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ON UNIQUELY REMOTAL SETS

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In this note, we derive some necessary and sufficient condition for a uniquely remotal set to be singleton. We show that in an inner product space the farthest point map is pseudo-contractive, moreover we prove the uniqueness of center for remotal subsets of inner product spaces.

1. INTRODUCTION

Let (E, d) be a metric space and T be a non empty bounded subset of E. The set-Valued map defined by

$$Q_T(x) = \{q_T(x) \in T : d(x, q_T(x)) = \sup_{t \in T} d(x, t)) \}$$

is called the farthest point map of T. If for any $x \in E$, the set $Q_T(x)$ is not empty (resp. is singleton) then T is said to be remotal (resp. uniquely remotal). If T is uniquely remotal set, then $x \to d(x, q_T(x))$ is uniformly continuous. One can easily verifiy the continuity of the farthestpoint map for compact uniquely remotal set (Ahuja *et al.* 1975), and upper (K)-semi continuity of the farthest point map for closed and remotal subset of E (Narang 1978). It follows from Mazur-Tychonoff theorems that a compact uniquely remotal set in a normed space is singleton (Klee 1961). The following well-known question is not solved yet: If T is a uniquely remotal subset of a normed space, then can we conclude that T is singleton ? There are some affirmative answers to this question in Asplund (1967), Blatter (1969), Bosznay (1979, 1981), Narang (1977) and Panda *et al.* (1978). It is easy to show that the "unique farthest point property" of a set in a normed space is inherited by its convex hull and also (in any Hilbert Space) by its closure. An affirmative solution to above problem implies that every chebyshev set in a Hilbert space is convex (Klee (1961)). It is easy to see that the farthest point map of a uniquely remotal subset of a metric space is closed, i.e, if $x_n \to x$ and $q_T(x_n) \to y$, then $y = q_T(x)$. It follows that T is singleton if and only if there is an element x_0 such that $q_T^n(x_0)$ is convergent. One can see that the map $x \to \| q_T(x) \|$ is continuous

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for any uniquely remotal subset of a normed space. Thus the following problem poses itself: Can we conclude that the map $x \to q_T(x)$ is continuous? This map is continuous if T is compact. If the farthest point map of a uniquely remotal subset of a Banach space is continuous, then T is singleton (Blatter 1969). Panda and Kapoor used the idea of Chebyshev center and obtained the Blatter result for spaces admitting centers (Panda *et al.* 1978). A center (or chebyshev center) of a bounded, non-empty set T in a normed space E is an element c in E for which

$$\sup_{t\in T} \| c-t \| = \inf_{x\in E} \sup_{t\in T} \| x-t \|.$$

The number $\sup_{t\in T} \| c-t \|$ is called the chebyshev radius of T and is denoted by $r(T)$. In this note we show that the farthest point map of a uniquely remotal subset of an inner product space $(H, <,>)$ is pseudo-Contractive, i.e.

$$\| q_T(x)-q_T(y) \|^2 \leqslant \| x-y \|^2 + \| x-y+q_T(y)-q_T(x) \|^2$$

then we give a proof for Blatter result in Hilbert spaces. We also show that an element c of an inner product space H is a center of T if and only if $\| x-q_T(x) \|^2 \geqslant \| x-c \|^2 + \| c-q_T(c) \|^2$ $(x\in H)$. It follows that if T admits center, then it has a unique center. It follows also that if $c = 0$ and $(\mathrm{Re} <x_0, q_T(x_0)>) > 0$ for some $x_0 \in H$, then T is singleton. We notice that in a uniformly rotund Banach space, every bounded subset has a unique center (Holmes 1982).

2. The Results

The following results may be useful tools towards the solution of the farthest point problem in Hilbert spaces.

Theorem 1—Let T be a uniquely remotal subset of a normed space $(E, \| \cdot \|)$. Then a necessary and sufficient condition that T be singleton is that $\| x-q_T(x) \| = \| y-q_T(y) \|$ implies $q_T(x) = q_T(y)$.

Proof: If T is singleton, then $q_T(x) = q_T(y)$ $(x, y \in E)$. Suppose now that $\| x-q_T(x) \| = \| y-q_T(y) \| \Rightarrow q_T(x) = q_T(y)$. We show that for any $x, y \in E$, $q_T(x) = q_T(y)$ and hence T is singleton. For let $x_0 \neq y_0$ and

$$\| x_0-q_T(x_0) \| \neq \| y_0-q_T(y_0) \| \neq 0.$$

We can assume without loss of generality that $\| x_0-q_T(x_0) \| < \| y_0-q_T(y_0) \|$.

Taking $z_0 = q_T(x_0) + \dfrac{\| y_0-q_T(y_0) \|}{\| x_0-q_T(x_0) \|} (x_0-q_T(x))$, it follows, that,

$$\| z_0-q_T(x_0) \| = \| y_0-q_T(y_0) \|$$

and

$$\begin{aligned} \| y_0 - q_T(y_0) \| &= \| z_0 - q_T(x) \| \\ &\leqslant \| z_0 - q_T(z_0) \| \\ &\leqslant \| z_0 - x_0 \| + \| x_0 - q_T(z_0) \| \\ &\leqslant \| z_0 - x_0 \| + \| x_0 - q_T(x_0) \| = \| y_0 - q_T^{-}(y_0) \| \end{aligned}$$

Hence $\| y_0 - q_T(y_0) \| = \| z_0 - q_T(x_0) \| = \| z_0 - q_T(z_0) \|$ and thus

$$q_T(y_0) = q_T(z_0) = q_T(x_0).$$

Let T be a remotal subset of an inner product space $(H, <,>)$. For any $x, y \in H$ let $q_T(x) \in Q_T(x)$, $q_T(y) \in Q_T(y)$. We need the following lemma which can be found in (Papini (1982)) due to the referees remarks.

Lemma 2—$\operatorname{Re} < x-y, q_T(x)-q_T(y) > \leqslant 0$.

PROOF : Let $t \in T$, then $\| x-t \| \leqslant \| x-q_T(x) \|$ $(x \in H)$. An easy computation shows that

$$\| q_T(x) - t \|^2 + 2 \operatorname{Re} < x-q_T(x), q_T(x)-t > \leqslant 0$$

and therefore.

$$\operatorname{Re} < 2x-t-q_T(x), q_T(x)-t > \leqslant 0 \quad (*)$$

Take $t = q_T(y)$ in (*) and replace x to y and t to $q_T(x)$ we get

$$\operatorname{Re} < 2x-q_T(y)-q_T(x), q_T(x)-q_T(y) > \leqslant 0.$$

and

$$\operatorname{Re} < 2y-q_T(x)-q_T(y), q_T(y)-q_T(x) > \leqslant 0.$$

Hence

$$\operatorname{Re} < x-y, q_T(x)-q_T(y) \geqslant \leqslant 0 \ (x, y \in H).$$

It follows from the above lemma that

$$\| q_T(x)-q_T(y) \| \leqslant \| x-y+q_T(y)-q_T(x) \|$$

for any $x, y \in H$ and $q_T(x) \in Q_T(x)$, $q_T(y) \in Q_T(y)$. Thus we have

Proposition 3—The farthest point map of a uniquely remotal subset of an inner product space is pseudo-Contractive.

Corollary 4—Let T be a uniquely remotal subset of a Hilbert space. Then T it singleton if and only if the farthest point map is continuous on the closure of T.

PROOF : We can assume without loss of generality that T is closed and convex. Applying Theorem 2 of Reinermann *et al.* (1976) we obtain a fixed point for q_T, since q_T is Pseudo-contractive continuous map on T and

$$\lim_{\lambda \to 0^+} \frac{\inf \{ \| (1-\lambda)\, x + \lambda q_T(x) - t \| \}}{\lambda} = 0 \quad (x \in T).$$

Let T be a uniquely remotal subset of a normed space E. We define a real-valued function f_T on E by $f_T(x) = \| x - q_T(x) \|$ and call it the farthest distance of T. Let x_0 be an element in E and $E_{x_0} = \{x \in E : f_T(x) \geqslant f_T(x_0)\}$. One can see easily that the farthest point map q_T is continuous at x_0 if and only if the restriction of q_T to E_{x_0} is continuous at x_0.

Theorem 5—Let T be a remotal subset of an inner product space $(H, <,>)$. Then an element $c \in H$ is a center of T if and only if

$$\| x - q_T(x) \|^2 \geqslant \| x - c \|^2 + \| c - q_T(c) \|^2 \quad (x \in H). \qquad (**)$$

PROOF : If (**) holds it is easy to see that c is a center of T. Now suppose that c is a center of T. We can assume that T is not singleton, otherwise (**) is obvious. By replacing T to $G = \{c - t : t \in T\}$. One can assume that $c = 0$, since G is remotal, $\| q_G(0) \| = \| c - q_T(c) \|$ and $q_G(x) = c - q_T(c - x)$ $(x \in H)$.

Lemma 2 shows that $\mathrm{Re} < x - y, q_T(x) - q_T(y) > \leqslant 0$ $(x, y \in H)$. It follows that if $0 < \epsilon \leqslant 1$, then

$$\mathrm{Re} < x, q_T(\epsilon x) > \geqslant \mathrm{Re} < x, q_T(x) > \quad (x \in H).$$

We show that for any $x \in H$, $\mathrm{Re} < x, q_T(x) > \leqslant 0$. For,

$$\begin{aligned} \| q_T(0) \|^2 \leqslant \| x - q_T(x) \|^2 &= \| x \|^2 + \| q_T(x) \|^2 - 2\mathrm{Re} < x, q_T(x) > \\ &\leqslant \| x \|^2 + \| q_T(0) \|^2 - 2\mathrm{Re} < x, q_T(x) > \end{aligned}$$

hence $2\mathrm{Re} < x, q_T(x) > \leqslant \| x \|^2$ $(x \in H)$. It follows that

$$2\mathrm{Re} < x, q_T(x) > \leqslant \epsilon \| x \|^2$$

and therefore $\mathrm{Re} < x, q_T(x) > \leqslant 0$. Thus we have

$$\begin{aligned} \| q_T(0) \|^2 \leqslant \| \epsilon x - q_T(\epsilon x) \|^2 &= \| \epsilon x - x + x - q_T(\epsilon x) \|^2 \\ &= (1-\epsilon)^2 \| x \|^2 + \| x - q_T(\epsilon x) \|^2 \\ &\quad + 2\mathrm{Re} < \epsilon x - x, x - q_T(\epsilon x) > \\ &\leqslant \| x - q_T(x) \|^2 - (1-\epsilon^2) \| x \|^2 \\ &\quad + 2(1-\epsilon)\, \mathrm{Re} < x, q_T(\epsilon x) > \\ &\leqslant \| x - q_T(x) \|^2 - (1-\epsilon^2) \| x \|^2. \end{aligned}$$

Hence

$$\| x-q_T(x) \|^2 \geqslant \lim_{\epsilon \to 0^+} (\| q_T(0) \|^2 + (1-\epsilon^2) \| x \|^2) = \| x \|^2 + \| q_T(0) \|^2.$$

Corollary 6—Let c be a center of a remotal subset of an inner product space. Then c is unique.

PROOF : If c, c_0 are two centers of a remotal set T, then by Theorem 5 we have

$$\| c_0-q_T(c_0) \|^2 \geqslant \| c_0-c \|^2 + \| c-q_T(c) \|^2 = \| c_0-c \|^2 + \| c_0-q_T(c_0) \|^2$$

Thus $c_0 = c$.

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