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ON THE EFFECT OF INITIAL STRESSES IN TWO DIMENSIONAL FINITE ELEMENT STRESS ANALYSIS

S. DEY

Department of Physics and Mathematics, Indian School of Mines, Dhanbad 826004

AND

P. PAL ROY

Jhargram Raj College, Jhargram, West Bengal

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In the present paper an attempt has been made to obtain the solutions of initially stressed materials using finite element method. In contrast to other traditional techniques the finite element method can handle such complicated aspects very easily. The formulations used here eliminate most of the difficulties which were present in earlier classical formulations. It has been found that the total stiffness matrix becomes non-symmetric when there are initial stresses and it becomes symmetric when there are no initial stresses and this changing nature of the stiffness matrix is due to the fact that the medium does not remain isotropic in the presence of initial stresses. A computer programme based on FORTRAN-IV has also been given. The general formulations have been strongly supported by two very usable problems in the later section of the paper.

INTRODUCTION

Remarkable development has been made in recent years in the application of the finite element methods in routine solutions of two and three dimensional elasticity problems. However, attempts to extend this method to initially stressed elasticity problems have not been made so extensively. This is because of the antisymmetric nature of the total stiffness matrix and due to the new concept of incremental stresses developed by Biot (1965). Since no assumptions are introduced regarding the physical process of generating initial stresses it becomes possible to apply the concept of finite element technique to advocate the approximate but acceptable solutions for two dimensional isotropic materials under initial stresses.

Hansteen (1967) analysed the stress distribution in shear walls and gave a nice technique on the application of this method to plate and shell problems. Many other authors like Petyt (1965), Helbing (1969), Humphrey (1970) discussed the

applicability to stress and structural analysis. Problems of this type have been treated extensively in the context of engineering and particularly in branch of science termed as strength of material. The theory of Biot, which introduces a local rotation varies from point to point and provides a separation of the purely geometric properties of the deformation field from those which depend on the physics of the material, resulting the incremental stresses and strains with respect to the locally rotated frame, has been discussed. Since due to the existence of the initial stresses the stress-strain relations are effected, this effect will create an abrupt change in the nature of the total stiffness matrix.

The general formulations of plane stress and plane strain problems have been done and those are applied in some problems.

A. Relations Between Strain and Incremental Stress

1. *Two-Dimensional Relations*

Let us consider an elastic continuum in an initial state of stress whose components in x-y plane are

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}. \qquad \ldots(1.1)$$

The incremental stresses referred to locally rotated axes are

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}. \qquad \ldots(1.2)$$

Biot (1965) has shown when both the incremental stresses and the strains are referred to the rotated axes then if the deformation is small the incremental stress components (1.2) must be functions only of the strain components (e_{xx}, e_{yy}, e_{xy}) and it eliminates the influence of the rotation on the stress strain relation.

For small incremental deformations the incremental stress-strain relations, are assumed to be linear. We may write them as

$$\left.\begin{aligned} s_{11} &= A_{11}^{11}\, e_{xx} + A_{11}^{22}\, e_{yy} + 2\, A_{11}^{12}\, e_{xy} \\ s_{22} &= A_{22}^{11}\, e_{xx} + A_{22}^{22}\, e_{yy} + 2\, A_{22}^{12}\, e_{xy} \\ s_{12} &= A_{12}^{11}\, e_{xx} + A_{12}^{22}\, e_{yy} + 2\, A_{12}^{12}\, e_{xy}. \end{aligned}\right\} \qquad \ldots(1.3)$$

The coefficients A_{11}^{11}, A_{11}^{22} etc. may be either adiabetic or isothermal when there exists a strain energy potential.

2. *Three-Dimensional Relations*

Let us now consider a state of three-dimensional stress system. The six initial stress components referred to x, y, z axes are

$$(S_{ij}) = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \qquad \ldots(2.1)$$

with the condition

$$S_{ij} = S_{ji} \ (i, j = 1, 2, 3). \qquad \ldots(2.2)$$

The incremental stresses are

$$(s_{ij}) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \qquad \ldots(2.3)$$

where

$$s_{ij} = s_{ji}. \qquad \ldots(2.4)$$

The infinitesimal strain components referred to the rotated axes are

$$(e_{ij}) = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} \qquad \ldots(2.5)$$

When referred to the initial areas the alternative stress components are given by t_{ij}. These stress components t_{ij} are conjugate to ϵ_{ij}, the strain components defined in terms of first and second order quantities (Biot 1965).

The linear relations between t_{ij} and e_{ij} are

$$t_{ij} = C_{ij}^{lm} \, e_{lm} \qquad \ldots(2.6)$$

with

$$C_{ij}^{lm} = C_{ji}^{lm} = C_{ij}^{ml} \, . \qquad \ldots(2.7)$$

We may also use the incremental stresses s_{ij} and write linear relations between the incremental stresses and the strains e_{ij} as

$$s_{ij} = A_{ij}^{lm} \, e_{lm} \qquad \ldots(2.8)$$

with

$$A_{ij}^{lm} = A_{ji}^{lm} = A_{ij}^{ml} \, . \qquad \ldots(2.9)$$

The relations between A_{ij}^{lm} and C_{ij}^{lm} are given by

$$A_{ij}^{lm} = C_{ij}^{lm} - S_{ij} \, \delta_{lm} + \frac{1}{4} (S_{im} \, S_{lj} \, S_{mj} + S_{jm} \, \delta_{li} + S_{jl} \, \delta_{mi}) \qquad \ldots(2.10)$$

δ being the Kronecker δ-function.

When there exists a strain energy potential the coefficients A_{ij}^{lm} will satisfy the conditions

$$A_{ij}^{lm} - A_{lm}^{ij} = S_{lm}\,\delta_{ij} - S_{ij}\,\delta_{lm}. \qquad \ldots(2.11)$$

We now confine ourselves by considering the unstressed medium to be either of orthotropic or isotropic symmetry in the original unstressed condition. We also assume that the principal initial stress components S_{11}, S_{22}, S_{33} are directed along the planes of elastic symmetry of the medium. Which in turn will not effect the symmetry of the material i.e. the symmetry of the material is not modified by the initial stress and applies equally to the incremental deformation.

Since the incremental stress-strain relations possess orthotropic symmetry, they must necessarily assume the form

$$\left.\begin{aligned} s_{11} &= A_{11}\,e_{xx} + A_{12}\,e_{yy} + A_{13}\,e_{zz} \\ s_{22} &= A_{21}\,e_{xx} + A_{22}\,e_{yy} + A_{23}\,e_{zz} \\ s_{33} &= A_{31}\,e_{xx} + A_{32}\,e_{yy} + A_{33}\,e_{zz} \\ s_{23} &= Q_1\,e_{yz},\ s_{31} = Q_2\,e_{zx},\ s_{12} = Q_3\,e_{xy} \end{aligned}\right\}. \qquad \ldots(2.12)$$

When there exists a strain energy potential the coefficients will satisfy relations

$$\left.\begin{aligned} A_{12} - A_{21} &= S_{22} - S_{11} \\ A_{23} - A_{32} &= S_{33} - S_{22} \\ A_{31} - A_{13} &= S_{11} - S_{33}. \end{aligned}\right\} \qquad \ldots(2.13)$$

The stress-strain relations (2.12) may also be written in the following form for practical use :

$$\left.\begin{aligned} s_{11} &= C_{11}\,e_{xx} + (C_{12} - S_{11})\,e_{yy} + (C_{31} - S_{11})\,e_{zz} \\ s_{22} &= (C_{12} - S_{22})\,e_{xx} + C_{22}\,e_{yy} + (C_{23} - S_{22})\,e_{zz} \\ s_{33} &= (C_{31} - S_{33})\,e_{xx} + (C_{23} - S_{33})\,e_{yy} + C_{33}\,e_{zz} \\ s_{23} &= Q_1\,e_{yz},\ s_{31} = Q_2\,e_{zx},\ s_{12} = Q_3\,e_{xy}. \end{aligned}\right\} \qquad \ldots(2.14)$$

3. Two Dimensional Plane Stress Analysis

In this case $s_{33} = 0$ $S_{33} = 0$

we get

$$e_{zz} = -\frac{C_{31}}{C_{33}}\,e_{xx} - \frac{C_{23}}{C_{33}}\,e_{yy}$$

which when substituted in the first two equations of (2.14) we get,

$$s_{11} = \left[C_{11} - (C_{31} - S_{11})\frac{C_{31}}{C_{33}}\right] e_{xx} + \left[(C_{12} - S_{11}) - \frac{C_{23}}{C_{33}}(C_{31} - S_{11})\right] e_{yy}$$

$$s_{22} = \left[(C_{12} - S_{22}) - (C_{23} - S_{22})\frac{C_{31}}{C_{33}}\right] e_{xx} + \left[C_{22} - (C_{23} - S_{22})\frac{C_{23}}{C_{33}}\right] e_{yy} \quad \ldots(3.1)$$

$s_{12} = Qe_{xy}$ [replacing Q_3 by Q in (2.14)].

Now, we know the relations

$$\bar{s}_{ij} = \lambda V \delta_{ij} + 2\mu e_{ij} \quad \ldots(3.2)$$

where V, the dialatation

$$= e_{xx} + e_{yy} + e_{zz} \quad \ldots(3.3)$$

λ, μ are Lames contants.

Hence we get $C_{ij} = \lambda$ when $i \neq j$ $(i, j = 1, 2, 3)$ ...(3.4)

$$= \lambda + 2\mu \text{ when } i = j$$

and $Q = \mu$.

Then

$$s_{11} = \left[\frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} + S_{11}\frac{\lambda}{\lambda+2\mu}\right] e_{xx} + \left[\frac{2\lambda\mu}{(\lambda+2\mu)} - \frac{2\mu}{(\lambda+2\mu)} S_{11}\right] e_{yy}$$

$$s_{22} = \left[\frac{2\lambda\mu}{(\lambda+2\mu)} - \frac{2\mu}{(\lambda+2\mu)} S_{22}\right] e_{xx} + \left[\frac{4\mu(\lambda+\mu)}{(\lambda+2\mu)} + S_{22}\frac{\lambda}{(\lambda+2\mu)}\right] e_{yy} \quad \ldots(3.5)$$

$s_{12} = \mu e_{xy}$.

Using

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \quad \ldots(3.6)$$

we get,

$$s_{11} = \frac{E}{(1-\nu^2)}\left[1 + \frac{S_{11}}{E}(1+\nu)\nu\right] e_{xx} + \frac{E}{(1-\nu^2)}\left[\nu - \frac{S_{11}}{E}(1+\nu)(1-2\nu)\right] e_{yy}$$

$$s_{22} = \frac{E}{(1-\nu^2)}\left[\nu - \frac{S_{22}}{E}(1-2\nu)(1+\nu)\right] e_{xx} + \frac{E}{(1-\nu^2)}\left[1+\frac{S_{11}}{E}(1+\nu)\nu\right] e_{yy} \quad \ldots(3.7)$$

$$s_{12} = \frac{E}{2(1+\nu)} e_{xy}.$$

Case (3 A)—When $S_{11} \neq 0$, $S_{22} = 0$, then

$$\frac{P}{2\mu} = Q \text{ (say)} = \frac{S_{22}-S_{11}}{2\mu}, \text{ so } S_{11} = -\frac{E}{(1+\nu)} Q. \quad \ldots(3.8)$$

Then (3.7) reduces to

$$s_{11} = \frac{E}{(1-\nu^2)}[1-\nu Q] e_{xx} + \frac{E}{(1-\nu^2)}[\nu + (1-2\nu) Q] e_{yy}$$

$$s_{22} = \frac{E\nu}{(1-\nu^2)} e_{xx} + \frac{E}{(1-\nu^2)} e_{yy}$$

$$s_{12} = \frac{E}{2(1+\nu)} e_{xy}. \quad \ldots(3.9)$$

In the matrix form (3.2) can be written as

$$[S_{INC}] = [D_A][E_{INC}] \quad \ldots(3.10)$$

where

$$\left.\begin{aligned}[S_{INC}]^T &= [s_{11} \quad s_{22} \quad s_{12}] \\ [E_{INC}]^T &= [e_{xx} \quad e_{yy} \quad e_{xy}]\end{aligned}\right\} \quad \ldots(3.11)$$

and

$$D_A = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1-\nu Q & \nu+(1-2\nu) Q & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \ldots(3.12)$$

Case (3 B)—When $S_{11} = 0$ $S_{22} \neq 0$ and $S_{22} = 2\mu Q = \frac{E}{(1-\nu)} Q$. In this case the elasticity matrix is given by

$$[D_B] = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu(1-2\nu) Q & 1+\nu Q & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad \ldots(3.13)$$

Case (3 C)— When both S_{11}, $S_{22} \neq 0$, then the elasticity matrix can at once be written as

$$[Dc] = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1+\frac{S_{11}}{E}(1+\nu)\,\nu & \nu - \frac{S_{11}}{E}(1+\nu)(1-2\nu) & 0 \\ \nu - \frac{S_{22}}{E}(1-2\nu)(1+\nu) & 1+\frac{S_{22}}{E}(1+\nu)\,\nu & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}. \quad \ldots(3.14)$$

4. *Two Dimensional Plane Strain Analysis*

Putting $e_{zz} = e_{yz} = e_{zx} = 0$ into equations (2.12) and writing Q instead of Q_3, we obtain

$$\left.\begin{aligned} s_{11} &= A_{11}\, e_{xx} + A_{12}\, e_{yy} \\ s_{22} &= A_{21}\, e_{xx} + A_{22}\, e_{yy} \\ s_{12} &= \bar{Q}\, e_{xy} \end{aligned}\right\}. \quad \ldots(4.1)$$

These can be supposed to be the general form of the incremental stress in a compressible medium with two dimensional orthotropy. The principal stresses are S_{11} and S_{22} with principal directions along the coordinate axes. But we known from (2.13)

$$\left.\begin{aligned} & A_{12} + S_{11} = A_{21} + S_{22} \\ \text{or} \quad & \\ & A_{21} + P = A_{12} \end{aligned}\right\} \quad \ldots(4.2)$$

where

$$P = S_{22} - S_{11}. \quad \ldots(4.3)$$

Substituting the following

$$\left.\begin{aligned} A_{11} &= K + N + P,\ A_{12} = K - N + P \\ A_{21} &= K - N,\ A_{22} = K + N \end{aligned}\right\} \quad \ldots(4.4)$$

into equations (4.1) we get

$$\left.\begin{aligned} s_{11} &= (K + N + P)\, e_{xx} + (K - N + P)\, e_{yy} \\ s_{22} &= (K - N)\, e_{xx} + (K + N) e_{yy} \\ s_{21} &= \bar{Q} e_{xy}. \end{aligned}\right\} \quad \ldots(4.5)$$

Incremental Isotropy—When the medium under initial stress is isotropic for incremental plane strain we can express the situation by substituting $N = \bar{Q} = \mu$, $K = \lambda + \mu$ in (4.5) which yields

$$\begin{aligned} s_{11} &= (\lambda + 2\mu + P)\, e_{xx} + (\lambda + P)\, e_{yy} \\ s_{22} &= \lambda\, e_{xx} + (\lambda + 2\mu)\, e_{yy} \\ s_{12} &= \mu\, e_{xy}. \end{aligned} \quad \ldots(4.6)$$

Substituting $\mu = \frac{E}{2(1+\nu)}$, $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$

we get

$$\begin{Bmatrix} s_{11} \\ s_{22} \\ s_{12} \end{Bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1+P_1 & \frac{\nu}{1-\nu}+P_1 & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{xy} \end{Bmatrix} \quad \ldots(4.7)$$

where

$$P_1 = \frac{P(1-2\nu)(1+\nu)}{E(1-\nu)}. \quad \ldots(4.8)$$

In the matrix form (4.7) can be written as

$$[S_{INC}] = [D_R][E_{INC}] \quad \ldots(4.9)$$

where $[S_{INC}]$, $[E_{INC}]$ are defined in (3.10) and

$$[D_R] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1+P_1 & \frac{\nu}{1-\nu}+P_1 & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2(1-\nu)} \end{bmatrix}. \quad \ldots(4.10)$$

B. General Formulation of the Problem

The general formulation of the problem consists of a few parts as discussed below. We have discussed it very briefly as they have been mentioned widely in Desai and Abel (1972), Ural (1973), and Zienkiewicz and Cheung (1967). The finite element method idealizes a structure or body by a set of elements to which the stiffness matrix method of structural analysis can be directly applied. When we consider the plane stress case the state of stress is defined by the stress components s_{11}, s_{22}, s_{12} in the locally rotated $x-y$ coordinate system whereas in the case of plane strain, the state of strain in the $x-y$ locally rotated coordinate system is defined by the strain components e_{xy}, e_{yy}, e_{xy}.

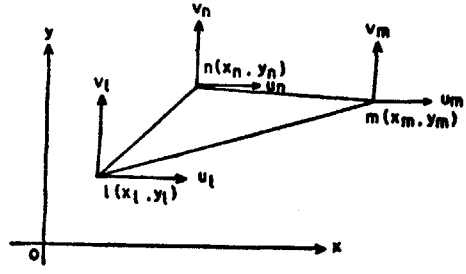


FIG. 1. A typical triangular element having three nodes at the vertices and each node has two degrees of freedom.

The analysis of plane stress and strain will follow similar logic except the variation of the elasticity matrix.

Suppose the considered structure is divided into N number of element and M number of nodal points in which each node point has two degrees of freedom, one in the x-direction another in the y-direction.

If (u_1, v_1), (u_2, v_2) ... (u_M, v_M) are the nodal displacements at the respective nodes then the matrix $[\delta]^T = [u_1\ v_1\ u_2\ v_2\ \ldots\ u_M, v_M]$ may be considered to be the basic unknowns of the analysis.

The general strains expression is written in the form

$$[E_{INC}] = [B_i]\,[\delta_i] \qquad \ldots(B.1)$$

where the matrix B_i and δ_i are for a typical element i ($i = 1, 2 \ldots N$) having l, m, n nodal points and (u_1, v_1), (u_m, v_m), (u_n, v_n) as the corresponding nodal displacements are given by

$$[B_i] = \frac{1}{2A_i}\begin{bmatrix} y_m-y_n & 0 & y_n-y_1 & 0 & y_1-y_m & 0 \\ 0 & x_n-x_m & 0 & x_1-x_m & 0 & x_m-x_1 \\ x_n-x_m & y_m-y_n & x_1-x_n & y_n-y_1 & x_m-x_1 & y_1-y_m \end{bmatrix} \qquad \ldots(B.2)$$

$$[\delta_i]^T = [u_1\ v_1\ u_m\ v_m\ u_n\ v_n]. \qquad \ldots(B.3)$$

The general stress expression is related to the strain relation of the element by a transformation matrix D_i, known as the elasticity matrix whose nature in the case of plane-stress is given by (3.14) and in the case of plane strain by (4.10) can be written as

$$[S_{INC}] = [D_i]\,[E_{INC}] \qquad \ldots(B.4)$$

using (B.1) we get at once

$$[S_{INC}] = [D_i]\,[B_i]\,[\delta_i]. \qquad \ldots(B.5)$$

The nodal displacements which are the basic unknowns are determined by solving the the system of linear equation generated by satisfying the equilibrium conditions at nodes. The stationary of the total potential energy principle may be regarded as the foundation of this analysis.

We now define the following :

(a) Internal strain energy

$$[I_{SE}] = \frac{1}{2}\int_V [E_{INC}]^T\,[S_{INC}]\,dV$$

$$= \frac{1}{2}\int_V [E_{INC}]^T\,[D]\,[E_{INC}]\,dV \qquad \ldots(B.6)$$

(b) Potential energy due to body forces

$$[P_{BF}] = \int_V \rho\,[\varphi]^T\,[F_B]\,dV \qquad \ldots(B.7)$$

where F_B and φ^T are respectively the body forces and the general displacement functiou of the system.

(c) Potential energy due to nodal forces

$$[P_{NF}] = [\delta]^T [R_{NF}] \qquad ...(B.8)$$

where $[\delta]^T$ and $[R_{NF}]$ are respectively the nodal displaeements and external nodal loads.

(d) Potential enei gy due to the surface traction

$$[P_{ST}] = \int_{\text{area}} [\varphi]^T [S_T]\, dA \qquad ...(B.9)$$

which can be rewritten using the position matrix N as

$$[P_{ST}] = \int_{\text{area}} \frac{1}{2A} [\delta]^T [N]^T [S_T]\, dA \qquad ...(B.10)$$

The total potential energy of the system is given by

$$[T_{PE}] = [I_{SE}] - [P_{BE}] - [P_{NF}] - [P_{ST}]. \qquad ...(B.11)$$

Let $[T_{LS}]$ represent the total loading system of the analysis and $[K] = \sum_{i=1}^{n} [k]_i$, the total stiffness matrix of the system. The concept of the principle of minimum potential energy will generate the necessary system of linear equations expressing the nodal equilibrium conditions and finally we get

$$[K_{\text{total}}] [\delta] = [T_{LS}] \qquad ...(B.12)$$

when the geometric boundary conditions are prescribed at some nodes its effect can be explained mathematically by partitioning the global equilibrium equations (B.12) as follows

$$\begin{bmatrix} K_{rr} & \vdots & K_{rs} \\ \cdots & \cdots & \cdots \\ K_{rs}^T & \vdots & K_{ss} \end{bmatrix} \begin{bmatrix} \delta_r \\ \cdots \\ \delta_s \end{bmatrix} = \begin{bmatrix} T_{LS}^r \\ \cdots \\ T_{LS}^s \end{bmatrix} \qquad ...(B.13)$$

where δ_r is the unconstrained or free displacement vector and δ_s is the prescribed displacement vector.

From (B.13) we then have

$$[K_{rr}] [\delta_r] = [T_{LS}^r] - [K_{rs}] [\delta_s] \qquad ...(B.14)$$

and

$$[T^s_{LS}] = [K_{rs}]^T [\delta_r] + [K_{ss}] [\delta_s]. \quad \ldots(B.15)$$

The matrix $[\delta_s]$ becomes null in the case of homogeneous boundary conditions. (B.14) will give the unknown nodal displacements and (B.15) will give unknown loadings.

C. Numerical Problems

Two simple problems have been selected to examine the effect of initial stresses. The results are divided into some parts relative to the variations of the ratio $P/2\mu = Q$. Their actions are analysed independently without complications. If desired automatically superimposition can be made. Some remarkable observations are made at the end of each problem.

C.1 *For Plane Stress Analysis*

Suppose a thin plate 5 ft wide, 6 ft long, 1.5 inch thick is fixed along one short side and is acting under pure tensile loads, 10 kips each, applied at two corners. E the Youngs modulus is 29000 ksi and ν, the Poissons ratio is 0.3.

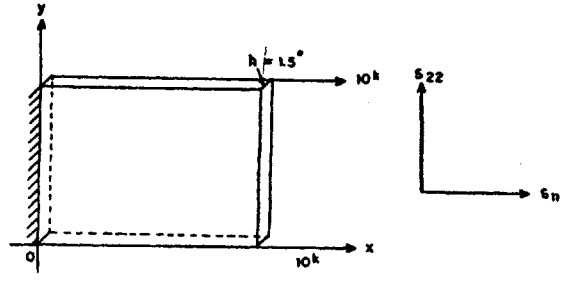


Fig. 2 Considered thin plate with S_{11}, S_{22} as principal initial stresses along the planes of elastic symmetry.

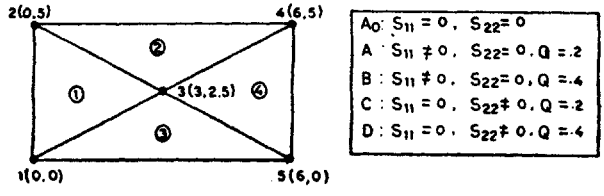


Fig. 3 Finite element idealisation.

For different combinations of S_{11}, S_{22} and Q following matrices are computed.

Case A_0 : $S_{11} = 0$ $S_{22} = 0$ $N = 4$

The total stiffness matrix as given in Ural (1973).

$$[k]_{A_0} = 15934 \begin{bmatrix} 28.8 & 9.75 & 3.1 & .375 & -18.8 & -9.75 & 0 & 0 & -3.1 & -.375 \\ & 22.375 & -3.75 & -6.81 & -9.75 & -22.375 & 0 & 0 & .375 & 6.81 \\ & & 18.8 & -9.75 & -18.8 & 9.75 & -3.1 & .375 & 0 & 0 \\ & & & 22.375 & 9.75 & 22.375 & -3.75 & 6.81 & 0 & 0 \\ & & & & 75.2 & 0 & -18.8 & -9.75 & -18.8 & 9.75 \\ & & & & & 89.5 & -9.7 & -22.375 & 9.75 & -22.375 \\ & & & & & & 18.8 & 9.75 & 3.1 & .375 \\ & & & \text{Symmetry} & & & & 22.375 & -.375 & -6.8 \\ & & & & & & & & 18.8 & 22.375 \\ & & & & & & & & & 22.375 \end{bmatrix}$$

And the matrix to be inverted (as u_1, v_1, u_2, v_2 are all zeros)

$$(A_0) = 1593.4 \begin{bmatrix} 75.2 & 0 & -18.8 & -9.75 & -18.8 & 9.75 \\ & 89.5 & -9.75 & -22.375 & 9.75 & -22.375 \\ & & 18.8 & 9.75 & 3.1 & .375 \\ & & & 22.375 & -.375 & -6.81 \\ & & \text{Symmetry} & & 18.8 & -9.75 \\ & & & & & 22.375 \end{bmatrix}$$

Case A—$S_{11} \neq 0 \quad S_{22} = 0 \quad Q = .2 \quad N = 4.$

The total stiffness matrix

$$[K_A] = 1593.4 \begin{bmatrix} 18.05 & 10.95 & 2.725 & -.225 & -18.05 & -10.95 & 0 & 0 & -2.725 & .225 \\ 9.75 & 22.375 & -.375 & -6.8125 & -9.75 & -22.375 & 0 & 0 & .375 & 6.8125 \\ 2.725 & .225 & 18.05 & -10.95 & -18.05 & 10.96 & -2.725 & -225 & 0 & 0 \\ 0.375 & -6.8125 & -9.75 & 22.375 & 9.75 & -22.375 & -.375 & 6.8125 & 0 & 0 \\ -18.05 & -10.95 & -17.15 & 10.20 & 70.05 & 0 & -17.15 & -10.32 & -18.05 & 10.95 \\ -9.75 & -22.375 & 9.75 & -22.375 & 0 & 89.5 & -9.75 & -11.375 & 9.75 & -22.375 \\ 0 & 0 & -2.725 & .225 & -18.05 & -10.95 & 18.05 & 10.95 & 2.725 & -.225 \\ 0 & 0 & .375 & 6.8125 & -9.75 & -22.375 & 9.75 & 22.375 & -.375 & -6.8125 \\ -2.725 & -.225 & 0 & 0 & -18.05 & 10.95 & 2.725 & .225 & 18.05 & -10.95 \\ -.375 & 6.8125 & 0 & 0 & 9.75 & -22.375 & .375 & -6.8125 & -9.75 & 22.375 \end{bmatrix}$$

And the matrix to be inverted

$$(A) = 1593.4 \begin{bmatrix} 70.5 & 0 & -17.15 & -10.2 & -18.05 & 10.95 \\ 0 & 89.5 & -9.75 & -22.375 & 9.75 & -22.375 \\ -18.05 & -10.95 & 18.05 & 10.95 & 2.725 & -.225 \\ -9.75 & -22.375 & 9.75 & 22.375 & -.375 & -6.8125 \\ -18.05 & 10.95 & 2.725 & .225 & 18.05 & -10.95 \\ 0.75 & -22.375 & .375 & -6.8125 & -9.75 & 22.375 \end{bmatrix}$$

Case $B - S_{11} \neq 0 \quad S_{22} = 0 \quad Q = .4$

The total stiffness matrix is

$$[k_B] = 1593.4 \begin{bmatrix} 17.3 & 12.15 & 12.15 & -.825 & -17.3 & -12.15 & 0 & 0 & -2.35 & .825 \\ 9.75 & 22.375 & -.375 & -6.8125 & -9.75 & -22.375 & 0 & 0 & .375 & 6.8125 \\ 2.35 & .825 & 17.3 & -12.15 & -17.3 & 12.15 & -2.35 & -.825 & 0 & 0 \\ .375 & -6.8125 & -9.75 & 22.375 & 9.75 & -22.375 & -.375 & 6.8125 & 0 & 0 \\ -17.3 & -12.15 & -17.3 & 12.15 & 69.2 & 0 & -17.3 & -12.15 & -17.3 & 12.15 \\ -9.75 & -22.375 & 9.75 & -22.375 & 0 & 89.5 & -9.75 & -22.375 & 9.75 & -22.375 \\ 0 & 0 & -2.35 & .825 & -17.3 & -12.15 & 17.3 & 12.15 & 2.35 & -.825 \\ 0 & 0 & .375 & 6.8125 & -9·75 & -22.375 & 9.75 & 22.375 & -.375 & -6.8125 \\ -2.35 & -.825 & 0 & 0 & -17.3 & 12.15 & 2.35 & .825 & 17.3 & -12.15 \\ -.375 & 6.8125 & 0 & 0 & 9.75 & -22.375 & .375 & -6.8125 & -9.75 & 22.375 \end{bmatrix}$$

The matrix to be inverted

$$(B) = 1593.4 \begin{bmatrix} 69.2 & 0 & -17.3 & -12.15 & -17.3 & 12 15 \\ 0 & 89.5 & -9.75 & -22.375 & 9.75 & -22.375 \\ -17.3 & -12.15 & 17.3 & 12.15 & 2.35 & -.825 \\ -9.75 & -22.375 & 9.75 & 22.375 & -.375 & -6.8125 \\ -17.3 & 12.15 & 2.35 & .825 & 17.3 & -12.15 \\ 9.75 & -22.375 & .375 & -6.8125 & -9.75 & 22.355 \end{bmatrix}$$

Case C— $S_{11} = 0 \quad S_{22} \neq 0 \quad Q = .2$

The total stiffness matrix is

$$[k_C] = 1593.4 \begin{vmatrix} 18.8 & 9.75 & 3.1 & .375 & -18.8 & -9.75 & 0 & 0 & -3.1 & -.375 \\ 8.55 & 23.815 & -.975 & -7.5325 & -8.55 & -23.815 & 0 & 0 & .975 & 7.5325 \\ 3.1 & -.375 & 18.8 & -9.75 & -18.8 & 9.75 & -3.1 & .375 & 0 & 0 \\ .975 & -7.5325 & -8.55 & 23.815 & 8.55 & -23.815 & -.975 & 7.5325 & 0 & 0 \\ -18.8 & -9.75 & -18.8 & 9.75 & 75.2 & 0 & -18.8 & -9.75 & -18.8 & 9.75 \\ -8.55 & -23.815 & 8.55 & -23.815 & 0 & 97.26 & -8.55 & -23.815 & 8.55 & -23.815 \\ 0 & 0 & -3.1 & -.375 & -18.8 & -9.75 & 18.8 & 9.75 & 3.1 & .375 \\ 0 & 0 & .975 & 7.5325 & 8.55 & -23.815 & 8.55 & 23.815 & -.975 & -7.5325 \\ -3.1 & .375 & 0 & 0 & -18.8 & 9.75 & 3.1 & -.375 & 18.8 & 9.75 \\ -.975 & 7.5325 & 0 & 0 & 8.55 & -23.815 & .975 & -7.5325 & -8.55 & 23.815 \end{vmatrix}$$

The matrix to be inverted

$$(C) = 1593.4 \begin{vmatrix} 75.2 & 0 & -18.8 & -9.75 & -18.8 & 9.75 \\ 0 & 95.26 & -8.55 & -23.815 & 8.55 & -23.815 \\ -18.8 & -9.75 & 18.8 & 9.75 & 3.1 & .375 \\ -8.55 & -23.815 & 8.55 & 23.815 & -.975 & -7.5325 \\ -18.8 & 9.75 & 3.1 & -.375 & 18.8 & -9.75 \\ 8.55 & -23.815 & .975 & -7.5325 & -8.55 & 23.815 \end{vmatrix}$$

Case D—When $S_{11} = 0 \quad S_{22} \neq 0 \quad Q = .4$

$$[k_D] = 1593.4 \begin{bmatrix} 18.8 & 9.75 & 3.10 & .375 & -18.8 & 9.75 & 0 & 0 & -3.1 & -.375 \\ 7.35 & 25.255 & -1.575 & -8.2525 & -7.35 & -25.225 & 0 & 0 & 1.575 & 8.2525 \\ 3.1 & -.375 & 18.8 & -9.75 & -18.8 & 9.75 & -3.1 & .375 & 0 & 0 \\ 1.575 & -8.2525 & -7.35 & -25.255 & 7.35 & -25.255 & -1.575 & 8.2525 & 0 & 0 \\ -18.8 & -9.75 & -18.8 & 9.75 & 75.2 & 0 & -18.8 & -9.75 & -18.8 & 9.75 \\ -7.35 & -25.255 & 7.35 & -25.255 & 0 & 101.2 & -7.35 & -25.255 & 7.35 & -25.255 \\ 0 & 0 & -3.1 & -.375 & -18.8 & -9.75 & 18.8 & 9.75 & 3.10 & .375 \\ 0 & 0 & 1.575 & 8.2525 & -7.35 & -25.255 & 7.35 & 25.255 & -1.575 & -8.2525 \\ -3.1 & .375 & 0 & 0 & -18.8 & 9.75 & 3.1 & .375 & 18.8 & -9.75 \\ -1.575 & 8.2525 & 0 & 0 & 7.35 & -25.255 & 1.575 & -8.2525 & -7.35 & 25.255 \end{bmatrix}$$

The matrix to be inverted

$$(D) = 1593.4 \begin{vmatrix} 75.2 & 0 & -18.8 & -9.75 & -18.8 & 9.75 \\ 0 & 101.2 & -7.35 & -25.255 & 7.35 & -25.255 \\ -18.8 & -9.75 & 18.8 & 9.75 & 3.1 & .375 \\ -7.35 & -25.255 & 7.35 & 25.255 & 1.575 & -8.2525 \\ -18.8 & 9.75 & 3.1 & .375 & 18.8 & -9.75 \\ 7.35 & -25.255 & 1.575 & -8.2525 & -7.35 & 25.255 \end{vmatrix}$$

The following table show the variations of the nodal displacements as computed for different combinatian

$E = 29000$ ksi, $\nu = 0.3$, h (thickness) = 1.5 inches

Case	N	S_{11}	S_{22}	Q	u_1	v_1	u_2	v_2	u_3	v_3	u_4	v_4	u_5	v_5
A_0	4	0	0	X	0	0	0	0	.000243881	0	.00053458	−.00009025	.00053458	.0000902472
A	4	$\neq 0$	0	.2	0	0	0	0	.000260841	0	.000582485	−.00009996	.000582485	.0000999604
B	4	$\neq$	0	.4	0	0	0	0	.000281068	0	.000640735	−.000111913	000640735	.000111913
C	4	0	$\neq 0$	.2	0	0	0	0	.000246332	0	.000523417	−.0000592946	.000523417	.0000592946
D	4	0	$\neq 0$	.4	0	0	0	0	.000248324	0	.00051437	−.0000341654	.00051437	.0000341654

The effect of the corresponding initial stresses on elemental stresses are shown in the following tables.

Case A_0

Element	s_{11} (in ksi)	s_{22} (in ksi)	s_{12} (in ksi)	s_{max} (in ksi)	s_{min} (in ksi)
1	.21600	.06460	0	.21600	.06460
2	.22200	.02310	−.00527	.22225	.022850
3	.22200	.02310	.0053	.22225	.022850
4	.22800	−.01864	0	.22800	−.01864

Case A

1	.2168314	.0692015	0	.2168314	.0692015
2	.2219483	.0242277	−.00417809	.2220342	.0241372
3	.2219483	.0242277	.00417809	.2220365	.0241394
4	.2270653	−.0207415	0	.2270653	−.0207415

Case B

1	.2187322	.0745678	0	.2187322	.0745678
2	.2219999	.0256123	−.0027231	.2220376	.0255745
3	.2219999	.0256123	.00272308	.2220376	.0255745
4	.2252677	−.0233431	0	.2252677	−.0233431

Case C

1	.2178413	.0479251	0	.2178413	.0479251
2	.2219999	.0169376	−.00346556	.2220583	.016879
3	.2219999	.0169376	.00346556	.2220583	.016879
4	.2261585	−.0140497	0	.2261585	−.0140497

Case D

1	.2196025	.0307443	0	.2196025	.0307443
2	.2219999	.0108125	−.00199634	.2220187	.0107936
3	.2219999	.0108125	.00199634	.2220187	.0107936
4	.2243862	−.0009162	0	.2243862	−.009162

Observations

(1) When $S_{11} \neq 0$, $S_{22} = 0$ it is clear from the stiffness matrices $[K_A]$ and $[K_B]$ that their 2nd, 4th, 8th and 10th rows i.e. even rows are equivalent to the corresponding rows of $[K_{A_0}]$, the stiffness matrix when $S_{11} = 0$, $S_{22} = 0$. But the corresponding odd rows are changing. This changing nature of the odd rows is due to the fact that S_{11} appears only in the 1st row (odd row) of the elasticity matrix $| D |$ defined in (3.13).

(2) When $S_{22} \neq 0$, $S_{11} = 0$ it is clear that the odd rows of $[K_C]$ and $[K_D]$ are equivalent to the corresponding rows of $[K_{A_0}]$ and the even rows are changing. This is due to the appearance of S_{22} in the 2nd row (even row) of the elasticity matrix (D) defined in (3.13).

(3) When both $S_{11} \neq 0$ and $S_{22} \neq 0$ then obviously all the rows of the corresponding total stiffness matrix will change than that of the rows of $[K_{A_0}]$

(4) The maximum and minimum stresses of the elements are changing due to the existence of the initial stresses. But the maximum and minimum stresses for elements in each case are equal as it happened when there were no initial stresses.

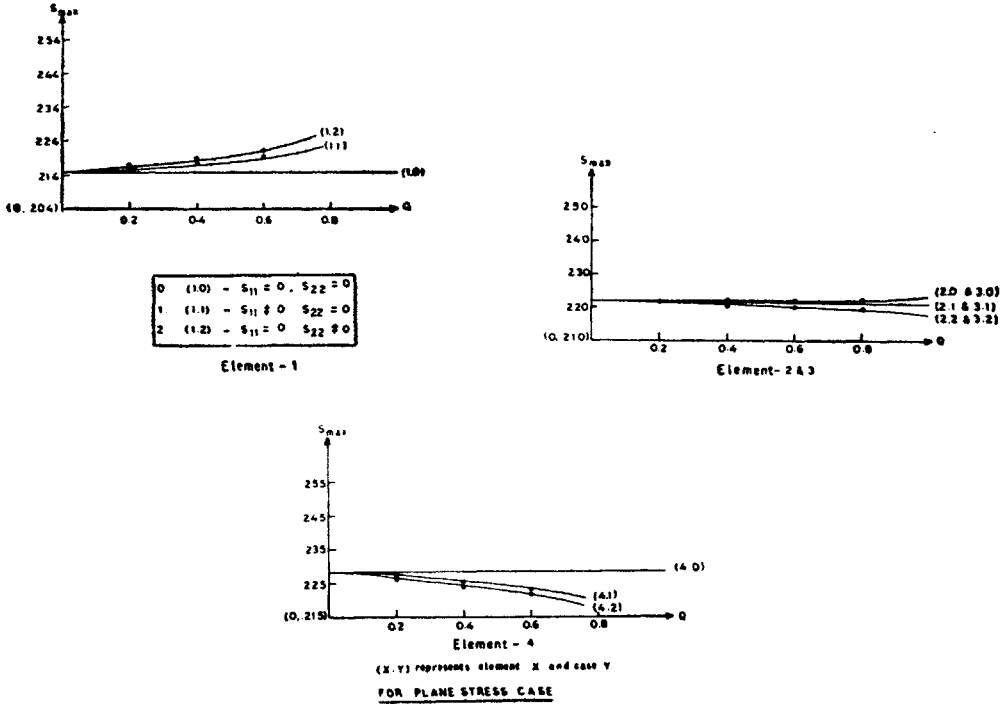


FIG. a Variation of S_{max} over Q for each element.

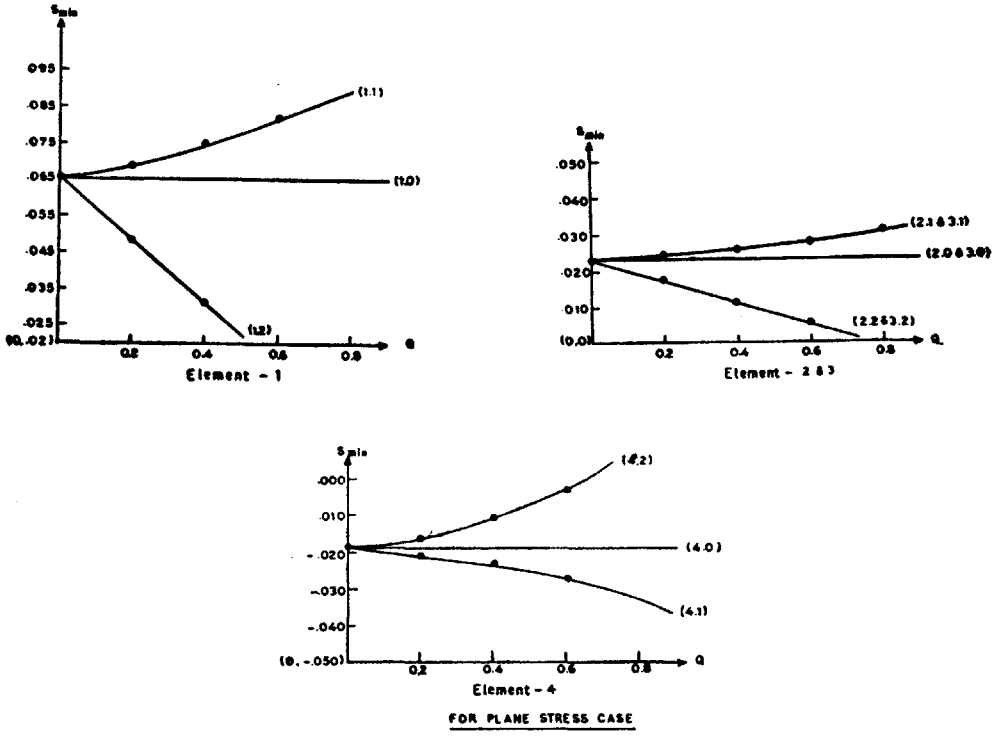


FIG. b. Variation of S_{min} over Q for each element.

C.2. *Plane Strain Analysis*

Let us consider a thin plate of 3 inches wide 8 inches long, 1.5 inches thick is lying on a support of (8/3) inches long and is compressed by the forces P_2 kips/unit inch and p_3 kips/unit inch from two sides and p_1 kips from the above. The material has an initial stress of compressive nature along the direction of x given by $S_{22} = 0$, $P/2\mu = -S_{11} = 0.5$. The following set is given

$E = 300000$ k si, $\nu = 0.3$

$p_1 = 6$ kips per inch

$p_2 = 4$ kips per inch

$p_3 = 8$ kips per inch.

Using plane-strain analysis we like to find out the nodal displacements and the elemental stresses. The following table shows the nature of idealization of the structure.

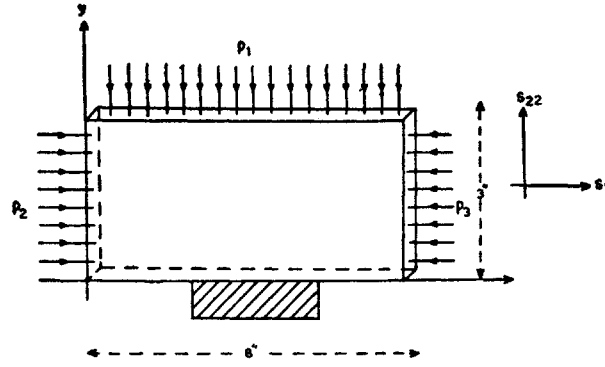
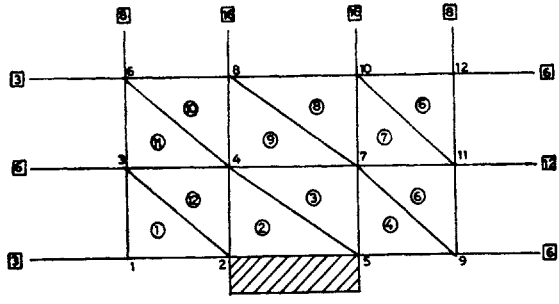


FIG. 4 Cosidered thin plate under specified stresses.



(X) : Represents the load X at the corresponding node.
(Y) : Represents element Y.

FIG. 5 Finite element idealisation.

Element specification	lth node x_l	y_l	mth node x_m	y_m	nth node x_n	y_m	(l, m, n) in anticlockwise sense
1	0	0	2.666667	0	0	1.5	(1, 2, 3)
2	2.666667	0	5.333333	0	2.666667	1.5	(2, 5, 4)
3	2.666667	1.5	5.333333	0	5.333333	1.5	(4, 5, 7)
4	5.333333	0	8	0	5.333333	1.5	(5, 9, 7)
5	5.333333	1.5	8	0	8	1.5	(7, 9, 11)
6	5.333333	3	8	1.5	8	3	(10, 11, 12)
7	5.333333	1.5	8	1.5	5.333333	3	(7, 11, 10)
8	2.666667	3	5.333333	1.5	5.333333	3	(8, 7, 10)
9	2.666667	1.5	5.333333	1.5	2.666667	3	(4, 7, 8)
10	0	3	2.666667	1.5	2.666667	3	(6, 4, 8)
11	0	1.5	2.666667	1.5	0	3	(3, 4, 6)
12	0	1.5	2.666667	0	2.666667	1.5	(3, 2, 4)

E=3000 ksi, h=0.5 inch ν=0.3 A_i (area of the ith element)=2 sq inch (i=1,2. ... 12).

The elasticity and displacement strain matrices are respectively given by (4.10) and (B.2).

The constant coefficient of the matrix multiplication $\int_V [B]^T [D] [B] \, dV$

$=2524.03846153$ for all elements. ...(C.2.1)

The total stiffness matrix

$$[K]_{24\times 24} = 2524.03846153 \left(\sum_i [k_i] \right).$$

The global equilibrium equations can we written as

$$[K] [\delta] = [T_{LS}] \qquad \text{...(C.2.2)}$$

where $[\delta]^T = [u_1 \; v_1 \; u_2 \; v_2 \; \ldots \; u_{12} \; v_{12}]$.

$[T_{LS}]$ is the matrix of the total loading system.

As $u_2 = v_2 = u_5 = v_5 = 0$ for the above problem (C.2.1) reduces to

$$[K] [\delta] = [\bar{T}_{LS}] \qquad \text{...(C.2.3)}$$

where $[\bar{K}]$ is a (20×20) matrix, $[\bar{\delta}]$ is a (20×1)matrix of unknown nodal displacement and $[\bar{T}_{LS}]$ is the known loading matrix. The form of $|\bar{T}_{LS}|$ for the above problem is

$$[\bar{T}_{LS}]^T = [3 \quad 0 \quad 6 \quad 0 \quad 0 \quad 3 \quad -8 \quad 0 \quad 0 \quad 0 \quad -16 \quad -6 \quad 0 \quad 0 \quad -16 \quad -12 \quad 0 \quad -6 \quad -8]$$

The computed nodal displacements are as follows :

u_1	v_1	u_2	v_2	u_3	v_3	u_4
.00075148	.00325818	0	0	.00298579	.00318091	.0025877
v_4	u_5	v_5	u_6	v_6	u_7	v_7
.000198404	0	0	.00510889	.00298177	.00358265	−.00184892
u_8	v_8	u_9	v_9	u_{10}	v_{10}	u_{11}
.005129	.00007879	−.0010901	−.0155824	.00780391	−.0030223	.00397354
v_{11}	u_{12}	v_{12}				
−.0161525	.0107083	−.0185167				

The following table shows the elemental stresses as computed for the problem considered above.

Element	s_{11} (in ksi)	s_{22} (in ksi)	s_{12} (in ksi)
1	−16.11814442	−6.95774064	3.08910792
2	3.81538366	5.34153713	19.90537964
3	−16.18411875	−43.32126087	18.69992773
4	−56.78166310	−56.85378988	−39.86531194
5	−3.35158527	−12.81155552	−22.93940859
6	11.08643423	−44.80093824	−15.23687814
7	−14.95437957	−29.05546755	−29.41890702
8	29.51705131	−14.23195124	19.05301400
9	17.07183447	3.23707364	10.68986050
10	−1.90862869	−3.08974800	6.98749283
11	−11.58088750	−7.94521942	3.42644916
12	−3.93588654	2.75778192	7.00029576

Observation — In each of the cases

i) $S_{11} \neq 0, \quad S_{22} = 0$

ii) $S_{11} = 0, \quad S_{22} \neq 0$

iii) $S_{11} \neq 0, \quad S_{22} \neq 0$

the only odd rows of the total stiffness matrix will differ than that of the total stiffness matrix when $S_{11} = 0$, $S_{22} = 0$ i.e. when there is no initial stress.

Conclusion

The concept of finite element technique seems very suitable to solve initially stressed problems. All the results we obtained with this method demonstrate its very good accuracy. On the other hand in concerning the finite element method, we have the results, those we obtained are good enough but with a non-symmetrical stiffness-matrix. We saw that there is a problem still not resolved to restore symmetrical matrix.

References

Biot, M. A. (1965). Mechanics of Incremental Deformation. John Wiley and Sons, Inc. New York.

Desai, C. S., and Abel, J. F. (1972). Introduction to Finite Element Method. VNR Company, New York.

Hansteen, O. E. (1967). Analysis of stress distribution is shear walls by the finite-element method. 4th International Congress on Application of Mathematics Practice of Engineering, Weimar, Germany.

Helbing, T. C. (1969). A study of plane stress or strain finite element analysis for solution of stress distribution in plane elastic continuna. Coll. of Apch. and Design, Langley Research Centre, NAT Aeronautical and Space Administration, Langley Stn. VA, Report no. NASA-TM-X-62848, p. 161 (N. T. I. S.-N70-25364).

Humphey, A. T. (1970). Application of finite element technique to stress and structural analysis. *Marconi Review* **33**, no. 179, 315.

Petyty, M. (1965). The application of finite element technique to plate and shell problems. Southampton University, England, AD-640, 839 p. 58.

Ural, O. (1973). Finite Element Method. The Intext series in Civil Engineering. Intext Educational publishers, New York.

Wilson, E. L., and Bathe, K. K. (1978). Numerical Methods in Finite Element Analysis. Prentice-Hall of India Pvt. Ltd., New Delhi.

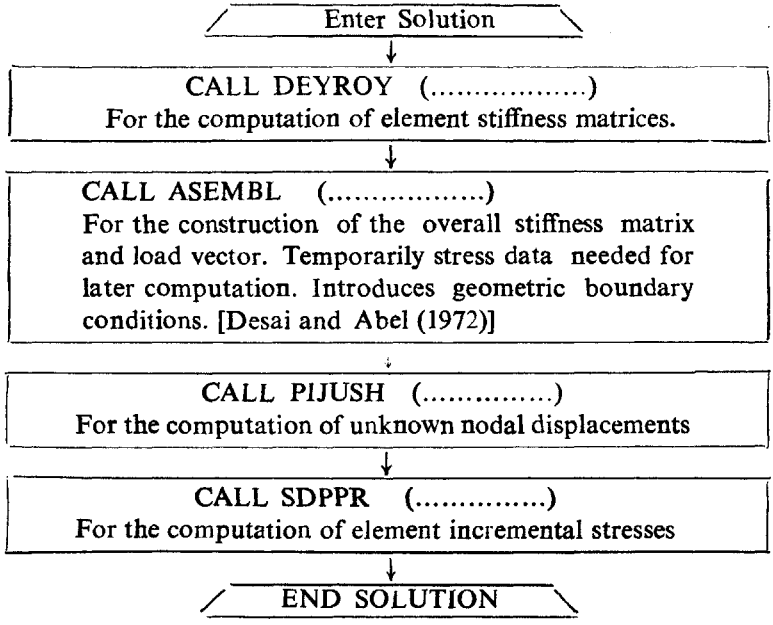
Zienkiewicz, D. C., and Cheung, Y. K. (1967). The Finite Element Method in structural and Continum Mechanics. McGraw-Hill, Book Co., Inc., New York.

APPENDIX

The following subroutines from the solution package.

i) DEYROY
ii) ASEMBL [Desai and Abel (1972)]
iii) PIJUSH
iv) SDPPR

The principal subroutines are called by the main programme in a sequence shown by the following flow diagram.



```
SUBROUTINE DEYROY (E, PR, AR, H, Q, XL, YL, XM, YM, XN, YN,S)
DIMENSION A (6, 6), B (3, 6), D (3, 3), C (6, 3), P (6, 3), S (6, 6)
P1 = (E * (1. -PR) * H)/(4. * AR * (1. + PR) * (1.-2. * PR)
B (1, 1) = YM-YN
B (1, 2) = 0.0
B (1, 3) = YN-YL
B (1, 4) = 0.0
B (1, 5) = YL-YM
B (1, 6) = 0.0
B (2, 1) = 0.0
B (2, 2) = XN-XM
B (2, 3) = 0.0
B (2, 4) = XL-XN
B (2, 5) = 0.0
B (2, 6) = XM-XL
B (3, 1) = XN-XM
B (3, 2) = YM-YN
B (3, 3) = XL-XN
B (3, 4) = YN-YL
B (3, 5) = XM-XL
B (3, 6) = YL-YM
P2 = (Q * (1-2. * PR))/(1.-PR)
D (1, 1) = 1. + P2
D (1, 2) = PR/(1.-PR) + P2
D (1, 3) = 0.0
D (2, 1) = PR/(1.-PR)
D (2,2) = 1·0
D (2, 3) = 0.0
D (3, 1) = 0.0
D (3, 2) = 0.0
D (3, 3) = P2/(2. * Q)
D0 10 I = 1, 6
D0 10 J = 1, 3
C (I, J) = 0.0
```

```
      P (I, J) = B (J, L)
      D0 10 K = 1, 3
10    C (I, J) = C (I, J) + P (I, K) * D (K, J)
      D0 20 I = 1, 6
      D0 20 J = 1, 6
      A (I, J) = 0.0
      D0 20 K = 1, 3
20    A (I, J) = A (I, J) + C (I, K) * B (K, J)
      D0 60 I = 1, 6
      D0 60 J = 1, 6
60    S (I, J) = A (I, J) * P1
      RETURN
      END
      SUBROUTINE ASEMBL (.....)
```

[DESAI AND ABEL (1973)].

```
C     INPUTS ARE T(N, N) OBTAINED FROM THE SUBROUTINE ASEMBL
C     1 AND THE LOAD MATRIX D (N, 1) ASSOCIATED WITH THE GIVEN
C     PROBLEM OUTPUTS ARE P (N, 1) THE DISPLACEMENTS
      SUBROUTINE PIJUSH (T, D, N, P)
      DIMENSION T (N, 2N), D (N, 1), B (N, N), P (N, 1)
      D0 70 I = 1, N
      D0 70 J = N + 1, 2N
      IF (J-I. EQ. N) G0 T0 6
      T (I, J) = 0.0
6     G0 T0 70
6     T (I, J) = 1.0
70    CONTINUE
      D0 10 I = 1, N
      IP = I + 1
      IPN = I + N
      D0 10 J = 1, N
C     J IS THE PIVOT ROW AND I IS THE ELIMINATION ROW
      IF (J-I) 14, 15, 14
```

```
14    F = - T(J, I)/T(I, I)
      D0 30 K = IP, IPN
30    T(J, K) = T(J, K) + F * T(I, K)
15    CONTINUE
10    CONTINUE
      D0 40 I = 1, N
      D0 40 J = N + 1, 2N
40    T(I, J) = T(I, J)/T(I, I)
      D0 35 I = 1, N
      D0 35 J = 1, N
35    B(I, J) = T(I, J + N)
      D0 25 I = 1, N
      P(I, 1) = 0.0
      D0 25 K = 1, N
25    P(I, 1) = P(I, 1) + B(I, K) + B(I, K) * D(K, I)
      RETURN
      END
      SUBROUTINE SDPPR (PR, E, H, AR, U, V, I, XL, YL, YM, XN, YN, YN,
                                                        LL, MM, NN, S)
C     PR AND AR ARE RESPECTIVELY POISSONS RATIO AND AREA OF
      THE ELEMENT
C     E IS THE YOUNGS MODULUS AND H IS THE THICKNESS OF THE
      ELEMENT
C     X AND Y ARE THE ELEMENT SPECIFICATIONS
C     U AND V ARE THE NODAL DISPLACEMENTS
C     I IS THE TOTAL NUMBER OF NODAL POINTS
      DIMENSION S(3, 1), D(3, 3), B(3, 6), W(6, 1), G(3, 6), U(I), V(I)
      Q = (E * (1. -PR))/(1. -PR-2. * PR * 2)
      R = H(1-2. * PR)/(1. -PR)
      D(1, 1) = Q * (1. + R)
      D(1, 2) = (Q * PR)/(1. -PR) + R * Q
      D(1, 3) = 0.0
      D(2, 1) = (PR * Q)/(1.-PR)
      D(2, 2) = Q
      D(2, 3) = 0.0
      D(3, 1) = 0.0
      D(3, 3) = (Q * (1.-2) * PR)/(2. * (1.-PR))
      B(1, 1) = (YM-YN)
```

```
      B (1, 2) = 0.0
      B (1, 3) = (YN−YL)
      B (1, 4) = 0.0
      B (1, 5) = (YL−YM)
      B (1, 6) = 0.0
      B (1, 6) = 0.0
      B (2, 1) = 0.0
      B (2, 2) = (XN−XM)
      B (2, 3) = 0.0
      B (2, 4) = (XL−XN)
      B (2, 5) = 0.0
      B (2, 6) = (XM−XL)
      B (3, 1) = (XN−XM)
      B (3, 2) = (YM−YN)
      B (3, 2) = (XL−XN)
      B (3, 4) = (YN−YL)
      B (3, 5) = (XM−XL)
      B (3, 6) = (YL−YM)
      A = 2. * AR
      D0 10 I = 1, 3
      D0 10 J = 1, 6

10    B (I, J) = B (I, J)/A
      W (1, 1) = U (LL)
      W (2, 1) = V (LL)
      W (3, 1) = U (MM)
      W (4, 1) = V (MM)
      W (5, 1) = U (NN)
      W (6, 1) = V (NN)
      D0 10 I = 1, 3
      D0 10 J = 1, 6
      C (I,J) = 0.0
      D0 10 K = 1, 3
10    G (I, J) = G (I, K) + D (I, K) * B (K, J)
      D0 20 I = 1,3
      S (I, 1) = 0.0
      D0 20 N = 1, 6
20    S (I, 1) = S (I, 1) + G (I, N) * W (N, 1)
      RETURN
      END.
```