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G-RINGS WITH FEW ZERO-DIVISORS AND THEIR QUOTIENT RINGS

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Quasi-valuation rings among *G*-rings with few zero-divisors have been characterised. It has been shown that a strong *G*-ring with few zero-divisors can be obtained as a finite intersection of quasi-valuation rings each of which is strong *G*-ring.

1. INTRODUCTION

Rings in which there are only a finite number of prime ideals maximal with respect to not containing any regular elements are called rings with few zero-divisors. These have been defined by Davis (1964). Rings with few zero-divisors in which for any two regular elements a and b either $(a) \subset (b)$ or $(b) \subset (a)$ are called quasi-valuation rings (Davis 1964). Adams (1974) has called a ring R a G-ring if the total quotient ring of it is of the form $R[u^{-1}]$ for some regular element u of R. We consider here those rings R with few zero-divisors for which every overring S of R contained in its total quotient ring is of the form $R[u^{-1}]$. Any ring with the above property is termed a strong G-ring. This concept of a strong G-ring is analogous to that of a strong G-domain defined in Revati and Viswanathan (1976).

2. PRINCIPAL QUOTIENT RING PROPERTY

A ring R is said to have principal quotient ring (pqr) property if $R_{S(P)}$ for every prime ideal P or R is a principal quotient ring $R_{(t)}$ for some regular element t or R. In this section we show that if a ring with few zero-divisors has pqr property, then it is semi-local.

Proposition 2.1—Let R be a ring with few zero-divisors, P a prime ideal of R and t a regular element of $R \setminus P$. Then the following are equivalent:

(1) $R_{S(P)} = R_t$

(2) If b is a regular element of $R \setminus P$, then t belongs to the radical of the ideal (b).

(3) t belongs to every regular prime ideal not contained in P.

PROOF : (3) → (2)—Consider $b \in P$. The prime ideals of R containing b are not contained in P. So, by (3), t belongs to each of them and consequently to their intersection also.

(2) → (1)—Let $x = a/b \in R_{S(P)}$. By (2), $t^n \in (b)$ for some positive integer n. Let $t^n = bc$ for $c \in R$. Since t is regular c is also regular. Hence $ac/bc = ac/t^n \in R_t$. Thus we have $R_{S(P)} \subset R_t$. Also, by choice of t, $R_t \subset R_{S(P)}$.

(1) → (3)—Let Q be a regular prime ideal of R not contained in P. Consider some regular element $q \in Q \setminus P$. $1/q \in R_{S(P)}$. By (1) $R_{S(P)} = R_t$. Hence $1/q = a/t^n$ for fome $a \in R$ and $n \geqslant 1$. That is, $t^n = aq \in Q$ or $t \in Q$.

Corollary 2.2—Let R be a ring with few zero-divisors having *pqr* property. Then every regular prime ideal P of R is a G-ideal i.e. R/P is a G-ring (Adams 1974).

Theorem 2.3—Every ring R with few zero-divisors and having *pqr* property is semi-local.

PROOF : Let $m_1, m_2, \ldots, m_k$ be the prime ideals of R maximal with respect to not containing any regular elements. Let $\{M_i\}_{i \in I}$ be the set of regular maximal ideals of R. Let $R_{S(M_i)} = R_{t_i}$. By Proposition 2.1, t_i belongs to every regular prime ideal of R not contained in M_i. Let X_{t_i} denote the set of those prime ideals of R which do not contain t_i. If, for the regular prime ideal P, we denote by Y_P the set $\{Q \subset P : Q$ is a regular prime ideal of $R\}$ and for the irregular prime ideal m_i we denote by y_{m_i} the set $\{Q \subset m_i : Q$ is a prime ideal of $R\}$, then $X_{t_i} = Y_{M_i} \cup y_{m_i} \ldots \cup y_{m_k}$. X_{t_i} is the complement of the closed set $V(t_i)$ which consists of all prime ideals of R containing t_i. Hence X_{t_i} is an open subset of spec R and $\bigcup_{i \in I} X_{t_i}$ is an open covering of the compact space spec R. Therefore it has a finite subcovering $Y_{M_1} \cup Y_{M_2} \cup \ldots \cup Y_{M_n} \cup y_{m_1} \cup \ldots \cup y_{m_k}$. Thus R is a semi-local ring.

3. STRONG G-RINGS

Strong G-rings defined earlier in the introduction form a subclass of G-rings. The results given here are on strong G-rings with few zero-divisors.

We first give a Lemma.

Lemma 3.1—Let R be a ring with few zero-divisors and $\{t_i\}_{i=1}^n$ be a family of regular elements of R. Then there exist elements $u_i \in R$, $i = 1, \ldots, n$ such that

$$R_{t_i} \cap R_{t_2} \cap \ldots \cap R_{t_n} \subset R_{t_1 + u_2 t_2 + u_n t_n}$$

and

$$t = t_1 + u_2 t_2 + \ldots + u_n t_n \text{ is regular in } R.$$

PROOF : Let $0 \neq y \in \bigcap_{i=1}^{n} R_{t_i}$. There exist elements $a_1, \ldots, a_n$ and positive integers $m_1, \ldots m_n$ such that $y = a_1/t_1^{m_1} = a_2/t_2^{m_2} = \ldots \ldots = a_n/t_n^{m_n}$. Thus $a_i = y\, t_i^{m_i}$, $i = 1, 2, \ldots, n$. Now, t_1 and t_2 regular in R implies there is a u_2 in R such that $t_1 + u_2\, t_2$ is regular in (Davis 1964, Lemma B), and $t_1 + u_2\, t_2$ and t_3 are regular implies $t_1 + u_2\, t_2 + u_3\, t_3$ is regular for some $u_3 \in R$ and so on. Each term of the multinomial $y\,(t_1 + \ldots + u_n\, t_n)^m$ is in R and so

$$y\,(t_1 + u_2\, t_2 + \ldots + u_n\, t_n)^m = r \in R \text{ which gives } y \in R_{t_1 + u_2 t_2 + \ldots + u_n t_n}.$$

Theorem 3.2—A quasi-valuation ring V with few zero-divisors is a strong G-ring if and only if the family of regular prime ideals of V is well ordered with respect to inclusion.

PROOF : Let V be a quasi-valuation strong G-ring with few zero-divisors and F be any nonempty family of regular prime ideals P of V. The regular prime ideals of V being linearly ordered, $Q = \bigcap_{P \in F} P$ is a regular prime ideal of V. If $Q \in F$, then Q is the first element of F with respect to inclusion. On the other hand let $Q \notin F$. V has pqr property and Q is a regular prime ideal of V. Hence by Corollary 2.2, Q is a G-ideal. Thus V/Q is a G-ring which is a quasi-valuation ring. The intersection of all regular prime ideals of V/Q is a non-zero regular prime ideal Q'/Q. So we have $Q \subset Q' \subset P$ for every $P \subset F$. Thus $Q = Q'$ which is a contradiction. Therefore Q must belong to F.

Now, suppose V is a quasi-valuation ring the family of prime ideals of which is well ordered. Let $V' = V_{S(P)}$ be any overring of V. If P is the maximal regular ideal of V we have $V = V_{S(P)}$. If not, let Q be a prime ideal which is minimal among all regular prime ideals strictly containing P. Q exists by assumption. Pick a regular element $t \in Q \setminus P$. By virtue of Proposition 2.1, we have $V_{S(P)} = V_t$. Hence V is a strong G-ring.

Theorem 3.3—Let $V_1, V_2, \ldots, V_n$ be a finite number of incomparable non-trivial quasi-valuation strong G-rings having the same total quotient ring. Then $R = \bigcap_{i=1}^{n} V_i$ is such that every overring of R is also a finite intersection of incomparable quasi-valuation strong G-rings.

PROOF : Let $M_1, M_2, \ldots, M_n$ be the regular maximal ideals of R and $V_i = R_{S(M_i)}$ for $i = 1, \ldots, n$ (Harui 1968). The regular maximal ideals being finite in number, any family of mutually incomparable regular prime ideals of R is finite. This together with Theorem 3.2 gives us that any nonempty family of regular prime

ideals has minimal elements and these are finite in number. Davis (1964) has shown that any overring R' of R is of the form $\bigcap_{P \in F} R_{S(P)}$ where F is a family of pairwise comaximal regular prime ideals. F is finite by the above discussion. Also, each $R_{S(P)} \supset R_{S(M_i)}$ for some $i = 1, \ldots, n$. Hence $R_{S(P)}$ is a quasi-valuation strong G-ring for every $P \in R$, thus establishing the theorem.

Theorem 3.4—Let R be a ring with few zero-divisors. If R is a strong G-ring then R is a finite intersection of incomparable, nontrivial quasi-valuation strong G-rings having the same total quotient ring.

PROOF : Suppose R is a strong G-ring. Then R as also all its overrings are integrally closed and so R is a Prufer ring (Davis 1964). Also, by Theorem 2.3, R has a finite number of regular maximal ideals $M_1, M_2, \ldots, M_n$. $R_{S(M_i)}$ for each $i = 1 \ldots, n$ is a quasi-valuation ring. Since $R = \bigcap_{i=1}^{n} R_{S(M_i)}$ it is enough to show each $R_{S(M_i)}$ is a strong G-ring. Any prime ideal of $R_{S(M_i)}$ is an extension P^e of a prime ideal P of R to $R_{S(M_i)}$. $(R_{S(M_i)})_{S(Pe)} = R_{S(P)}$, and $R_{S(P)} = R_t$ for some regular element t of R, by hypothesis on R. These give

$(R_{S(M_i)})_{S(Pe)} = (R_{S(M_i)})_t$, which makes $R_{S(M_i)}$ a strong G-ring. Hence the theorem.

Proposition 3.5—A Noetherian integrally closed G-ring R with few zero-divisors is a strong G-ring.

PROOF : R being a Noetherian G-ring has a finite number of regular prime ideals $P_1, P_2, \ldots, P_n$ (Adams 1974). Also, R being integrally closed, every regular prime ideal of R contains a regular element which is not in any other prime ideal (Davis 1964, Remarks, p. 210). Let $t_i \in P_i$ such that $t_i \notin P_j$ for $i \neq j$. This implies by Proposition 2.1 that $R_{S(P)} = R_t$. Similarly $R_{S(P_i)}$ for $i = 2, \ldots, n$ are also principal quotient rings. Every overring of R is a finite intersection of $R_{S(P_i)}$ for $i = 1, 2, \ldots, n$. This follows from (Davis 1964, Cor. 1 and Th. 4). So we will be done if we show that the overring $T = R_{S(P_i)} \cap R_{S(P_j)}$, $i \neq j$ is a principal quotient ring of R. Let $R_{S(P_i)} = R_{S_i}$ and $R_{S(P_j)} = R_{S_j}$ where $s_i = t_1 . t_2 \ldots t_{i-1} . t_{i+1} \ldots t_n$ and $s_j = t_1 . t_2 \ldots t_{j-1} . t_{i+1} \ldots \ldots t_n$. s_i and s_j are regular elements of R and R is a ring with few zero-divisors. Therefore, if we choose u in R such that u is in all the prime ideals maximal with respect to not containing regular elements, then $s_i + u s_j$ is a regular element of R. Also each $R_{S(P_i)}$ being a discrete valuation ring (Davis 1964, Th. 5), we have $u \notin P_i$ for any i. Therefore $s_i + us_j \notin P_i$ and P_j, giving $R_{s_i + us_j} \subset R_{S(P_i)} \cap R_{S(P_j)}$. Application of Lemma 3.1 gives $R_{S(P_i)} \cap R_{S(P_j)} \subset R_{s_i + us_j}$. Thus T is a principal quotient ring of R. Therefore R is a strong G-ring.

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