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ON QUASI-METRIZABILITY

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In this paper three sets of necessary and sufficient conditions for a bitopological space to be quasi-metrizable are derived.

In this paper three theorems on quasi-metrizability of bitopological spaces are derived. There are not very many results, to the best of author's knowledge, giving necessary and sufficient conditions for quasi-metrizability of bitopological spaces. The reason for this, probably, is that bases with σ-local finiteness property do not lend themselves easily to be developed in the bitopological setting. One can say the same thing about notions like full normality, paracompactness, bases which are σ-closure preserving etc. However, an analogue of the classical metrization theorem of Alexandroff and Urysohn has been proved by Reilly (1970, 2.3, p. 51). We will be using this result of Reilly to prove a theorem on quasi-metrizability of bitopological spaces and derive two other related results.

We write 'nbd' for neighbourhood. If A is a subset of X and $\mathcal{T}_i$ is a topology on X, then $\mathcal{T}_i$-cl A ($\mathcal{T}_i$-int A) is the closure (interior) of A in the space $(X, \mathcal{T}_i)$. The letters m, n, m_i, n_i, m_j, n_j all these are integers (> 0). The letters i, j, are always taken to vary as $i, j = 1, 2; i \neq j$. Whenever i appears as a subscript of a set, then the set is a $\mathcal{T}_i$-nbd (open or otherwise as the context will make it clear). For example $S_{n\,i}(x)$ is a $\mathcal{T}_i$-nbd of x. We write b. t. s to mean bitopological space.

Let us now give some preliminary definitions and references. By a quasi-metric p_1 on X we mean a non-negative real-valued function $p_1 : X \times X \to R$ such that

(i) $p_1(x, y) = 0$ if and only if $x = y$

(ii) $p_1(x, y) + p_1(y, z) \geqslant p_1(x, z)$ for all $x, y, z \in X$. If we write $p_2(x, y) = p_1(y, x)$, we get a quasi-metric p_2 on X. p_2 is called the 'conjugate of p_1'. The concept of quasi-metric spaces was first introduced by Wilson (1931). Then the fact that a quasi-metric leads to a conjugate quasi-metric was noticed by Kelly (1963)

leading to the study of bitopological spaces. A b. t. s $(X, \mathcal{T}_1, \mathcal{T}_2)$ is 'quasi-metrizable' if and only if there exists a quasi-metric p_1 on X such that the topology induced by p_1 is precisely $\mathcal{T}_1$ and $\mathcal{T}_2$ is the topology induced by p_2, the conjugate of p_1. It was proved by Kelly (1963) that "if $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise regular and $(X, \mathcal{T}_i)$ is second countable ($i = 1, 2,$) then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is quasi-metrizable". Later on Lane (1967) obtained sufficient conditions for $(X, \mathcal{T}_1, \mathcal{T}_2)$ to be quasi-metrizable and this result (Lane (1967; Theorem 3.1)) is a partial generalization of the celebrated metrization theorem of Nagata (1950).

A family $\mathcal{V}$ of subsets of $X \times X$ is called a 'quasi-uniformity' on X if (i) $V \in \mathcal{V}$ implies $\Delta \subset V$ (where $\Delta = \{(x, x) \mid x \in X\}$, the diagonal of $X \times X$) (ii) $V, V' \in \mathcal{V}$ implies that there is a $V'' \in \mathcal{V}$ such that $V'' \subset V \cap V'$ (iii) for each $V \in \mathcal{V}$ there is a $V' \in \mathcal{V}$ such that $V' \circ V' \subset V$ (iv) $V \in \mathcal{V}$ and $V \subset V'$ implies $V' \in \mathcal{V}$. A quasi-uniformity $\mathcal{V}$ on X induces a topology $\mathcal{T}(= \mathcal{T}(\mathcal{V}))$ on X by setting $U (\subset X)$ to be open if and only if for each $x \in U$ there is a $V \in \mathcal{V}$ such that $V[x] \subset U$, where $V[x] = \{y \mid (x, y) \in V\}$. Indeed, according to Pervin (1963), every topology $\mathcal{T}$ on X can be obtained in this way in the sense that there exists a quasi-uniformity $\mathcal{V}$ which induces $\mathcal{T}$. If we write $V^{-1} = \{(x,y) \mid (y, x) \in V\}$ and $\mathcal{V}^{-1} = \{V^{-1} \mid V \in \mathcal{V}\}$ then $\mathcal{V}^{-1}$ is a quasi-uniformity in its own right and it is called the conjugate of $\mathcal{V}$. $(X, \mathcal{T}_1, \mathcal{T}_2)$ is called a quasi-uniform space if and only if there exists a quasi-uniformity $\mathcal{V}$ such that $\mathcal{T}_1 = \mathcal{T}_1(\mathcal{V})$ and $\mathcal{T}_2 = \mathcal{T}_2(\mathcal{V}^{-1})$. $(X, \mathcal{T}_1, \mathcal{T}_2)$ is a quasi-uniform space if and only if the space is pairwise completely regular. $\mathcal{B}$ is called a base for the quasi-uniformity $\mathcal{V}$ if every member of $\mathcal{V}$ contains a member of $\mathcal{B}$. For a detailed study of quasi-uniform spaces one may refer to Murdeshwar and Naimpally (1966) and Stoltenberg (1967).

For definitions of pairwise Hausdorff, pairwise regular, pairwise normal and pairwise completely regular, one may refer to Kelly (1963) and Lane (1967). The quasi-metric analogue of the metrization theorem of Alexandroff and Urysohn referred to earlier is the following :

Theorem 1 (Reilly 1970)—A pairwise Hausdorff quasi-uniform space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is quasi-metrizable if and only if its quasi-uniformity has a countable base.

Let us now give the first result on quasi-metrizability.

Theorem 2—A pairwise Hausdorff b. t. s. $(X, \mathcal{T}_1, \mathcal{T}_2)$ is quasi-metrizable if and only if for each element $x \in X$, one can assign $\mathcal{T}_i$-nbd bases $\{U_{n\,i}(x) \mid n = 1, 2, ..\}$ $(i = 1, 2)$ such that for every n and each point x there are $\mathcal{T}_i$-nbds $S_{n\,i}(x)$ satisfying :

(i) $U_{n+1\,i}(x) \subset S_{n\,i}(x) \subset U_{n\,i}(x)$

(ii) $y \notin U_{n\,i}(x) \Rightarrow S_{n\,i}(x) \cap S_{n\,j}(y) = \phi$

(iii) $y \in S_{n\,i}(x) \Rightarrow S_{n\,i}(y) \subset U_{n\,i}(x)$

$(i, j = 1, 2, i \neq j)$

PROOF : Let us first prove that the conditions are sufficient. Let us now show that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise regular. If $S_{n\,i}(x) \cap S_{n\,j}(y) = \phi$, then $S_{n\,i}(x) \subset X - S_{n\,j}(y)$ so that $\mathscr{T}_j$-cl $S_{n\,i}(x) \subset X - \mathscr{T}_j$-int $S_{n\,j}(y)$. Thus, if $y \notin U_{n\,i}(x)$, then $y \notin \mathscr{T}_j$-cl $S_{n\,i}(x)$ so that $x \in S_{n\,i}(x) \subset \mathscr{T}_j$-cl $S_{n\,i}(x) \subset U_{n\,i}(x)$.

Further the space is pairwise normal. Indeed if A and B are $\mathscr{T}_1$-closed and $\mathscr{T}_2$-closed subsets (of X) respectively such that $A \cap B = \phi$ and $y \in B$, then there exists an integer $n(y)$ (> 0) such that $A \cap \mathscr{T}_2$-cl $U_{n(y)\,1}(y) = \phi$. Since $x \notin U_{n(y)\,1}(y)$ for each $x \in A$,

$$S_{n(y)\,1}(y) \cap S_{n(y)\,2}(x) = \phi \text{ for all } x \in A.$$

If $Q_{n(y)} = \cup \{\mathscr{T}_2\text{-int } S_{n(y)\,2}(x) \mid x \in A\}$, then $Q_{n(y)} \supset A$ and $Q_{n(y)} \cap \mathscr{T}_2$-cl $U_{n(y)\,1}(y) = \phi$. If we write $\cup \{\mathscr{T}_1\text{-int } S_{n(y)\,1}(y) \mid n(y) = k\} = W_{k\,1}$, then $Q_k \cap \mathscr{T}_2$-cl $W_{k\,1} = \phi$ so that we get a $\mathscr{T}_1$-open covering $\{W_{k\,1} \mid k = 1, 2.....\}$ of B such that $A \cap \mathscr{T}_2$-cl $W_{k\,1} = \phi$ for each k. Similarly we can form a $\mathscr{T}_2$-open covering $\{W_{k\,2} \mid k = 1, 2....\}$ of A such that $B \cap \mathscr{T}_1$-cl $W_{k\,2} = \phi$ for each k. Then a standard argument (Kelley 1955, p. 113) produces disjoint sets $W_1 \in \mathscr{T}_1$ and $W_2 \in \mathscr{T}_2$ such that $W_1 \supset B$ and $W_2 \supset A$.

Thus the space is pairwise completely regular so that there exists a quasi-uniformity $\mathscr{V}$ such that $\mathscr{T}_1 = \mathscr{T}_1(\mathscr{V})$ and $\mathscr{T}_2 = \mathscr{T}_2(\mathscr{V}^{-1})$ where $\mathscr{V}^{-1}$ is the conjugate of $\mathscr{V}$.

Now let us define two sets as follows :

$$S_i(x, \mathscr{K}_i^{(m)}) = \cup \{\mathscr{T}_i\text{-int } S_{m\,i}(y) \mid x \in \mathscr{T}_j\text{-int } S_{m\,j}(y)\}$$

where

$$\mathscr{K}_i^{(m)} = \{\mathscr{T}_i\text{-int } S_{m\,i}(y) \mid y \in X\}.$$

Let $\mathscr{B}_i(x) = \{S_i(x, \mathscr{K}_i^{(m)}) \mid m = 1, 2,....\}$. We claim $\mathscr{B}_i(x)$ is a $\mathscr{T}_i$-local base at x. If x is fixed initially and $U_i(x)$ are arbitrary $\mathscr{T}_i$-nbds of x, then there exist n_i such that $x \in U_{n_i\,i}(x) \subset U_i(x)$. Consider $m = \max(n_1 + 1, n_2 + 1)$. Then clearly $U_{m\,i}(x) \subset S_{n_i\,i}(x)$. In order to avoid confusion with subscripts, let us prove specifically $\mathscr{B}_2(x)$ is a local base at x. Let y be such that $x \in \mathscr{T}_1$-int $S_{m\,1}(y)$. Then $S_{m\,1}(y) \cap S_{m\,2}(x) \neq \phi$ so that $y \in U_{m\,2}(x) \subset S_{n_2\,2}(x)$. Hence $S_{n_2\,2}(y) \subset U_{n_2\,2}(x)$. Since $m \geqslant \max(n_1 + 1, n_2 + 1)$, $\mathscr{T}_2$-int $S_{m\,2}(y) \subset U_{n_2\,2}(x) \subset U_2 x$. Thus $\mathscr{B}_2(x)$ is a $\mathscr{T}_2$-local base at x. Similarly we see that $\mathscr{B}_1(x)$ is a $\mathscr{T}_1$-local base at x.

The proof of the theorem is complete if we exhibit a countable base for the quasi-uniformity. This will be true when we establish that the coverings $\mathscr{K}_i^{(m)} = \{\mathscr{T}_i\text{-int } S_{m\,i}(x) \mid x \in X\}$ admit refinements which resemble Δ-refinements.

If $x \in \mathscr{T}_1\text{-int } S_{n+2\,1}(y)$, then $S_{n+2\,1}(y) \cap S_{n+2\,2}(x) \neq \phi$ so that by (ii) and (i) $y \in U_{n+2\,2}(x) \subset S_{n+1\,2}(x)$. Hence by (iii) $S_{n+1\,2}(y) \subset U_{n+1\,2}(x)$, so that $\cup \{\mathscr{T}_2\text{-int } S_{n+2\,2}(y) \mid x \in \mathscr{T}_1\text{-int } S_{n+2\,1}(y)\} \subset U_{n+12}(x)$. Hence $S_2(x, \mathscr{K}_2^{(n+2)}) \subset U_{n+1\,2}(x) \subset \mathscr{T}_2\text{-int } S_{n\,2}(x)$. If we define $\mathscr{K}_i^{(m)\,\Delta} = \{S_i(x, \mathscr{K}_i^{(m)}) \mid x \in X\}$, then we see that $\mathscr{K}_i^{(n+2)\,\Delta} < \mathscr{K}_i^{(n)}$ for all $n = 1, 2, 3 \ldots\ldots$. If we write $V_{m\,i} = \cup \{\mathscr{T}_j\text{-int } S_{m\,j}(y) \times \mathscr{T}_i\text{-int } S_{m\,i}(y) \mid y \in X\}$, then $(x, y) \in V_{m+2\,i} \circ V_{m+2\,i}$ implies, for some $zd \in X$, $(x, z) \in V_{m+2\,i}$ and $(z, y) \in V_{m+2\,i}$. Indeed $x \in V_{m+2\,j}[z] \subset \mathscr{T}_j\text{-int } S_{m\,j}(z)$ an $y \in V_{m+2\,i}[z] \subset \mathscr{T}_i\text{-int } S_{m\,i}(z)$ so that $(x, y) \in V_{m\,i}$.

Thus the conditions are sufficient.

The necessary part of the theorem is true as follows. Let p_1 be the quasimetric that induces $\mathscr{T}_1$ and $\mathscr{T}_2$ be induced by the conjugate p_2. Let us write

$$U_{n\,i}(x) = \left\{y \mid p_i(x, y) < \frac{1}{2^{2n-1}}\right\}$$

$$S_{n\,i}(x) = \left\{y \mid p_i(x, y) < \frac{1}{2^{2n}}\right\}$$

$(i = 1, 2; n = 1, 2\ldots\ldots)$. Then, if $z \notin U_{n\,i}(x)$ and $S_{n\,i}(x) \cap S_{n\,j}(z) \neq \phi$, there exists $y \in X$ such that $p_i(x, y) < 1/2^{2n}$ and $p_j(x, y) < 1/2^{2n}$. Hence $p_i(x, z) \leqq p_i(x, y) + p_i(y, z) < 1/2^{2n-1}$, a contradiction. Also if $z \in S_{n\,i}(x)$ and $y_0 \in S_{n\,i}(z)$, then $p_i(x, z) < 1/2^{2n}$ and $p_i(z, y_0) < 1/2^{2n}$ so that $p_i(x, y_0) < 1/2^{2n-1}$. Hence $y_0 \in U_{n\,i}(x)$.

Theorem 3—A pairwise Hausadorff space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is quasi-metrizable if and only if for each $x \in X$ one can assign $\mathscr{T}_i$-nbd bases $\{U_{n\,i}(x) \mid n = 1, 2,\ldots\ldots\}$ $(i = 1, 2)$ such that for every n and each point x there are $\mathscr{T}_i$-nbds $S_{n\,i}(x)$ satisfying:

(i) $U_{n+1\,i}(x) \subset S_{n\,i}(x) \subset U_{n\,i}(x)$

(ii) $y \notin U_{n\,i}(x) \Rightarrow S_{n\,i}(x) \cap S_{n\,j}(y) = \phi$

(iii) $y \in S_{n\,i}(x) \Rightarrow x \in S_{n\,j}(y)$ $(i, j = 1, 2; i \neq j)$

PROOF: We have to verify only the condition (iii) of the Theorem 2. Now $y \notin U_{n\,i}(x)$ implies $S_{n\,i}(x) \cap S_{n\,j}(y) = \phi$ so that if $z \in S_{n\,i}(x)$, then $z \notin S_{n\,j}(y)$. Thus $y \notin S_{n\,i}(z)$.

The necessity is obvious

Theorem 4—A pairwise Hausdorff space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is quasi-metrizable if and only if for each $x \in X$ one can assign $\mathcal{T}_i$-nbd bases $\{U_{n\,i}(x) \mid n = 1, 2, \ldots\}$ $(i = 1, 2)$ such that for every n and each point x there are $\mathcal{T}_i$-nbds $S_{n\,i}(x)$ satisfying :

(i) $U_{n+1\,i}(x) \subset S_{n\,i}(x) \subset U_{n\,i}(x)$

(ii) $y \in S_{n\,i}(x) \Rightarrow S_{n\,i}(y) \subset U_{n\,i}(x)$

(iii) $y \in S_{n\,i}(x) \Rightarrow x \in S_{n\,j}(y)$ $(i, j = 1, 2; i \neq j)$

PROOF : We have to verify only the condition (ii) of the Theorem 2. If $S_{n\,i}(x) \cap S_{n\,j}(y) \neq \phi$, then there is an $z \in S_{n\,i}(x)$ and $z \in S_{n\,j}(y)$. Since $z \in S_{n\,i}(x)$, $S_{n\,i}(z) \subset U_{n\,i}(x)$. Also $z \in S_{n\,j}(y)$ implies $y \in S_{n\,i}(z)$. Thus $y \in U_{n\,i}(x)$.

The necessity is obvious.

If we take $\mathcal{T}_1 = \mathcal{T}_2$ in Theorems 2, 3 and 4 we get a theorem on metrization as follows :

Corollary 5—Let X be a T_1 space. X is metrizable if and only if at each $x \in X$ there is an open nbd base $\{U_n(x) \mid n = 1, 2, \ldots\}$ and for each x and each integer n there is nbd $S_n(x)$ satisfying (1), (2) and (3) or (1), (2) and (4) or (1), (3) and (4) :

(1) $U_{n+1}(x) \subset S_n(x) \subset U_n(x)$

(2) $y \notin U_n(x) \Rightarrow S_n(x) \cap S_n(y) = \phi$

(3) $y \in S_n(x) \Rightarrow S_n(y) \subset U_n(x)$

(4) $y \in S_n(x) \Rightarrow x \in S_n(y)$

If we write down the proof of this Corollary taking verbatim $\mathcal{T}_1 = \mathcal{T}_2$ in the proof of Theorem 2, we notice that the proof of the Corollary 5 does not involve proving the space to be fully normal (or equivalently paracompact). Thus in this Corollary—though could be thought of as a consequence of Theorem 1 or 7 of Nagata (1957)—the extra condition (1) in its statement helps us steer clear of σ-local finiteness, σ-discreteness etc. in its proof. This is probably the importance of Corollary 5. Thus condition (i) of Theorems 2, 3 and 4 helps us to dispense off the necessity of generalizing the concepts of σ-local finitess, σ-discreteness etc., in the quasi-metric setting which is rather not easy in view of the very delicate stability of the quasi-metric spaces.

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