Indian J. pure appl. Math., **15**(8) : 879–884, August 1984

HOMOGENEOUS BANACH SPACES OF DISTRIBUTIONS

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(*Received* 6 *April* 1984)

It is shown that if E is a Banach space of distributions, then E and E^* are $(C, 1)$ complementary iff E is homogeneous. It is also shown that if a homogeneous Banach space of distributions is reflexive, then its dual is also homogeneous. A relationship between convolution and translation operators on these spaces is also established.

1. INTRODUCTION

The space of distributions on $G = R/2\pi Z$ is considered in Edwards (1967, vol. II, p. 50). In this paper we consider Banach spaces of distributions on G. *BK* spaces, which are nothing but generalizations of Banach spaces of distributions, were considered in Goes (1961, 1976). Using a result of Goes (1976) we can easily show that a Banach space of distributions is homogeneous iff the set of all trigonometric polynomials is dense in it. We show that the dual space of a Banach space of distributions is $(C, 1)$ complementary iff the given space is homogeneous. We also show that if a homogeneous Banach space of distributions is reflexive, then its dual space is also homogeneous. A relationship between the convolution and translation operators is also established for these spaces.

2. DEFINITIONS, NOTATIONS AND PRELIMINARY RESULTS

We refer to Edwards (1967) for all the standard definitions, notations and assumptions. In particular D will denote the space of all distributions on G.

Throughout the paper E will denote a Banach space of distributions satisfying the following properties :

(2.1) $C^\infty \subset E \subset D$.

(2.2) The inclusion map $i : E \to D$ is continuous.

(2.3) $f \in E \Rightarrow T_x f \in E$ and $\| T_x f \|_E = \| f \|_E$ for all real x, where $\| f \|_E$ denotes the norm of f in E.

(2.4) $f \in E \Rightarrow \check{f} \in E$ and $\|\check{f}\|_E = \|f\|_E$, where $\check{f}(u) = f(\check{u})$ for all $u \in C^\infty$ and $\check{u}(t) = u(-t)$ for all $t \in [0, 2\pi]$.

It can be checked easily that the spaces C, M, C_m, D_m as defined in Edwards (1967, Vol. 11) are Banach spaces of distributions satisfying the above properties. Orlicz spaces, as defined in Zygmund (1968), also satisfy the above properties.

E will be called homogeneous if for every $f \in E$, the function $x \to T_x f$ is continuous from R to E.

We first establish some useful properties of E and E^* in the following two lemmas :

Lemma 1—(a) The inclusion map $i : C^\infty \to E$ is continuous.

(b) If, for $f \in C^\infty$, we define $j(f)(u) = u(f)$ for all $u \in E$ then $j : C^\infty \to E^*$ is one-one and continuous.

PROOF : (a) Let $u_n \to u$ in C^∞ and $i(u_n) = u_n \to v$ in E as $n \to \infty$. By (2.2), $u_n \to v$ in D and hence $\hat{u}_n(k) \to \hat{v}(k)$ for all $k \in Z$. But $\hat{u}_n(k)$ also tends to $\hat{u}(k)$ for all k. Therefore $\hat{u}(k) = \hat{v}(k)$ for all k and hence $u = v$. By the closed graph theorem, i is continuous.

(c) We can easily show that $j(f) \in E^*$ for every $f \in C^\infty$. To show the continuity of j, let $f_n \to f$ in C^∞ and $j(f_n) \to g$ in E^* as $n \to \infty$. Then, $g(u) = \lim_{n\to\infty} j(f_n)(u) = \lim_{n\to\infty} u(f_n) = u(f)$ for all $u \in E$. Hence $g = j(f)$. By the closed graph theorem, j is continuous.

Now suppose $j(f_1) = j(f_2)$ for f_1, f_2 in C^∞. Then $j(f_1)(u) = j(f_2)(u)$ for all $u \in E$ i. e. $u(f_1) = u(f_2)$ for all $u \in E$. Taking $u = e_k$, we get $\hat{f}_1(k) = \hat{f}_2(k)$ for all k. Hence $f_1 = f_2$. Therefore j is one-one.

Remark : Given any $f \in C^\infty$, in view of the above lemma, we can identify $j(f)$ with f, and hence, we can write $u(f) = f(u)$ for all $u \in E$.

Lemma 2—(a) P, the set of all trigonometric polynomials, is dense in E iff C^∞ is dense in E.

(b) If C^∞ is dense in E, then $j : E^* \to D$, defined by $j(F)(u) = F(u)$ for all $u \in C^\infty$, is one-one and continuous. So, in this case, E^* can also be thought of as a Banach space of distributions satisfying (2.1) and (2.2).

PROOF : (a) Necessity is obvious as $P \subset C^\infty$. Sufficiency can be proved easily using the Lemma 1 (a) and the fact that P is dense in C^∞.

(b) Since C^∞ is dense in E, every continuous linear functional on E is uniquely determined by its restriction to C^∞. This shows that j is one-one. The continuity of j can be proved using the closed graph theorem.

Theorem 1—Let P be dense in E. Then, for $\mu \in M, f \in E$ and $F \in E^*$,

(a) $F^* f \in C$ and $\| F^* f \|_\infty \leqslant \| F \|_{E_*} \| f \|_E$.

(b) $\mu^* F$, defined for each u in E by $\mu^* F(u) = \mu ((F^* \check{\check{u}}))$, is a continuous linear functional on E and

$$\| \mu^* F \|_{E_*} \leqslant \| \mu \|_1 \| F \|_{E_*}.$$

(c) $\mu^* f \in E$ and $\| \mu^* f \|_E \leqslant \| \mu \|_1 \| f \|_E$.

PROOF : (a) Since E^* can be thought of a Banach space of distributions $F^* f$ is defined as a distribution [Edwards 1967, Vol. II, p. 73].

Let us define $g(x) = F(T_x \check{f})$ for all x.

Find $\{u_n\} \subset P$ such that $\| u_n - f \|_E \to 0$ as $n \to \infty$.

Now, $F^* u_n (x) = F(T_x \check{u}_n)$ for all n. Hence

$$| F^* u_n (x) - g(x) | \leqslant \| F \|_{E_*} \| T_x (u_n - f) \|_E$$

$$= \| F \|_{E_*} \| u_n - f \|_E$$

for all x. Therefore

$$\| F^* u_n - g \|_\infty \leqslant \| F \|_{E_*} \| u_n - f \|_E.$$

The last expression tends to zero as $n \to \infty$. Therefore $F^* f = \lim_{n \to \infty} F^* u_n = g \in C$.

(b) Clearly $\mu^* F$ is a linear functional on E. Now for each $u \in E$,

$$| \mu^* F(u) | = | \mu ((F^* \check{\check{u}})) | \leqslant \| \mu \|_1 \| F^* u \|_\infty$$

$$\leqslant \| \mu \|_1 \| F \|_{E_*} \| u \|_E$$

by part (a). Hence

$$\| \mu^* F \|_{E_*} \leqslant \| \mu \|_1 \| F \|_{E_*}.$$

(c) First we note that, for any $G \in E^*$,

$G(\mu^* e_n) = G(\hat{\mu}(n) e_n) = \hat{\mu}(n) G(-n) = \check{\mu}^* G(e_n)$; and hence, for any $t \in P$,

$$| G(\mu^* t) | = | \check{\mu}^* G(t) | \leqslant \| \mu \|_1 \| G \|_{E_*} \| t \|_E.$$

Therefore $\| \mu^* t \|_{E^{**}} \leqslant \| \mu \|_1 \| t \|_E$. But $\mu^* t \in P \subset E$, hence

$$\| \mu^* t \|_E = \| \mu^* t \|_{E^{**}} \leqslant \| \mu \|_1 \| t \|_E.$$

Since P is dense in E, t can be replaced by f in the above result.

3. Homogeneous Banach Spaces of Distributions

Every Banach space of distributions satisfying the axioms (2.1) to (2.4) is a translation invariant BK space as defined in Goes (1976). So the Theorem 4.1 (ii) of Goes (1976) can be stated in the following way.

Theorem 2 (Goes 1976) — If E is a Banach space of distributions, then E is homogeneous iff for every $f \in E$, $\sigma_n f \to f$ in E as $n \to \infty$.

If E is homogeneous then, from the above theorem, P will be dense in E. The converse is also true. So we have the following :

Theorem 3— E is homogeneous iff P is dense in E.

Proof : One part is clear from Theorem 2. Conversely, suppose P is dense in E. For any $g \in P$, $T_a g \to T_{a_0} g$ in C^∞ as $a \to a_0$, and therefore, $T_a g \to T_{a_0} g$ in E as $a \to a_0$. Now, using the denseness of P and the translation invariance of the E-norm, we can easily show the above property for every $g \in E$. Hence E is homogeneous.

The following theorem is a simple consequence of the above theorem and Theorem 5 of Goes (1961). A proof is also supplied for the sake of completeness.

Theorem 4 — E is homogeneous iff E and E^* are $(C, 1)$-complementary, i.e., for any $f \in E$ and $F \in E^*$, the series $\Sigma_{k \in Z} \hat{F}(k) \hat{f}(-k)$ is $(C, 1)$ summable to $F(f)$.

Proof : First suppose E is homogeneous $f \in E$ and $F \in E^*$. By Theorem 2, $\sigma_n f \to f$ in E as $n \to \infty$. Hence $F(\sigma_n f) \to F(f)$. But $F(\sigma_n f)$ is nothing but the nth Cesaro sum of $\Sigma \hat{F}(k) \hat{f}(-k)$. Hence the given series is $(C, 1)$ summable to $F(f)$.

Conversely, suppose that for all $f \in E$ and $F \in E^*$, $\Sigma \hat{F}(k) \hat{f}(-k)$ is $(C, 1)$ summable to $F(f)$, i. e., $\lim_{n \to \infty} F(\sigma_n f) = F(f)$. This means that $\sigma_n f$ converges weakly to f in E for every f in E. Hence P is weakly dense in E. Since P is convex, P is strongly dense in E by (Rudin 1973). Hence, by Theorem 3, E is homogeneous.

Corollary — If E is homogeneous and reflexive, then E^* is also homogeneous.

Proof : We just have to check that if E is homogeneous, i.e., if P is dense in E, then E^* is a Banach space of distributions satisfying the axioms (2.1) to (2.4) [see Lemma 2 (b) also]. The rest is easy.

If E is homogeneous, then, as remarked above, E^* can be thought of as a Banach space of distributions. In this case, the, following theorem gives a necessary and sufficient condition for any distribution to be in E^*.

Theorem 5—Let E be homogeneous. Then any distribution F is in E^* iff $\Sigma_{k \in Z} \hat{F}(k)\hat{f}(-k)$ is $(C, 1)$ summable for all f in E.

PROOF: One part is already proved in the previous theorem. To show the other part, let us suppose that the given series is $(C, 1)$ summable, i.e., $\lim_{n\to\infty} \sigma_n F(f)$ exists for all f in E. But $\sigma_n F$ is a bounded linear functional on E for each n. Hence, by the uniform boundedness principle, there exists a number A such that $\| \sigma_n F \|_{E^*} \leqslant A$ for all n. Define G on E by $G(f) = \lim_{n\to\infty} \sigma_n F(f)$. Then G is a continuous linear functional on E with $\| G \|_{E^*} \leqslant A$. Now we can easily check that $\hat{G}(k) = \hat{F}(k)$ for all $k \in Z$. Hence $G = F$ and therefore $F \in E^*$.

4. Convolution and Translation

Theorem 6—Let E be a homogeneous Banach space of distributions and $g \in E$. If $\mu \in M$, then $\mu^* g$ is the limit in E of finite linear combinations of translates of g.

PROOF: Denote by $\bar{V}_g$ the closed linear subspace of E generated by $T_a g$ where a is any real number. We shall first prove the above theorem for $f \in L^1$ instead of μ. Let S denote the set of those $f \in L^1$ for which $f^* g \in \bar{V}_g$. Clearly S is a linear subspace of L^1; it is closed by Theorem 1.

Take $f = \chi_{[a,b]}$ where $0 < a < b < 2\pi$. Let $I = [a, b]$. Since E is homogeneous, the vector valued integral $\int_I T_y g \, dy$ is defined and belongs to E. Comparing the Fourier coefficients we can easily check that

$$f^* g = \chi_I^* g = \frac{1}{2\pi} \int_I T_y g \, dy.$$

Partition I by a finite number of subintervals I_k whose lengths $| I_k |$ are majorized by a number δ to be chosen later. Choose and fix a point a_k in each I_k. Now

$$f^* g - \frac{1}{2\pi} \Sigma \, | I_k | \, T_{a_k} g = \frac{1}{2\pi} \int_I T_y g \, dy - \Sigma \frac{1}{2\pi} \int_{I_k} T_{a_k} g \, dy$$

$$= \frac{1}{2\pi} \Sigma \, h_k$$

where $h_k = \int_{I_k} (T_y g - T_{a_k} g) \, dy$.

Given $\epsilon > 0$, let $\delta > 0$ be such that

$$| y - a_k | < \delta \Rightarrow \| T_y g - T_{a_k} g \|_E < \epsilon.$$

Then $\| h_k \|_E \leqslant | I_k | \, \epsilon$ for each k.

Hence $\| f^* g - \frac{1}{2\pi} \Sigma \, | I_k | \, T_{a_k} g \|_E \leqslant \epsilon \, | I | \, / \, 2\pi$.

Therefore $f * g \in \bar{V}_g$ and hence $f \in S$ where $f = \chi_I$. As finite linear combinations of such functions are dense in L^1 and S is a closed linear subspace of L^1, we can infer that $S = L^1$.

Now $\sigma_n \mu \in L^1$ for all n. Hence $\sigma_n \mu * g \in \bar{V}_g$ for all n.

But

$$\| \sigma_n \mu * g - \mu * g \|_E = \| \mu * (\sigma_n g - g) \|_E \leqslant \| \mu \|_1 \| \sigma_n g - g \|_E.$$

$$\to 0 \text{ as } n \to \infty.$$

Hence $\mu * g \in \bar{V}_g$.

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