Indian J. pure appl. Math., 15(9) : 1020–1035, September 1984

STATE SPACE APPROACH TO THERMOELASTIC AND MAGNETOTHERMOELASTIC PROBLEMS

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(*Received* 8 *September* 1983)

The fundamental equations of the general one dimensional problems of thermoelasticity and magnetothermoelasticity have been written in the form of an inhomogeneous matrix differential equation and solved in the Laplace transform domain by following the state space approach of the modern control theory. It is pointed out that a broad class of problems of thermoelasticity and magnetothermoelasticity can be solved by this method. Applications to problems pertaining to an infinite solid due to instantaneous heat sources are presented.

NOMENCLATURE

The nomenclature is as follows :

$\mathbf{H}$ = magnetic field vector

$\mathbf{B}$ = magnetic induction vector

$\mathbf{j}$ = current density vector

$\mathbf{E}$ = electric field vector

$\mathbf{u}$ = mechanical displacement vector

$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}$, $\mathbf{H}_0 = [H_1, H_2, H_3]$, $\mathbf{h} = [h_x, h_y h_z]$

$\mathbf{H}_0$ = primary magnetic field

$\mathbf{h}$ = perturbation vector of $\mathbf{H}$

τ_{ij} = mechanical stress tensor

e_{ij} = strain tensor

T = differential temperature distribution

μ_e = magnetic permeability

$\nu_H = \dfrac{1}{\eta\,\mu_e}$, magnetic viscosity

η = electrical conductivity

c = velocity of light in vacuum

λ, μ = Lame constants

ρ = mass density

α = Coefficient of linear thermal expansion

T_0 = reference temperature of the body when unstressed

C_v = specific heat at constant volume

$K_1 = \dfrac{K}{\rho\, C_v}$, thermal diffusivity

K = thermal conductivity

Q = heat source

F = body force component

$\delta(t)$ = Dirac delta function of t

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}$$

$$b_1 = \frac{\beta}{\rho C_V}$$

$$\beta = (3\lambda + 2\mu)\,\alpha$$

$$d_1 = \frac{K_1}{T_0\,\rho C_V\, C_1^2}$$

$$a_1 = \frac{\beta\, T_0}{\lambda + 2\mu}$$

$$b = \frac{H_3^2}{4\pi(\lambda + 2\mu)}$$

$$d = \frac{4\pi K_1}{C_1^2\, \nu_H}$$

$$\epsilon = \frac{a_1 \beta}{\rho C_V}.$$

INTRODUCTION

In recent papers Bahar and Hetnarski (1978, 1979, 1980) have extended the state space approach developed by Bahar in boundary value problems [Bahar (1974)] and heat conduction [Bahar (1974)] to problems of coupled thermoelasticity. Following Bahar and Hetnarski (1978), Das *et al.* (1981) applied the state space approach

to magnetothermoelasticity as a further extension of Bahar and Hetnarski (1978). But in all these papers, they have neglected the body forces in the equations of motion, the heat source in the equation of heat conduction etc. As a result, the matrix differential equation involved become homogeneous.

In this paper we have taken into account of an instantaneous heat source that acts on a certain plane of the infinite solid. Thus the resulting matrix differential equation become inhomogeneous and the solution of which would give further extension of the state space approach to thermoelasticity and mngnetothermoelasticity. The results obtained in the Laplace transform domain, are then compared with the existing literature, vide, Paria (1967, 1968), who obtained the results by the method of Laplace-Fourier transforms.

Development of the Method

Instantaneous Heat Source in an Infinite Thermoelastic Solid

The coupled one dimensional equation of heat conduction and motion are given by

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right) T = b_1 \frac{\partial^2 u}{\partial x\, \partial t} - d_1 Q(x, t) \qquad \ldots(1)$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) u = a_1 \frac{\partial T}{\partial x} - \frac{K_1}{C_1(\lambda + 2\mu)} F(x, t). \qquad \ldots(2)$$

The equations are written in the nondimensional form, where the position x, the displacement u, the normal stress τ_{xx}, the time t and the temperature T are related to their dimensional counterparts x_1, u_1 $(\tau_1)_{xx}$. t_1 and T_1 by the relations $x = C_1 x_1/K_1$, $u = C_1 u_1/K_1$, $\tau_{xx} = 1/\lambda + 2\mu\ (\tau_1)_{xx}$ $t = C_1^2\, t_1/K_1$ and $T = T_1/T_0$ respectively.

For the solution of eqns. (1) and (2), we shall use Laplace transforms defined by

$$\bar{u} = \int_0^\infty u \exp(-pt)\, dt, \quad \bar{T} = \int_0^\infty \bar{T} \exp(-pt)\, dt.$$

Taking Laplace transforms of equations (1) and (2) and writing the resulting equations in the matrix form, results in

$$\frac{d}{dx}\begin{Bmatrix} \bar{T}(x,p) \\ \bar{u}(x,p) \\ \bar{T}'(x,p) \\ \bar{u}'(x,p) \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p & 0 & 0 & b_1 p \\ 0 & p^2 & a_1 & 0 \end{bmatrix} \begin{Bmatrix} \bar{T}(x,p) \\ \bar{u}(x,p) \\ \bar{T}'(x,p) \\ \bar{u}'(x,p) \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -d_1 \bar{Q}(x,p) \\ -\dfrac{K_1}{C_1(\lambda+2\mu)} \bar{F}(x,p) \end{Bmatrix} \qquad \ldots(3)$$

where we have used the conditions that u, $\partial u/\partial t$, and T are zero initially. The prime indicates differentiation with respect to x.

Equation (3) can be written in contracted form as

$$\frac{d\mathbf{v}}{dx} = \mathbf{A}\,\mathbf{v} + \mathbf{f} \qquad \ldots(4)$$

where $\mathbf{v}(x, p)$ denotes the state vector in the transformed domain whose components consist of the transformed temperature, displacement as well as their gradients.

Equation (4) can be integrated by means of the matrix exponential to yield, vide, Bellman

$$\mathbf{v}(x, p) = [\exp(\mathbf{A}x)]\,\mathbf{C} + \int_{x=x_0}^{x} (\exp\{\mathbf{A}(x-s)\}]\,\mathbf{f}(s)\,ds \qquad \ldots(5)$$

where

$$\mathbf{C} = \{\exp(-\mathbf{A}\,x_0)\}\,\mathbf{v}(x_0).$$

The characteristic equation corresponding to the matrix $\mathbf{A}$ in (4) is given by

$$R^4 - p(p + 1 + \epsilon)\,R^2 + p^3 = 0 \qquad \ldots(6)$$

where $\epsilon = a_1 b_1$, is the coupling parameter, and R is a characteristic root. Since the matrix A must satisfy its own characteristic equation (Cayley-Hamilton theorem), it follows that

$$\mathbf{A}^4 - p(p + 1 + \epsilon)\,\mathbf{A}^2 + p^3\,\mathbf{I} = 0. \qquad \ldots(7)$$

Equation (7) shows that $\mathbf{A}^4$ and all higher powers of $\mathbf{A}$ can be expressed in terms of $\mathbf{A}^3$, $\mathbf{A}^2$, $\mathbf{A}$ and $\mathbf{I}$, the unit matrix of order four. The matrix exponential can now be written in the form

$$\exp(\mathbf{A}\,x) = S_0\,\mathbf{I} + S_1\,\mathbf{A} + S_2\,\mathbf{A}^2 + S_3\,\mathbf{A}^3. \qquad \ldots(8)$$

The scalar co-efficients of (8) are now evaluated by replacing the matrix $\mathbf{A}$ by its characteristic roots $\pm R_1$, $\pm R_2$ which are the roots of (6). This leads to the system of equations

$$\exp(R_1 x) = S_0 + S_1 R_1 + S_2 R_1^2 + S_3 R_1^3 \qquad \ldots(9a)$$

$$\exp(-R_1 x) = S_0 - S_1 R_1 + S_2 R_1^2 - S_3 R_1^3 \qquad \ldots(9b)$$

$$\exp(R_2 x) = S_0 + S_1 R_2 + S_2 R_2^2 + S_3 R_2^3 \qquad \ldots(9c)$$

$$\exp(-R_2 x) = S_0 - S_1 R_2 + S_2 R_2^2 - S_3 R_2^3. \qquad \ldots(9d)$$

Addition and subtraction of equations (9a) and (9b) as well as (9c) and (9d) yield the following results :

$$S_0 = \frac{R_2^2 \cosh R_1 x - R_1^2 \cosh R_2 x}{R_1^2 - R_2^2},$$

$$S_1 = \frac{-(R_2^2/R_1) \sinh R_1 x - (R_1^2/R_2) \sinh R_2 x}{R_1^2 - R_2^2},$$

$$S_2 = \frac{\cosh R_1 x - \cosh R_2 x}{R_1^2 - R_2^2},$$

$$S_3 = \frac{(1/R_1) \sinh R_1 x - (1/R_2) \sinh R_2 x}{R_1^2 - R_2^2}. \qquad \ldots(10)$$

Using (10), the matrix exponential given by (8) can now be computed as

$$\exp(\mathbf{A}x) = \begin{bmatrix} S_0 + S_2 p & b_1 S_3 p^3 & S_1 + S_3 p(1+\epsilon) & b_1 S_2 p \\ a_1 S_3 p & S_0 + S_2 p^2 & a_1 S_2 & S_1 + S_3 p(p+\epsilon) \\ [S_1 + S_3 p(1+\epsilon)]p & b_1 S_2 P^3 & S_0 + S_2 p(1+\epsilon) & b_1 p[S_1 + S_3 p(p+1+\epsilon)] \\ a_1 S_2 p & [S_1 + S_3 p(p+\epsilon)]p^2 & a_1[S_1 + S_3 p(p+1+\epsilon)] & S_0 + S_2 p(p+\epsilon) \end{bmatrix} \qquad \ldots(11)$$

$$= [L_{ij}(x, p)] \text{ (say)}, \; i, j = 1, 2, 3, 4.$$

We may now use (5) to solve a broad class of problems of coupled thermoelasticity in the Laplace transform domain. Application to an infinite space due to instantaneous heat source, is presented to illustrate the foregoing development.

Application I

Thermoelastic Interactions in an Infinite Solid

We consider an infinite elastic solid body which is unstrained and unstressed initially, but has a uniform temperature distribution T_0. It is then subjected to instantaneons heat sources distributed over a plane area. The problem is to determine the subsequent distribution of temperature, stresses and deformations, as well as the interaction between the temperature and deformation fields.

Let $x_1 = K_1 x/C_1 = 0$ represent the plane area over which the heat sources are situated and the solid occupies the whole space $-\infty < x < \infty$. We assume that the heat source is instantaneous and acts on the plane $x = 0$. We may represent it as

$$Q\left(\frac{K_1}{C_1} x, \frac{K_1}{C_1^2} t\right) = q\,\delta\left(\frac{K_1}{C_1} x\right)\delta\left(\frac{K_1}{C_1^2} t\right) \qquad \ldots(12)$$

so that

$$\bar{Q}\left(\frac{K_1}{C_1} x, \frac{K_1}{C_1^2} p\right) = q\,\frac{C_1^2}{K_1}\,\delta\left(\frac{K_1}{C_1} x\right)$$

where q is the constant strength of the source, $\delta\,(K_1/C_1\,x)$ is the Dirac's delta function.

Using (11) and (12) and neglecting the body force, we may now write down the solution (5) in the form

$$\mathbf{v}\,(n,\,p) = \left\{\begin{array}{l}\bar{T}(x,\,p)\\ \bar{u}\,(x,\,p)\\ \bar{T}'\,(x,\,p)\\ \bar{u}'\,(x,\,p)\end{array}\right\} = \int_{-\infty}^{x} \left\{\begin{array}{l} -\dfrac{q\,c_1^2\,d_1}{K_1}\,L_{13}\,(x-s,\,p)\,\delta\left(\dfrac{K_1}{C_1}\,s\right)\\ -\dfrac{q\,c_1^2\,d_1}{K_1}\,L_{23}\,(x-s,\,p)\,\delta\left(\dfrac{K_1}{C_1}\,s\right)\\ -\dfrac{q\,c_1^2\,d_1}{K_1}\,L_{33}\,(x-s,\,p)\,\delta\left(\dfrac{K_1}{C_1}\,s\right)\\ -\dfrac{q\,c_1^2\,d_1}{K_1}\,L_{43}\,(x-s,\,p)\,\delta\left(\dfrac{K_1}{C_1}\,s\right)\end{array}\right\} ds, \quad x > 0 \qquad \ldots(13)$$

where we have used the condition that $\mathbf{v}\,(x_0,\,p) = 0$ for $x_0 = -\infty$, i.e. $\bar{T}$, $\bar{u}$ $\bar{T}'$, $\bar{u}'$ are zero for $x_0 = -\infty$.

Thus using (10) and (11), the temperature $\bar{T}\,(x,\,p)$ and the deformation $\bar{u}\,(x,\,p)$ can be determined from (13) as follows :

$$\bar{T}\,(x,\,p) = \frac{qC_1^2\,d_1}{2K_1^2\,(R_1^2-R_2^2)\,R_1R_2}\,[R_2\,(R_1^2-p^2)\,e^{-R_1x}-R_1\,(R_2^2-P^2)\,e^{-R_2x}], \quad x > 0 \qquad \ldots(14)$$

$$\bar{u}\,(x,\,p) = \frac{qa_1\,C_1^2\,d_1}{2K_1^2\,(R_1^2-R_2^2)}\,[e^{-R_2x}-e^{-R_1x}],\ x > 0. \qquad \ldots(15)$$

Equations (14) and (15), when written in the dimensional form, are in couplete agreement with eqns. (3.5) and (4.1) of Paria (1968) in different notations.

Application II

Magnetothermoelastic Interaction in an Infinite Solid

In the absence of displacement current and charge density, the Maxwell's equations reduce to vide Paria (1967)

$$\text{rot}\,\mathbf{H} = \frac{4\pi}{c}\,\mathbf{j} \quad \text{rot}\,E = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t_1} \qquad \ldots(16)$$

$$\text{div}\,\mathbf{B} = 0 \quad \mathbf{B} = \mu_e\,\mathbf{H}$$

while Ohm's law is given by

$$\mathbf{j} = \eta\left[\mathbf{E} + \frac{1}{c}\left(\frac{\partial\,\mathbf{u}_1}{\partial t_1} \times \mathbf{B}\right)\right] \qquad \ldots(17)$$

where we have neglected the effect of temperature on current.

The equation of heat conduction, neglecting the effect of current on temperature, is given by

$$K\nabla^2 T_1 + Q = \rho C_v \frac{\partial T_1}{\partial t_1} + T_0 \beta \frac{\partial^2 u_1}{\partial x_1 \partial t_1}. \qquad \ldots(18)$$

Taking into account the Lorentz body force, the equation of motion becomes,

$$\rho \frac{\partial^2 (u_1)_i}{\partial t_1^2} = \frac{\partial (\tau_1)_{ik}}{\partial (x_1)_k} + \frac{1}{c}(\mathbf{j} \times \mathbf{B})_i + \mathbf{F}. \qquad \ldots(19)$$

The stress-strain relations are given by

$$(\tau_1)_{ij} = 2\mu (e_1)_{ij} - (\lambda e_1 - \beta T_1)\, \delta_{ij}$$

where

$$(e_1)_{ij} = \frac{1}{2}\left\{(u_1)_{ij} - (u_1)_{j,i}\right\}$$

$$e_1 = \text{div } \mathbf{u}_1$$

[Subscript '1' indicates the quantities involved are dimensional].

Assuming for simplicity, $\mu_e \approx 1$ (as is practically justified) the coupled one-dimensional equations of magnetothermoelasticity in the non-dimensional form are given by Das and Bhattacharya (1981), Paria (1967), as

$$\left(\frac{\partial^2}{\partial x^2} - d\frac{\partial}{\partial t}\right) h_y = 0 \qquad \ldots(20)$$

$$\left(\frac{\partial^2}{\partial x^2} - d\frac{\partial}{\partial t}\right)\left(\frac{h_z}{H_3}\right) = d\,\frac{\partial^2 u}{\partial x\, \partial t} \qquad \ldots(21)$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right) T = b_1 \frac{\partial^2 u}{\partial x\, \partial t} - d_1 Q\left(\frac{K_1}{C_1}x,\ \frac{K_1}{C_1^2}t\right) \qquad \ldots(22)$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) u = a_1 \frac{\partial T}{\partial x} + b\frac{\partial}{\partial x}\left(\frac{h_z}{H_3}\right) - \frac{K_1}{C_1(\lambda + 2\mu)} F\left(\frac{K_1}{C_1}x, \frac{K_1}{C_1^2}t\right). \qquad \ldots(23)$$

Equation (20) gives $h_y = 0$, since h_y is zero initially. Now for the solution of eqns. (21)-(23) for the variables u, T and h_z, we proceed as follows:

Taking the Laplace transform of eqns. (21)-(23) and writing the resulting equations in matrix form yields

$$\frac{d}{dx}\begin{Bmatrix} \bar{T}(x,p) \\ \bar{u}(x,p) \\ \bar{T}'(x,p) \\ \bar{u}'(x,p) \\ \bar{h}_z(x,p) \\ \bar{h}_z'(x,p) \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ p & 0 & 0 & b_1 p & 0 & 0 \\ 0 & p^2 & a_1 & 0 & 0 & b/H_3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & H_3\, pd\, pd & 0 & \end{bmatrix} \begin{Bmatrix} \bar{T}(x,p) \\ \bar{u}(x,p) \\ \bar{T}'(x,p) \\ \bar{u}'(x,p) \\ \bar{h}_z(x,p) \\ \bar{h}_z'(x,p) \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -d_1\bar{Q} \\ -\dfrac{K_1 F}{C_1(\lambda + 2\mu)} \\ 0 \\ 0 \end{Bmatrix} \qquad \ldots(24)$$

where the prime indicates differentiation with respect to x.

Equation (24) can be written in contracted form as

$$\mathbf{v}'(x, p) = \mathbf{A}(p)\,\mathbf{v}(x, p) + \mathbf{f} \qquad \ldots(25)$$

where we have used the conditions that u, $\partial u/\partial t$, T and h_z are all zero initially.

Equation (25) can be integrated by means of the matrix exponential to yield

$$\mathbf{v}(x, p) = [\exp(\mathbf{A}\,x)]\,\mathbf{C} + \int_{x=x_0}^{x} [\exp\{\mathbf{A}(x-s)\}]\,\mathbf{f}(s)\,ds \qquad \ldots(26)$$

where

$$\mathbf{C} = \{\exp(-\mathbf{A}\,x_0)\}\,\mathbf{V}(x_0).$$

The characteristic equation corresponding to the matrix $\mathbf{A}$ in (25) is given by

$$R^6 - p(p + 1 + \epsilon + d + bd)\,R^4 + p^2(p + pd + \epsilon d + bd + d)\,R^2 - dp^4 = 0 \qquad \ldots(27)$$

where R is a characteristic root. Since the matrix $\mathbf{A}$ must satisfy its own characteristic equation, it follows that

$$\mathbf{A}^6 - p(p + 1 + \epsilon + d + bd)\,\mathbf{A}^4 + p^2(p + pd + \epsilon d + bd + d)\,\mathbf{A}^2 - dp^4\,\mathbf{I} = 0. \qquad \ldots(28)$$

Equation (28) shows that $\mathbf{A}^6$ and all other higher powers of $\mathbf{A}$ can be expressed in terms of $\mathbf{A}^5$, $\mathbf{A}^4$, $\mathbf{A}^3$, $\mathbf{A}^2$, $\mathbf{A}$ and $\mathbf{I}$, the unit matrix of order six.

The matrix exponential can now be in the form

$$\exp(\mathbf{A}\,x) = S_0\,\mathbf{I} + S_1\,\mathbf{A} + S_2\mathbf{A}^2 + S_3\mathbf{A}^3 + S_4\mathbf{A}^4 + S_5\mathbf{A}^5. \qquad \ldots(29)$$

The scalar eo-efficient of (29) are now evaluated by replacing the matrix $\mathbf{A}$ by its characteristic roots $\pm R_1$, $\pm R_2$, $\pm R_3$ which are the roots of (27). This leads to the system of equations

$$\exp(R_1 x) = S_0 + S_1R_1 + S_2R_1^2 + S_3R_1^3 + S_4R_1^4 + S_5R_1^5 \qquad \ldots(30a)$$

$$\exp(-R_1 x) = S_0 - S_1R_1 + S_2R_1^2 - S_3R_1^3 + S_4R_1^4 - S_5R_1^5 \qquad \ldots(30b)$$

$$\exp(R_2 x) = S_0 + S_1R_2 + S_2R_2^2 + S_3R_2^3 + S_4R_2^4 + S_5R_2^5 \qquad \ldots(30c)$$

$$\exp(-R_2 x) = S_0 - S_1R_2 + S_2R_2^2 - S_3R_2^3 + S_4R_2^4 - S_5R_2^5 \qquad \ldots(30d)$$

$$\exp(R_3 x) = S_0 + S_1R_3 + S_2R_3^2 + S_3R_3^3 + S_4R_3^4 + S_5R_3^5 \qquad \ldots(30e)$$

$$\exp(-R_3 x) = S_0 - S_1R_3 + S_2R_3^2 - S_3R_3^3 + S_4R_3^4 - S_5R_3^5 \qquad \ldots(30f)$$

Equations (30a)-(30f) when solved for S_i, $i = 1, 2 \ldots 5$, yield the following results :

$$S_0 = \cosh R_1 x - S_2 R_1^2 - S_4 R_1^4 \qquad \ldots(31a)$$

$$S_1 = \frac{1}{R_1} (\sinh R_1 x - S_3 R_1^3 - S_5 R_1^5) \qquad \ldots(31b)$$

$$S_2 = \frac{(R_2^4 - R_3^4)(\cosh R_1 x - \cosh R_2 x) - (R_1^4 - R_2^4)(\cosh R_2 x - \cosh R_3 x)}{(R_1^2 - R_2^2)(R_2^4 - R_3^4) - (R_2^2 - R_3^2)(R_1^4 - R_2^4)} \qquad \ldots(31c)$$

$$S_3 = \frac{(R_2^4 - R_3^4)(1/R_1 \sinh R_1 x - 1/R_2 \sinh R_2 x) - (R_1^4 - R_2^4)(1/R_3 \sinh R_2 x - 1/R_3 \sinh R_3 x)}{(R_1^2 - R_2^2)(R_2^4 - R_3^4) - (R_2^2 - R_3^2)(R_1^4 - R_2^4)} \qquad \ldots(31d)$$

$$S_4 = \frac{(R_2^2 - R_3^2)(\cosh R_1 x - \cosh R_2 x) - (R_1^2 - R_2^2)(\cosh R_2 x - \cosh R_3 x)}{(R_1^4 - R_2^4)(R_2^2 - R_3^2) - (R_2^4 - R_3^4)(R_1^2 - R_2^2)} \qquad \ldots(31e)$$

$$S_5 = \frac{(R_2^2 - R_3^2)(1/R_1 \sinh R_1 x - 1/R_2 \sinh R_2 x) - (R_1^2 - R_2^2)(1/R_2 \sinh R_2 x - 1/R_3 \sinh R_3 x)}{(R_1^4 - R_2^4)(R_2^2 - R_3^2) - (R_2^4 - R_3^4)(R_1^2 - R_2^2)} \qquad \ldots(31f)$$

Computing $\mathbf{A}^2$, $\mathbf{A}^3$, $\mathbf{A}^4$ and $\mathbf{A}^5$ from $\mathbf{A}$ and using eqn. (29), the matrix exponential can be written as

$$\exp [\mathbf{A} x] = [L_{ij}], \; i, j = 1, 2, \ldots 6 \qquad \ldots(32)$$

where L_{ij} are the influence functions in the Laplace transform domain. The expressions for the influence functions L_{ij} can be found in Das and Bhattacharya (1981).

We may now use eqn. (26) to solve a broad class of problems of coupled magnetothermoclasticity in the Laplace transform domain. To illustrate the foregoing development, we shall consider the problems of a conducting infinite space due to instantaneous heat source acting on a certain plane.

(i) *Finite Conductivity*

We consider an infinite elastic solid without any initial stress or strain, but having an uniform temperature distribution T_0 throughout. The body is embeded in a uniform magnetic field of intensity H_3. The problem is to determine the distribution of stress, strain, temperature, electric and magnetic field in the body; if it is subjected to instantaneous heat sources distributed on a plane extending in all directions. Such heat sources will be called plane heat sources. Without any loss of generality, the plane containing the heat sources can be taken parallel to the initial magnetic field of intensity H_3, and the y_1 axis in the plane of the heat sources. The

axis of x_1 is then perpendicular to this plane such that x_1, y_1, z_1 form a right handed system of Cartesian co-ordinates.

As in (12), we may now represent the plane heat source in the non-dimensional co-ordinates as follows :

$$Q\left(\frac{K_1}{C_1}x, \frac{K_1}{C_1^2}t\right) = q\,\delta\left(\frac{K_1}{C_1}x\right)\delta\left(\frac{K_1}{C_1^2}t\right) \qquad \ldots(33)$$

so that

$$\bar{Q}\left(\frac{K_1}{C_1}x, \frac{K_1}{C_1^2}p\right) = q\,\frac{C_1^2}{K_1}\,\delta\left(\frac{K_1}{C_1}x\right).$$

Using (32) and (33) and neglecting the body force, we may now write down the solution (26) for $\bar{T}$, $\bar{u}$, $\bar{h}_z$ in the following form :

$$\begin{Bmatrix} \bar{T}(x,p) \\ \bar{u}(x,p) \\ \bar{h}_z(x,p) \end{Bmatrix} = -\int_{-\infty}^{x} \frac{q\,C_1^2\,d_1}{K_1} \begin{Bmatrix} L_{13}(x-s,p) \\ L_{23}(x-s,p) \\ L_{53}(x-s,p) \end{Bmatrix} \delta\left(\frac{K_1}{C_1}s\right) ds,\ x > 0 \qquad \ldots(34)$$

where

$$\left.\begin{aligned} L_{13}(x,p) &= S_1 + Z_1 S_3 + Z_2 S_5 \\ L_{23}(x,p) &= a_1 S_2 + Z_3 S_4 \\ L_{53}(x,p) &= Z_4 S_3 + Z_5 S_5 \end{aligned}\right\} \qquad \ldots(35)$$

and

$$Z_1 = p + a_1 b_1 p$$

$$Z_2 = p^2\{p + (1+\epsilon)^2 + \epsilon\, bd\}$$

$$Z_3 = a_1 p\,(p + 1 + \epsilon + bd)$$

$$Z_4 = a_1\, d\, H_3 p$$

$$Z_5 = a_1\, dH_3 P^2\,(p + 1 + \epsilon + d + bd).$$

Thus, using (31) and (35), the distribution of temperature, deformation and the perturbation of the magnetic field can be determined from (34) as follows :

$$\begin{aligned} \bar{T}(x,p) = \frac{q\,C_1^3\,d_1}{2\,K_1^2}\Bigg[& \frac{1}{R_1}e^{-R_1 x} - \frac{1}{\Delta}\Big\{(Z_2 - R_1^4)(R_2^2 - R_3^2) \\ & - (Z_1 - R_1^2)(R_2^4 - R_3^4)\Big\}\left(\frac{1}{R_2}e^{-R_2 x} - \frac{1}{R_1}e^{-R_1 x}\right) \\ & - \frac{1}{\Delta}\Big\{(Z_1 - R_1^2)(R_1^4 - R_2^4) - (Z_2 - R_1^4)(R_1^2 - R_2^2)\Big\} \\ & \left(\frac{1}{R_3}e^{-R_3 x} - \frac{1}{R_2}e^{-R_2 x}\right)\Bigg] \end{aligned} \qquad \ldots(36)$$

$$\bar{u}(x,p) = -\frac{q C_1^3 d_1}{2K_1^2 \Delta}\Big[(R_2^2-R_3^2)\Big\{Z_3-a_1(R_3^2+R_3^2)\Big\}(e^{-R_1 x}-e^{-R_1 x}) + (R_1^2-R_2^2)\Big\{-Z_3+a_1(R_1^2+R_2^2)\Big\}(e^{-R_2 x}-e^{-R_{\cdot} x})\Big] \quad \ldots(37)$$

$$\bar{h}_z(x,p) = -\frac{q C_1^3 d_1}{2K_1^2 \Delta}\Big[(R_2^2-R_3^2)\Big\{Z_5-Z_4(R_2^2+R_3^2)\Big(\frac{1}{R_2}e^{-R_2 x}-\frac{1}{R_1}e^{-R_1 x}\Big) + (R_1^2-R_2^2)\Big\{Z_4(R_1^2+R_2^2)-Z_5\Big\}\Big(\frac{1}{R_3}e^{-R_3 x}-\frac{1}{R_2}e^{-R_2 x}\Big)\Big] \quad \ldots(38)$$

where

$$\Delta = -(R_1^2-R_2^2)(R_2^2-R_3^2)(R_3^2-R_1^2)$$

and

$$x > 0.$$

(ii) *Infinite Conductivity*

Letting $\eta \to \infty$ i.e. $\nu_H = 0$ or $d \to \infty$, appropriate equations from (21)-(23) reduce in the Laplace transform domain to

$$\bar{h}_z = -H_3 \frac{d\bar{u}}{dx} \quad \ldots(39)$$

$$\Big(\frac{d^2}{dx^2}-p\Big)\bar{T} = b_1 p \frac{d\bar{u}}{dx} - d_1 \bar{Q}\Big(\frac{K_1}{C_1}x, \frac{K_1}{C_1^2}p\Big) \quad \ldots(40)$$

$$\Big(\frac{d^2}{dx^2}-p^2\Big)\bar{u} = a_1 \frac{d\bar{T}}{dx_1} + \frac{b}{H_3}\frac{d\bar{h}_z}{dx} - \frac{K_1}{C_1(\lambda+2\mu)}\bar{F}\Big(\frac{K_1}{C_1}x, \frac{K_1}{C_1^2}p\Big). \quad \ldots(41)$$

Using $\bar{h}_z$ from (39) in (41), the resulting equation and eqn. (40) can be written in the matrix form as

$$\frac{d}{dx}\begin{Bmatrix}\bar{T}(x,p)\\ \bar{u}(x,p)\\ \bar{T}'(x,p)\\ \bar{u}'(x,p)\end{Bmatrix} = \begin{Bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ p & 0 & 0 & b_1 p\\ 0 & p^2/1+b & a_1(1+b) & 0\end{Bmatrix}\begin{Bmatrix}\bar{T}(x,p)\\ \bar{u}(x,p)\\ \bar{T}'(x,p)\\ \bar{u}'(x,p)\end{Bmatrix} + \begin{Bmatrix}0\\ 0\\ -d_1\bar{Q}\\ -\frac{K_1\bar{F}}{C_1(\lambda+2\mu)}\end{Bmatrix} \quad \ldots(42)$$

which can be written in the contracted form as

$$\frac{d\mathbf{v}}{dx} = \mathbf{A}\,\mathbf{v} + \mathbf{f}. \quad \ldots(43)$$

As in (26), the solution of (43), in the absence of any body force, can be written as

$$\mathbf{v}(x, p) = [\exp(\mathbf{A}\, x)]\, \mathbf{C} + \int_{x=x_0}^{x} [\exp\{\mathbf{A}\,(x-s)\}]\, \mathbf{f}(s)\, ds \qquad \ldots(44)$$

where

$$\mathbf{C} = \{\exp(-\mathbf{A}\, x_0)\}\, \mathbf{v}(x_0).$$

The characteristic equation corresponding to the matrix **A** in (43) is given by

$$R^4 - \frac{p(p+1+\epsilon+b)}{1+b} R^2 + \frac{p^3}{1+b} = 0. \qquad \ldots(45)$$

Let $\pm R_1$ and $\pm R_2$ be the roots of (45). Proceeding as in the manner as previously, the matrix exponential can be expressed as

$$\exp[\mathbf{A}\, x] = [L_{ij}(x, p)],\ i, j = 1, 2, 3, 4. \qquad \ldots(46)$$

The expression for L_{ij} can be found in Das *et al.* (1981). Thus, in the absence of any body force, the temperature and the displacement field can be expressed from (44) as follows :

$$\begin{Bmatrix} \bar{T}(x, p) \\ \bar{u}(x, p) \end{Bmatrix} = -\frac{q C_1^2 d_1}{K_1} \int_{-\infty}^{x} \begin{Bmatrix} L_{13}(x-s, p) \\ L_{23}(x-s, p) \end{Bmatrix} \delta\left(\frac{K_1}{C_1} s\right) ds,\ x > 0 \qquad \ldots(47)$$

where

$$\left.\begin{aligned} L_{13}(x, p) &= \left[R_1\left(R_2^2 - \frac{p^2}{1+b}\right) e^{-R_2 x} - R_2\left(R_1^2 - \frac{p^3}{1+b}\right) e^{-R_1 x}\right] \Big/ [2R_1 R_2 (R_1^2 - R_2^2)] \\ L_{23}(x, p) &= \frac{a_1}{T_0 (1+b)}\left[\frac{e^{-R_1 x} - e^{-R_2 x}}{2(R_1^2 - R_2^2)}\right]. \end{aligned}\right\} \qquad \ldots(48)$$

Using (48) in (47) and performing the integration, we determine the temperature and displacement field as

$$\bar{T}(x, p) = \frac{q C_1^3 d_1}{2K_1^2 R_1 R_2 (R_1^2 - R_2^2)}\left[R_2\left(R_1^2 - \frac{p^2}{1+b}\right) e^{-R_1 x} - R_1\left(R_2^2 - \frac{p^2}{1+b}\right) e^{-R_2 x}\right],\ x > 0 \qquad \ldots(49)$$

$$\bar{u}(x, p) = \frac{q C_1^3 d_1 a_1}{2K_1^2 (R_1^2 - R_2^2)(1+b)} (e^{-R_2 x} - e^{-R_1 x}),\ x > 0. \qquad \ldots(50)$$

The perturbation of the magnetic field can now be determined from (49) and (50) as

$$\bar{h}_z(x, p) = -\frac{H_3 q C_1^3 d_1 a_1}{2(1+b) K_1^2 (R_1^2 - R_2^2)} (R_1 e^{-R_1 x} - R_2 e^{-R_2 x}),\ x > 0. \quad \ldots(51)$$

It may be noted that (i) in the absence of the magnetic field, i.e when $b = 0$, eqn. (45) is identical with (6) and, as expected, eqns. (49) and (50) are identical with eqns. (14) and (15) respectively.

Paria (1967) also solved the above problem by the method of Laplace-Fourier transforms and found out the expressions for $\bar{h}_z(x, p)$ in eqn. (10.36a) in the Laplace transform domain, but this expression is not in agreement with our expression (51) due to a mistake in his calculation in eqn. (10.15d).

Distribution of Stress and Induced Magnetic Field for Small Time

For small time i.e. for large p, $\sqrt{R}$, R_1 and R_2 can be calculated as

$$\sqrt{R} = K_1 p_1,\ R_1 = \frac{\sqrt{K_1}}{C_1 C_3} \xi_3,\ R_2 = \frac{K_1}{C_1} \xi_4 \quad \ldots(52)$$

where

$$\xi_3 = \sqrt{K_1 p_1} \left(1 + \frac{C_5^2}{4 K_1 p_1}\right),\ \xi_4 = \frac{C_5}{C_3} \sqrt{\frac{p_1}{2K_1}}$$

and p_1 is the dimensional counter part of non-dimensional p.

Equation (51) now reduces to

$$(\bar{h}_1)_z = -\frac{q H_3 \beta}{2 \rho^2 C_v C_3 K_1 p_1^2} \left[\left(p_1 + \frac{C_5^2}{4K_1}\right) \exp\left(-\frac{p_1 x_1}{C_3}\right) \exp\left(-\frac{C_5^2 x_1}{2K_1 C_3}\right) - C_5 \sqrt{\frac{p_1}{2K_1}} \exp\left(-\frac{C_5}{C_3} x_1 \sqrt{\frac{p_1}{2K_1}}\right)\right]. \quad \ldots(53)$$

Taking the inverse Laplace transform and then using the dimensionless quantities η, τ, R_M, ϵ_T defined as

$$\eta = \frac{C_1}{K_1} x_1,\ \tau^2 = \frac{C_1^2}{K_1} t_1,\quad \epsilon_T = \epsilon$$

$$R_M = 1 + R_H = 1 + b = \frac{C_2^2}{C_1^2} \quad \ldots(54)$$

we obtain

$$h_z(\eta, \tau) = \frac{q H_3 \beta}{2 \rho^2 C_v K_1 C_1 R_M} \varphi(\eta, \tau; R_M, \epsilon_T)$$

where

$$\varphi(\eta, \tau; R_M, \epsilon_T) = \{\sqrt{R_M} + \frac{1}{4}(R_M+\epsilon_T)(\tau^2\sqrt{R_M} - \eta)\}$$

$$H(\tau^2\sqrt{R_M} - \eta)\exp\left(-\frac{R_M+\epsilon_T}{4\sqrt{R_M}}\eta\right)$$

$$-\sqrt{2(R_M+\epsilon_T)R_M}\,\tau\,\text{ierfc}\left(\frac{\eta}{2\tau}\sqrt{\frac{R_M+\epsilon_T}{2R_M}}\right). \quad \ldots(55)$$

Here $H(\tau^2\sqrt{R_M} - \eta)$ and ierfc, denotes heavy side unite step function and associated complementary error function respectively. Table 1 gives the values of $\varphi(\eta)$ when $\tau = 0.5$, $R_M = 1.5$, and $\epsilon_T = 0$.

TABLE I

Values of $\varphi(\eta)$

η	0	0.1	0.2	0.3	0.4	0.5	0.6
$\varphi(\eta)$	1.003	0.961	0.927	0.890	−0.194	−0.163	−0.129

The normal stress-displacement relation in the dimensional form after taking Laplace transform, is given by

$$(\bar{\tau}_1)_{xx} = (\lambda + 2\mu)\frac{d\bar{u}_1}{dx_1} - \beta T_1. \quad \ldots(56)$$

Now making the equations (49) and (50) in the dimensional form and then using the results in (56) we get, after simplification

$$(\bar{\tau}_1)_{xx} = -\frac{q\beta}{2\rho C_v p_1^{3/2}}\left[\frac{A_1\xi_4}{C_1\sqrt{K_1}}\exp\left(-\frac{\xi_3 x_1}{C_3\sqrt{K_1}}\right) + B_1\xi_3\exp(-\xi_4 x_1)\right] \quad \ldots(57)$$

where,

$$A_1 = \frac{C_1^2}{p_1\sqrt{R}}[R_H\xi_3^2 - (1+R_H)p_1^2 K_1] \quad \ldots(58)$$

$$B_1 = -\frac{1}{p_1\sqrt{R}}[C_1^2 R_H \xi_4^2 - p_1^2].$$

Now making use of the approximations (52), non-dimensional quantities (54) and

$$(\bar{\tau}_1)_{xx} = (\lambda + 2\mu)\,\bar{\tau}_{xx}$$

we get from (57), after taking Laplace inversion,

$$\tau_{xx} = \frac{q\beta}{2\rho^2 C_v C_1 K_1}\,\psi(\eta, \tau; R_M, \epsilon_T)$$

where

$$\psi(\eta, \tau; R_M, \epsilon_T) = \frac{1}{\bar{R}_M}\sqrt{\frac{1}{2}(R_M + \epsilon_T)}\left\{1 - \frac{1}{2}\frac{R_H}{\sqrt{R_M}}(R_M + \epsilon_T)\right.$$

$$\left.(\tau^2\sqrt{R_M} - \eta)\right\}\exp\left(-\frac{1}{4}\frac{R_M + \epsilon_T}{\sqrt{R_M}}\eta\right)$$

$$H(\tau^2\sqrt{R_M} - \eta) - \frac{1}{\sqrt{\pi\tau}}\exp\left(-\frac{1}{8}\frac{R_M + \epsilon_T}{\sqrt{R_M}}\frac{\eta^2}{\tau^2}\right)$$

$$-\frac{1}{4}(1 - R_H)(R_M + \epsilon_T)$$

$$\frac{1}{R_M}\left\{\frac{2\tau}{\sqrt{\pi}}\exp\left(-\frac{1}{8}\frac{R_M + \epsilon_T}{R_M}\frac{\eta^2}{\tau^2}\right) - \sqrt{\frac{R_M + \epsilon_T}{2R_M}}\operatorname{erfc}\left(\sqrt{\frac{R_M + \epsilon_T}{2R_M}}\frac{\eta}{2\tau}\right)\right\}$$

which is same as (10.56) of Paria (1967).

Concluding Remarks

The foregoing development shows that the state space approach is a simple but very effective method to deal with the problems of interactions of the coupled fields of the most general type. Thus we may state that a broad class of problems of coupled fields can be solved in the Laplace transform domain by this approach. The real difficulty of the theoretical investigation of the problems lies in the final inversion of the Laplace transforms, depending upon the physical nature of the problem.

It is always possible to obtain the time domain information from the expressions in the Laplace transform domain and this can be accomplished through the use of initial and final value theorems.

$$\lim_{t \to 0} F(t) = \lim_{p \to \infty} p f(p)$$

$$\lim_{t \to \infty} F(t) = \lim_{p \to 0} p f(p).$$

Again, the inversion can also be made on a computer using Laplace transform inversion algorithms, vide, Olsen and Brockus (1976) Longman (1976, Luke (1978), Piessens (1975), Krylov and Skoblya (1969) Singhal and Vlach (1975) etc.

ACKNOWLEDGEMENT

We are thankful to the U.G.C.-D.S.A. programme of the Department of Mathematics, Jadavpur University, for the support of this work.

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