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CONSTRUCTION OF FIXED POINTS OF STRICTLY-PSEUDO CONTRACTIVE MAPPINGS IN A GENERALIZED HILBERT SPACE AND RELATED APPLICATIONS

R. N. MUKHERJEE AND T. SOM

Mathematics Section, School of Applied Sciences, Institute of Technology
Banaras Hindu University, Varanasi 221005

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A theorem regarding weak convergence of iterates of a strictly pseudo-contractive mapping to a fixed point of the mapping is proved in a generalized Hilbert space. This generalizes a result of Browder and Petryshyn (1967) for such a mapping on a Hilbert space. We also give an application of fixed point theory in Hilbert space for strictly pseudocontractive mappings related to proximity maps.

1. INTRODUCTION

Hicks and Huffman (1978) extended certain results of Browder (1965), Browder and Petryshyn (1967) for fixed points and their constructions for nonexpansive mappings on a 'generalized Hilbert space', (i.e., a T_2-locally convex space whoes generating family of semi-norms satisfies the Parallelogram law). We have continued the study of fixed point theory on a generalized Hilbert space in the present work and have obtained yet another generalization of a result of Browder and Petryshyn (1967) for fixed point of a strictly pseudocontractive mapping.

In section 3 we also give an application of fixed point theory of strictly pseudo-contractive maps for a result on proximity maps on a Hilbert space.

Let X be a T_2-locally convex space generated by a family $\{\rho_\alpha : \alpha \in \Delta\}$ of continuous semi-norms. The function $\rho : X \to R^\Delta$ is defined by

$$(\rho (x)) (\alpha) = \rho_\alpha (x), x \in X, \alpha \in \Delta.$$

ρ satisfies the axioms of a norm. The topology t_ρ, generated by ρ is the original topology where a t_ρ-neighbourhood of x is of the form $\Omega (x, U) = \{y : \rho (x-y) \in U\}$, U being a neighbourhood of zero in R^Δ.

Thus, ρ norms X over R^{Δ}. In an H-locally convex space (generalized Hilbert space) each ρ_α has a compatible semi-inner product that we will denote by $(,)_\alpha$. For $f, g \in R^{\Delta}$,

(1) $f \leqslant g$ means $f(\alpha) \leqslant g(\alpha)$ for all $\alpha \in \Delta$,

(2) $f < g$ means $f \leqslant g$ and their exists $\alpha \in \Delta$ such that $f(\alpha) < g(\alpha)$.

2. Strictly Pseudocontractive Mappings

Definition 2.1— A mapping $T : X \to X$ where X is a generalized Hilbert space is called nonexpansive if

$$\rho_\alpha (Tx-Ty) \leqslant \rho_\alpha (x-y)$$

for all $x, y \in X$ and each $\alpha \in \Delta$.

Definition 2.2— A mapping $T : X \to X$ (a generalized Hilbert space) will be called strictly pseudocontractive if there exists a constant $k < 1$ such that

$$\rho^2_\alpha (Tx-Ty) \leqslant \rho^2_\alpha (x-y) + k\, \rho^2_\alpha ((I-T)\, x - (I-T)\, y)$$

for all $x, y \in X$ and each $\alpha \in \Delta$.

Definition 2.3— A mapping $T : X \to X$ is said to in class M_2 if there exists a constant β, $0 < \beta < 1$ such that $(Tx-Ty, x-y)_\alpha \geqslant \beta\, \rho^2_\alpha (Tx-Ty)$, for all $x, y \in X$, and each $\alpha \in \Delta$.

Definition 2.4— A mapping W will be said to lie on the ray from the identity mapping I generated by U (denoted by Ray (U)), if there exists a constant $t > 0$ such that

$$W = I + t\,(U-I) \text{ or } W = tU + (4-t)\, I.$$

We need the following theorems which are extensions of Theorem 1 (part 2 & 4) and Theorem 2 of Browder and Petryshyn (1967).

Theorem 2.1— (a) A mapping $U : X \to X$ is strictly pseudocontractive if and only if $T = I - U$ lies in M_2. (b) U is strictly pseudocontractive implies that Ray (U) is strictly pseudocontractive.

Proof : (a) Suppose $T = I - U \in M_2$, i.e., for each $\alpha \in \Delta$

$$(Tx-Ty, x-y)_\alpha \geqslant \beta\, \rho^2_\alpha (Tx-Ty).$$

Then

$$\rho_\alpha^2 (Ux-Uy) = \rho_\alpha^2 ((I-T)x-(I-T)y)$$

$$= \rho_\alpha^2 (x-y) + \rho_\alpha^2 (Tx-Ty) - 2(Tx-Ty, x-y)_\alpha$$

$$\leqslant \rho_\alpha^2 (x-y) + \rho_\alpha^2 (Tx-Ty) - 2\beta \rho_\alpha^2 (Tx-Ty)$$

$$= \rho_\alpha^2 (x-y) + (1-2\beta) \rho_\alpha^2 (Tx-Ty)$$

Thus U is pseudocontractive with $K = 1-2\beta < 1$.

Conversely, suppose that $T = I - U$ with U pseudocontractive, i.e.,

$$\rho_\alpha^2 (Ux-Uy) \leqslant \rho_\alpha^2 (x-y) + k\rho_\alpha^2 (Tx-Ty)$$

then

$$\rho_\alpha^2 (Ux-Uy) = \rho_\alpha^2 ((I-T)x-(I-T)y)$$

$$= \rho_\alpha^2 (x-y) + \rho_\alpha^2 (Tx-Ty) - 2(Tx-Ty, x-y)_\alpha$$

$$\leqslant \rho_\alpha^2 (x-y) + k\rho_\alpha^2 (Tx-Ty)$$

therefore,

$$(Tx-Ty, x-y)_\alpha \geqslant \left(\frac{1-k}{2}\right) \rho_\alpha^2 (Tx-Ty).$$

This shows that $T \in M_2$ with $\beta = \frac{1-k}{2}$.

(b) U is strictly pseudocontractive implies that

$$I-U \in M_2 \Leftrightarrow t(I-U) = I-W \in M_2$$

$\Leftrightarrow$ W is strictly pseudocontractive by part (a).

Theorem 2.2— U is strictly pseudocontractive if and only if there exists an element $W \in$ Ray (U) such that W is nonexpansive.

PROOF: Suppose there exists $W \in$ Ray (U) such that W is nonexpansive. Let $T = I - U$ and $tT = I - W$. Since $W \in$ Ray (W), by Theorem (2.7 a), it suffices to show that W is strictly pseudocontractive. But it is an easy excercise to show that a nonexpansive map is strictly pseudocontractive, hence W is strictly pseudocontractive. Conversely, suppose U is strictly pseudocontractive. By Theorem 2.1,

$T = I - U \in M_2$ and hence $(Tx-Ty)(x-y)_\alpha, \geqslant \beta\, \rho_\alpha^2\ (Tx-Ty)$ for some $\beta > 0$ and each $\alpha \in \Delta$.

$\left(\text{In fact as was shown in Theorem 2.1 (a) that } \beta = \frac{1-k}{2}\right)$. Consider the mapping $U_t = I + t\,(U-I)$ in $R(U)$ for $t > 0$. Since $U_t = I - tT$, it follows that

$$\rho_\alpha^2\,(U_t\,x - U_t\,y) = \rho_\alpha^2\,((1-tT)\,x-(1-tT)\,y)$$

$$= \rho_\alpha^2\,(x-y) + t^2\,\rho_\alpha^2\,(Tx-Ty) - 2t\,(Tx-Ty, x-y)_\alpha$$

$$\leqslant \rho_\alpha^2\ (x-y) + (t^2-2t\,\beta)\,\rho_\alpha^2\ (Tx-Ty).$$

Hence for any fixed t with $0 < t \leqslant 2\beta$ $(2\beta = 1-k)$, the map $W = U_t$ has the property that $\rho_\alpha\,(Wx-Wy) \leqslant \rho_\alpha\,(x-y)$, (for each $\alpha \in \Delta$). i.e., W is nonexpansive.

3. Construction of Fixed Points of Pseudocontractive Mappings

Our main result is as follows :

Theorem 3.1—Let C be a bounded, closed, convex weakly sequentially compact subset of X, and U a strictly pseudocontractive mapping of C into C, i.e. there exists a constant $k < 1$ such that

$$\rho_\alpha^2\ (Ux-Uy) \leqslant \rho_\alpha^2\,(x-y) + k\,\rho_\alpha^2\,((I-U)\,x-(I-U)\,y)$$

for $x, y \in C$ and each $\alpha \in \Delta$. Then for any $x_0 \in C$ and any fixed γ such that $1 - k < \gamma < 1$, $U_\gamma^n\ x_0 \rightharpoonup y \in C$ (weakly) and y is a fixed point of U in C. If additionally we assume that U is demicompact, then $U_\gamma^n\,x_0 \to y$ (Strongly).

Proof : Since U is strictly contractive, Theorem 2.2 shows that for every fixed t such that $0 < t \leqslant k-1$, the mapping $U_t = t\,U + (I-t)\,I$ is nonexpansive. Hence by Theorem 2 in Hicks and Huffman (1978), U_t (and therefore U) has a fixed point in C and for any given $x_0 \in C$ and any fixed λ with $0 < \lambda < 1$, the sequence $\{x_n\}$ $= \{(U_t)^n\,x_0\}$ converges weakly to some point of y of U in C. (Hicks and Huffman 1978, Theorem 9). But

$$(U_t)_\lambda = \lambda I + (1-\lambda)\,U_t$$

$$= (1-\lambda)\,t\,U + 1 - (1-\lambda)\,t\,I$$

$$= \gamma\,I + (1-\gamma)\,U = U_\gamma$$

where $\gamma = 1 - (1-\lambda)\, t$. Since, as is easy to see, $\lambda\, t < k - 1$, for each fixed λ with $0 < \lambda < 1$ if and only if $\gamma > k$, the proof of the first part of our theorem is complete.

To prove the second part, by virtue of Theorem 7 of Hicks and Huffman (1978) it is sufficient to show that U is demi-compact. But this follows easily from demi-compactness of U and the equality $U_\gamma\, x - x = (1-\gamma)\,(Ux - x)$ which holds for every x in C.

Example — As an example of a generalized Hilbert space one can construct the following (see Schaefer 1966). Let Λ be an index set directed under a (reflexive, transitive anti-symmetric) relation "$\leqslant$" let $\{E_\alpha : \alpha \in \Lambda\}$ be a family of Hilbert spaces over a field K, and denote, for $\alpha \leqslant \beta$, by $g_{\alpha\beta}$ a continuous linear map of E_β into E_α. Let E be the subspace of $\Pi_\alpha\, E_\alpha$ whose elements $x = (x_\alpha)$ satisfy the relation $x_\alpha = g_{\alpha\beta}\,(x_\beta)$ whenever $\alpha \leqslant \beta$; E is called the projective limit of the family $\{E_\alpha : \alpha \in \Lambda)\}$ with respect to the mappings $g_{\alpha\beta}\,(\alpha, \beta \in \Lambda;\, \alpha \leqslant \beta)$, and is denoted by $\underleftarrow{\lim}\, g_{\alpha\beta}\, E_\beta$. The topology of E is the projective topology on E with respect to the family $\{(E_\alpha, F_\alpha, f_\alpha)\colon \alpha \in \Lambda\}$, where F_α denotes the topology of E_α, and f_α denotes the restriction to E of the projection map P_α of $\Pi_\beta\, E_\beta$ onto E_α. E becomes a locally convex T_2-space which is a generalized Hilbert space.

In fact a complete nuclear locally convex space is isomorphic to a generalized Hilbert space (Schaefer 1966, Corr. 3, p. 103).

As an example of a strictly pseudocontractive mapping we can first take a map $T : E \to E$ satisfying $(Tx-Ty,\, x-y)_\alpha \geqslant \beta\, \rho_\alpha^2\, (Tx-Ty)$, for $0 < \beta < 1$ and $\alpha \in \Lambda$, for $x, y \in E$, where $(\;,\;)_\alpha$ is the compatible semi-inner product with respect to the semi-norm ρ_α on E for each $\alpha \in A$. By Theorem 2.1 in our work it follows that $I-T$ is a strictly pseudocontractive mapping. Moreover we observe that it is definitely an extension of strictly pseudocontractive mapping definition of Browder and Petryshyn (1967) since E is not isomorphic to any Hilbert space.

4. Proximity Maps and Strictly Pseudocontractive Mappings in Hilbert Space

We need some preliminary definitions and results for this section.

Definition 4.1—Let X be a normed linear space and C a nonempty subset of X. Then for $x \in X$, define

$$d\,(x, C) = \inf_{y \in C} \| x-y \|$$

and

$$P_C\,(x) = \{ y \in C : \| x-y \| = d\,(x, C)\}.$$

The set valued map $P_C(x)$ is called a metric projection on C. In the case when C is Chebyshev set. $P_C(x)$ is a single-valued map called a proximity map.

Definition 4.2— Let C be a convex subset of a Hilbert space H. The point $b \in C$ is the 'nearest point' to $a \notin C$ if and only if $(x-b, b-a) \geqslant 0$ for all $x \in C$.

If additionally C is closed then, $P_C(x)$, the proximity map is nonexpansive i.e., $\| P_C(x) - P_C(y) \| \leqslant \| x-y \|$, for all $x, y \in H$. For details on proximity maps (see Cheney and Goldstein 1959). Cheney *et al.* (Preprint) proved the following result.

Theorem 4.1— Let C be a closed, convex subset of a Hilbert space H and $: C \to H$ be nonexpansive with $f(C)$ bounded. Then there exists a y in C such that

$$\| y - f(y) \| = d(f(y), C).$$

As an application of fixed point theory of strictly pseudocontractive maps we give a result analogous to Theorem 4.1.

Theorem 4.2—Let C be a closed bounded convex subset of a Hilbert space H and $f: C \to H$ is strictly pseudocontractive. Then there is a $\lambda \in (0, 1)$ and $y \in C$ such that if $g = \lambda f + (1-\lambda) I$, then

$$\lambda \| y - f(y) \| = d(g(y), C).$$

PROOF : Let $P: H \to C$ be the proximity map. Then we know that P is nonexpansive. Also from Theorem 2.2 we see that for a $\lambda \in (0, 1)$, $g = \lambda f + (1-\lambda) I$ is nonexpansive. Therefore $Pg: C \to C$ is also nonexpansive. Then from Browder [see Cheney *et al.* preprint] we see that Pg has a fixed point $y \in C$. Therefore,

$$\| y - g(y) \| = \| Pg(y) - g(y) \| = d(g(y), C)$$

from which it follows that

$$\lambda \| y - f(y) \| = d(g(y), C).$$

As a second application we prove the following result which is an extension of a result of Schöneberg (1976).

Theorem 4.3—Let H be a Hilbert space and C a closed, bounded convex subset of H. Let $f: C \to H$ be a pseudocontractive map. Then by Theorem 2.2 there is a $\lambda \in (0, 1)$ such that $g = \lambda f + (1-\lambda) I$ is nonexpansive. Assume for each x on the boundary of C

$$\| g(x) - y \| \leqslant \| x-y \| \text{ for some } y \in C.$$

Then f has a fixed point in C.

PROOF: From Schöneberg's (1976) result since g is nonexpansive, therefore there is an element $y_0 \in C$ such that $g(y_0) = y_0$. Now it is easy to see that y_0 in turn is also a fixed point of f.

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