns (13)-(16) satisfying the boundary conditions (17) are as follows:

$$u_0 = \frac{\cosh M - \cosh My}{M \sinh M} \qquad \ldots(18)$$

$$h_0 = \frac{\sinh My - y \sinh M}{M^2 \sinh M} \qquad \ldots(19)$$

$$u_1 = \left\{ \frac{x_1 \sinh b_2 \cosh b_1 y - x_2 \sinh b_1 \cosh b_2 y}{x_2 \sinh b_1 \cosh b_2 - x_1 \sinh b_2 \cosh b_1} + 1 \right\} / i\omega \qquad \ldots(20)$$

$$h_1 = \frac{\sinh b_2 \sinh b_1 y - \sinh b_1 \sinh b_2 y}{x_2 \sinh b_1 \cosh b_2 - x_1 \sinh b_2 \cosh b_1}. \qquad \ldots(21)$$

The constants x_1, x_2, b_1, b_2 are given in Appendix.

Substituting (18)–(21) in equations (6), we can get the expressions for the velocity and magnetic field.

In order to get the physical insight of the problem, we study the transient velocity and the transient magnetic field which are given by, for $\omega t = \pi/2$,

$$u = u_0 - \epsilon M_i \qquad \ldots(22)$$

$$h = h_0 - \epsilon h_i \qquad \ldots(23)$$

where $M_r + iM_i = u_1$, $h_r + ih_i = h_1$.

The transient and steady velocities are shown in Fig. 1 for $R_m/\mathrm{Re} = 0.02$, $\epsilon = 0.2$. We observe from this figure that an increase in ω or M leads to a decrease in the transient velocity and the velocity tends to flatten at the centre-line of the channel.

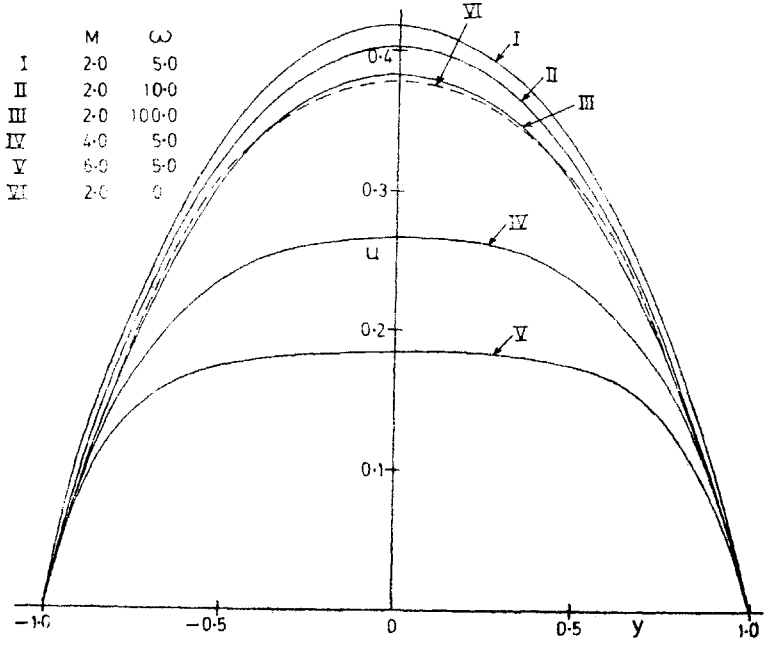

FIG. 1.

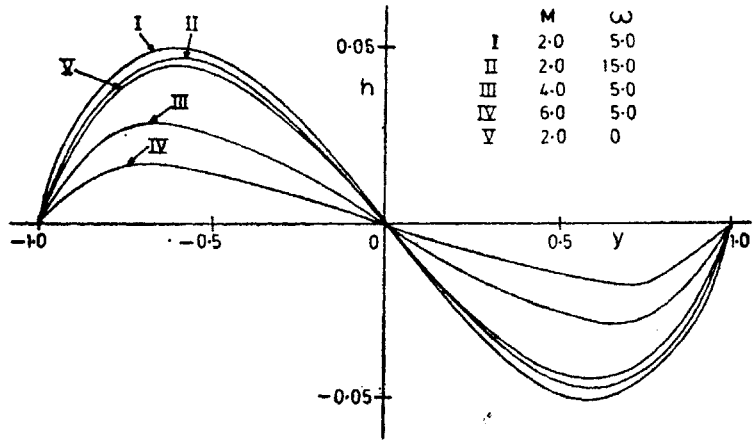

FIG. 2.

Also, at small value of ω, the transient velocity increases as compared to the steady state velocity and the transient velocity approaches the steady state as the frequency ω increases. In Fig. 2, the transient magnetic field is shown and we observe that the transient magnetic field decreases with increasing ω.

If we put $\theta = T' - T_1' / T_2' - T_1'$, where T_1' and T_2' are the temperatures of the lower and upper plates respectively, equation (4) becomes

$$\rho c_p \ (T_2' - T_1') \ \frac{\partial \theta}{\partial t'} = k \ (T_2' - T_1') \ \frac{\partial^2 \theta}{\partial y'^2} + \mu \left(\frac{\partial u'}{\partial y'} \right)^2 \frac{j_z^2}{\sigma}. \qquad \ldots(24)$$

We now assume

$$u = u_0 + \frac{\epsilon}{2} \left(e^{i\omega' t'} u_1 + e^{-i\omega' t'} \bar{u}_1 \right) \qquad \ldots(25)$$

$$\theta = \theta_0 + \frac{\epsilon}{2} \left(e^{i\omega' t'} \theta_1 + e^{-i\omega' t'} \bar{\theta}_1 \right) + \frac{\epsilon^2}{2} \left(e^{2i\omega' t'} \theta_2 + e^{-2i\omega' t'} \bar{\theta}_2 \right) \qquad \ldots(26)$$

$$j_z = j_0 + \frac{\epsilon}{2} \left[e^{i\omega' t'} j_1 + e^{-i\omega' t'} \bar{j}_1 \right]. \qquad \ldots(27)$$

The term in ϵ^2 is not taken into consideration in equation (25) because its contribution is negligibly small. Substituting eqns. (25)–(27) in equation (24), equating harmonic and non-harmonic terms, neglecting the coefficients of ϵ^2, we have

$$0 = k \Delta T' \ \frac{d^2 \theta_0}{dy'^2} + \mu \left\{ \left(\frac{du_0}{dy'} \right)^2 + \frac{\epsilon^2}{2} \left(\frac{du_1}{dy'} \right) \left(\frac{d\bar{u}_1}{dy'} \right) \right\} + \frac{1}{\sigma} \left\{ j_0^2 + \frac{\epsilon^2}{2} j_1 \bar{j}_1 \right\} \qquad \ldots(28)$$

$$\rho c_p \Delta T' i\omega' \theta_1 = k\Delta T' \frac{d^2\theta_1}{dy'^2} + 2\mu \frac{du_0'}{dy'} \frac{du_1'}{dy'} + \frac{2}{\sigma} j_0 j_1 \qquad \ldots(29)$$

$$\rho c_p \Delta T' i\omega' \theta_2 = \frac{1}{2} k\Delta T' \cdot \frac{d^2\theta_2}{dy'^2} + \frac{\mu}{4}\left(\frac{du_1'}{dy'}\right)^2 + \frac{1}{4\sigma} j_1^2 \qquad \ldots(30)$$

where $\Delta T' = T_2' - T_1'$.

In view of (12), eqns. (28)–(30) reduce to the following:

$$\frac{d^2\theta_0}{dy^2} + PE\left\{\left(\frac{du_0}{dy}\right)^2 + \frac{\epsilon^2}{2}\left(\frac{du_1}{dy}\right)\left(\frac{d\bar{u}_1}{dy}\right)\right\} + M^2 PE\left\{J_0^2 + \frac{\epsilon^2}{2} j_1 \bar{j}_1\right\} = 0 \qquad \ldots(31)$$

$$i\omega P\theta_1 = \frac{d^2\theta_1}{dy^2} + 2PE\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right) + 2M^2 PE\, j_0 \bar{j}_1 \qquad \ldots(32)$$

$$i\omega P\theta_2 = \frac{1}{2}\frac{d^2\theta_2}{dy^2} + \frac{1}{4} PE\left(\frac{du_1}{dy}\right)^2 + \frac{1}{4} M^2 PE\, J_1^2 \qquad \ldots(33)$$

where

$$P = \frac{\mu c_p}{k}, \text{ Prandtl number, } E = \frac{A^{*2}}{c_p \Delta T'}, \text{ Eckert number,}$$

$$J = j_z/\sigma\mu_e H_0 A^*$$

and

$$J_0 = -\frac{dh_0}{dy}, \; J_1 = -\frac{dh_1}{dy}$$

and the boundary conditions are

$$\theta_0(-1) = 0, \; \theta_0(+1) = 0, \; \theta_1(\pm 1) = 0, \; \theta_2 = (\pm 1) = 0. \qquad \ldots(34)$$

The solutions of equations (31)–(33), satisfying the boundary conditions (34) are as follows :

$$\theta_0 = \frac{y+1}{2} + P_1(y) - P_1(1) \qquad \ldots(35)$$

where

$$P_1(y) = -PE\left[P_{11}(y) + \frac{\epsilon^2}{2} P_{12}(y) + M^2\left\{P_{13}(y) + \frac{\epsilon^2}{2} P_{14}(y)\right\}\right]$$

$$P_{11}(y) = X_0^2 (\cosh 2My - 2M^2 y^2)/8$$

$$P_{12}(y) = X_8 \cosh b_5 y + X_9 \cosh b_6 y + X_{10} \cosh b_7 y + X_{11} \cosh b_8 y + X_{12} \cosh b_9 y + X_{13} \cosh b_{10} y + X_{14} \cosh b_{11} y + X_{15} \cosh b_{12} y$$

$$P_{13}(y) = X_0^2 \cosh 2My/8M^2 + 2X_0 \cosh My/M^4 + (X_0^2 M^4 + 2)\, y^2/4M^4$$

$$P_{14}(y) = X_{16} \cosh b_5 y + X_{17} \cosh b_6 y + X_{18} \cosh b_7 y + X_{19} \cosh b_8 y + X_{20} \cosh b_9 y + X_{21} \cosh b_{10} y + X_{22} \cosh b_{11} y + X_{23} \cosh b_{12} y$$

and the constants X_0, X_8,..., X_{23}, b_5, b_6,..., b_{12} are given in the Appendix.

$$\theta_1(y) = \{P_2(y) \cosh b_3 - P_2(1) \cosh b_3 y\}/\cosh b_3 \qquad \ldots(36)$$

where

$$P_2(y) = -2PE\{P_{21}(y) - M^2 P_{22}(y)\}$$

$$P_{21}(y) = X_{24} \cosh b_{13} y + X_{25} \cosh b_{14} y + X_{26} \cosh b_{15} y + X_{27} \cosh b_{16} y$$

$$P_{22}(y) = X_{28} \cosh b_{13} y + X_{29} \cosh b_{14} y + X_{30} \cosh b_{15} y + X_{31} \cosh b_{16} y + X_{32} \cosh b_1 y + X_{33} \cosh b_2 y$$

and the constants X_{24},..., X_{33}. b_{13},..., b_{16} are defined in the Appendix.

$$\theta_2(y) = \{P_3(y) \cosh b_4 - P_3(1) \cosh b_4 y\}/\cosh b_4 \qquad \ldots(37)$$

where

$$P_3(y) = -\frac{1}{2} PE\{P_{31}(y) + M^2 P_{32}(y)\}$$

$$P_{31}(y) = X_{34} \cosh 2b_1 y + X_{35} \cosh b_{17} y + X_{36} \cosh b_{18} y + X_{37} \cosh 2b_2 y + X_{38}$$

$$P_{32}(y) = X_{39} \cosh 2b_1 y + X_{40} \cosh b_{17} y + X_{41} \cosh b_{18} y + X_{42} \cosh 2b_2 y + X_{43}$$

and the constants X_{34},...,X_{43}, b_{17}, b_{18} are defined in the Appendix.

Substituting (35), (36) and (37) in eqn. (26), we get the expression for the temperature.

Knowing the velocity field, we can now calculate the skin-friction. It is given by

$$\tau' = -\mu \left.\frac{\partial u'}{\partial y'}\right|_{y'=\pm L} \qquad \ldots(38)$$

and in view of (12), (38) reduces to

$$\tau = -\tau'/\rho A^* = \left.\frac{du}{dy}\right|_{y=\pm 1} = \left.\frac{du_0}{dy}\right|_{y=\pm 1} + \epsilon e^{i\omega t} \left.\frac{du_1}{dy}\right|_{y=\pm 1}. \qquad \ldots(39)$$

The expression for the skin-friction can be expressed in terms of the amplitude and phase as

$$\tau = \tau_0 + \epsilon \mid B \mid \cos(\omega t + \alpha) \qquad \ldots(40)$$

where

$$\left.\begin{aligned} B = B_r + iB_i &= \left.\frac{du_1}{dy}\right|_{y=\pm 1} \\ \text{and} \qquad \tan\alpha &= B_i/B_r. \end{aligned}\right\} \qquad \ldots(41)$$

We have calculated the numerical values of $\mid B \mid$ and $\tan\alpha$ from (41) and these are tabulated in Table I. We observe from

TABLE I

Values of the amplitude $\mid B \mid$ *and phase* $\tan\alpha$

M	R_m/Rc	ω	$\mid B \mid$	$\tan\alpha$
2.0	0.02	5.0	0.5607	−0.9953
2.0	0.02	10.0	0.3528	−1.193
2.0	0.02	15.0	0.2730	−1.174
2.0	0.02	100.0	0.1002	−1.033
2.0	0.01	5.0	0.5607	−0.9942
2.0	0.01	15.0	0.2732	−1.172
4.0	0.02	5.0	0.6986	−0.8445

TableI that increase in ω leads to a decrease in the value of $\mid B \mid$ whereas an increase in M leads to an increase in the value of $\mid B \mid$. Also, there is always a phase-lag in case of the skin-friction.

From the temperature field, we now study the effects of ω on the rate of heat transfer. It is given by

$$q' = -k \left.\frac{\partial T'}{\partial y'}\right|_{y'=\pm L} \qquad \ldots(42)$$

and in view af (12), (42) reduces to

$$\begin{aligned} q = -\frac{q' L}{k\Delta T} &= \left.\frac{d\theta}{dy}\right|_{y=\pm 1} \\ &= \left.\frac{d\theta_0}{dy}\right|_{y=\pm 1} + \epsilon e^{i\omega t} \left.\frac{d\theta_1}{dy}\right|_{y=\pm 1} + \epsilon^2 e^{2i\omega t} \left.\frac{d\theta_2}{dy}\right|_{y=\pm 1}. \end{aligned} \qquad \ldots(43)$$

We have calculated the mean rate of heat transfer $q_m = d\theta_0/dy \mid_{y=\pm 1}$ and its values are tabulated in Table II. The mean rate of heat transfer is less at the upper plate

as compared to that at the lower plate. It decreases with increasing M, q can be expressed in terms of the amplitude and phase as

$$q = q_m + \epsilon \mid Q_1 \mid \cos(\omega t + \alpha_1) + \epsilon^2 \mid Q_2 \mid \cos(2\omega t + \alpha_2) \qquad \ldots(44)$$

TABLE II

Values of q_m, $\mid Q_1 \mid$, $\mid Q_2 \mid$, $\tan \alpha_1$, $\tan \alpha_2$

$P = 0.71$				q_m					
M	R_m/Re	ω	E	$y = -1$	$y = +1$	$\mid Q_1 \mid$	$\tan \alpha_1$	$\mid Q_2 \mid$	$\tan \alpha_2$
2.0	0.05	5.0	0.01	0.5019	0·4981	0.1509×10^{-2}	−3.074	0.1686×10^{-3}	1.495
2.0	0.02	10.0	0.01	0.5019	0.4981	0.7413×10^{-3}	−6.863	0.4947×10^{-4}	0.7764
2.0	0.02	15.0	0.01	0.5019	0.4981	0.4942×10^{-3}	−9.121	0.2350×10^{-4}	0.7077
2.0	0.02	100.0	0.01	0.5019	0.4981	0.7593×10^{-4}	−47.63	0.1120×10^{-5}	0.9066
2.0	0.02	5.0	0.02	0.5038	0.4962	0.3019×10^{-2}	−3.074	0.3372×10^{-3}	1.495
2.0	0.01	5.0	0.01	0.5019	0.4981	0.1509×10^{-2}	−3.066	0.1686×10^{-3}	1.500
4.0	0.02	5.0	0.01	0.5013	0.4987	0.1475×10^{-2}	−1.581	0.2272×10^{-3}	4.780

where

$$Q_1 = Q_{1r} + iQ_{1i} = \left.\frac{d\theta_1}{dy}\right|_{y=\pm1}$$

$$Q_2 = Q_{2r} + iQ_{2i} = \left.\frac{d\theta_2}{dy}\right|_{y=\pm1} \qquad \ldots(45)$$

$$\tan \alpha_1 = Q_{1i}/Q_{1r},\ \tan \alpha_2 = Q_{2i}/Q_{2r}.$$

The numerical values of $\mid Q_1 \mid$, $\tan \alpha_1$, $\mid Q_2 \mid$ and $\tan \alpha_2$ are tabulated in Table II. We observe from this table that an increase in ω leads to a decrease in the value of $\mid Q_1 \mid$ and $\mid Q_2 \mid$. Greater viscous dissipative heat causes an increase in the values of $\mid Q_1 \mid$, $\mid Q_2 \mid$. But $\mid Q_1 \mid$ decreases and $\mid Q_2 \mid$ increases with increasing M. In case of the first harmonic of the rate of heat transfer, there is always a phase-lag whereas in case of the second harmonic, there is always a phase-lead.

3. CONCLUSIONS

(1) The transient velocity decreases with increasing ω or M and at very large values of ω, the transient velocity is less than the steady state velocity. (2) The transient magnetic field decreases with increasing ω. (3) $\mid B \mid$ increases with increasing M and decreases with increasing ω. (4) There is always a phase-lag in case of the skin-friction. (5) The mean rate of heat transfer decreases with increasing M. (6) An increase in ω leads to a decrease in the value of $\mid Q_1 \mid$, $\mid Q_2 \mid$. (7) $\mid Q_1 \mid$, $\mid Q_2 \mid$ increase due to greater viscous dissipative heat. (8) $\mid Q_1 \mid$ decreases and $\mid Q_2 \mid$

increases with increasing M. (9) There is always a phase-lag in case of the first harmonic and a phase-lead in case of the second harmonic of the rate of heat transfer.

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APPENDIX

$$b_{1,2}^2 = \tfrac{1}{2}\{M^2 + i\omega(N+1) \pm \sqrt{[M^2 + i\omega(N+1)]^2 + 4\omega^2 N}\}$$

$$b_3 = \sqrt{i\omega P},\ b_4 = \sqrt{2i\omega P},\ b_5 = b_1 + \bar{b}_1,\ b_6 = b_1 - \bar{b}_1,$$

$$b_7 = b_1 + \bar{b}_2,\ b_8 = b_1 - \bar{b}_2,\ b_9 = b_2 + \bar{b}_1,\ b_{10} = b_2 - \bar{b}_1,$$

$$b_{11} = b_2 + \bar{b}_2,\ b_{12} = b_2 - \bar{b}_2,\ b_{13} = M + b_1,\ b_{14},\ = M - b_1$$

$$b_{15} = M + b_2,\ b_{16} = M - b_2,\ b_{17} = b_1 + b_2,\ b_{18} = b_1 - b_2$$

$$X_0 = -1/M \sinh M,\ X_1 = b_1(M^2 + i\omega N - b_1^2)$$

$$X_2 = b_2(M^2 + i\omega N - b_2^2),\ X_3 = X_2 \sinh b_1 \cosh b_2 - X_1 \sinh b_2 \cosh b_1$$

$$X_4 = b_1 X_2 \sinh b_2/X_3 i\omega, \qquad X_5 = -b_2 X_2 \sinh b_1/X_3 i\omega$$

$$X_6 = b_1 \sinh b_2/X_3, \qquad X_7 = -b_2 \sinh b_1/X_3$$

$$X_8 = X_4 \bar{X}_4/2b_5^2 \qquad X_9 = -X_4 \bar{X}_4/2b_6^2$$

$$X_{10} = X_2 \bar{X}_5/2b_7^2 \qquad X_{11} = -X_4 \bar{X}_5/2b_8^2$$

$$X_{12} = X_5 \bar{X}_4/2b_9^2 \qquad X_{13} = -X_5 \bar{X}_4/2b_{10}^2$$

$$X_{14} = X_5 \bar{X}_5/2b_{11}^2 \qquad X_{15} = -X\bar{X}_5 \bar{X}_5/2b_{12}^2$$

$$X_{16} = X_6 \bar{X}_6/2b_5^2 \qquad X_{17} = X_6 \bar{X}_6/2b_6^2$$

$$X_{18} = X_6 \bar{X}_7/2b_7^2 \qquad X_{19} = X_6 \bar{X}_7/2b_8^2$$

$$X_{20} = X_7 \bar{X}_6/2b_9^2 \qquad X_{21} = X_7 \bar{X}_6/2b_{10}^2$$

$$X_{22} = X_7 \bar{X}_7/2b_{11}^2 \qquad X_{23} = X_7 \bar{X}_7/2b_{12}^2.$$

In the following $f(t) = t^2 - i\omega P$ and $g(t) = t^2 - 2i\omega P$.

$$X_{24} = X_0 MX_4/2f(b_{13}) \qquad X_{25} = -X_0 MX_4/2f(b_{14})$$

$$X_{26} = X_0 MX_5/2f(b_{15}) \qquad X_{27} = -X_0 M X_5/2f(b_{16})$$

$$X_{28} = X_0 X_6/2f(b_{13}) \qquad X_{29} = X_0 X_6/2f(b_{14})$$

$$X_{30} = X_0 X_7/2f(b_{15}) \qquad X_{31} = X_0 X_7/2f(b_{16})$$

$$X_{32} = X_6/M^2 f(b_1) \qquad X_{33} = X_7/M^2 f(b_2)$$

$$X_{34} = X_4^2/2g(2b_1) \qquad X_{35} = X_4 X_5/g(b_{17})$$

$$X_{36} = -X_4 X_5/g(b_{18}) \qquad X_{37} = X_5^2/2g(2b_2)$$

$$X_{38} = (X_4^2 + X_5^2)/4i\omega P \qquad X_{39} = X_6^2/2g(2b_1)$$

$$X_{40} = X_6 X_7/g(b_{17}) \qquad X_{41} = X_6 X_7/2g(b_{18})$$

$$X_{42} = X_7^2/2g(2b_2) \qquad X_{43} = -(X_6^2 + X_7^2)/4i\omega P.$$