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A NOTE ON THE SUM OF APPELL FUNCTION F_2

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The main object of the present note is to give some corrections to certain results of Lal (1975) which is appeared in this journal.

1. INTRODUCTION

This note is concerned with a paper of Charanjit Lal (1975). In this paper a sum of terminating Appell's polynomial of second kind F_2 [Erdélyi *et al.* 1953, p. 224 (7)] was obtained in the form

$$F_2[a; -m, -n; 1+a-e, e; 1, 1] = \frac{(e-a)_{n-m}}{(e)_{n-m}} \qquad \ldots(1.1)$$

where

$$(a)_m = \Gamma(a+m)/\Gamma(a).$$

The result (1.1) has been applied to obtain some transformations of Horn's double hypergeometric functions G_1, G_2, G_3 and H_2 [Erdélyi *et al.* 1953, pp. 224-25 (10, 11, 12, 14)]. Two of his transformations are given by [Lal 1975, p. 625 (3.3, 3.4)]

$$G_3[e-a, 1-e: x, y] = (1+x)^{a-1} G_1\left[1-a, e-a, 1-e: \frac{x}{1+x}, \frac{x}{1+x}\right], \qquad \ldots(1.2)$$

$$H_2[1-e, e-a, e-a, e, e-a: x, y] = (1+y)^{-e} G_2\left[1-a, e, a, 1-e: -x, \frac{-y}{1+y}\right]. \qquad \ldots(1.3)$$

Use of (1.1) was also made to obtain two another transformations of triple hypergeometric functions G_A and G_B (Pandey 1963) in the form [Lal 1975, p. 626 (3.7, 3.8)]

$$G_A\left[\frac{1}{a},\ a,\ a : b,\ b',\ b :\ \frac{1}{c},\ c,\ c : x,\ y,\ z\right]$$

$$= (1-x-z)^{-b}\,(1-y)^{-b'}\,F_F\left[c-a,\ c-a,\ c-a : b,\ b',\ b : 1-a,\ c,\ c :\right.$$

$$\left.\frac{-x}{1-x-z},\ \frac{-y}{1-y},\ \frac{-z}{1-x-z}\right] \qquad \ldots(1.4)$$

$$G_B\left[\frac{1}{a},\ a,\ a' : b_1,\ b_2,\ b_3 :\ \frac{1}{c},\ c,\ c : x,\ y,\ z\right]$$

$$= (1-x)^{-b_1}\,(1-y)^{-b_2}\,(1-z)^{-b_3}\,F_G\left[c-a,\ c-a,\ c-a : b_1,\right.$$

$$\left.b_2,\ b_3 \times 1-a,\ c,\ c : \frac{-x}{1-x},\ \frac{-y}{1-y},\ \frac{-z}{1-z}\right] \qquad \ldots(1.5)$$

where F_F and G_G are Saran's (1954) functions.

It does not appear to have been observed previously that formulae (1.2) to (1.5) are incorrect. In this note we give the following correct forms of above transformations,

$$H_1\,[e-a,\ 1-a,\ a;\ e;\ y,\ x]$$

$$= (1+x)^{a-1}\,G_1\left[1-a,\ e-a,\ 1-e;\ \frac{-x}{1+x},\frac{-y}{1+x}\right] \qquad \ldots(1.6)$$

$$H_2\,[1-e,\ 1-a,\ e-a,\ e;\ 1-a;\ x,\ y]$$

$$= (1+y)^{-e}\,G_2\left[1-a,\ e,\ a,\ 1-e;\ -x,\frac{-y}{1+y}\right] \qquad \ldots(1.7)$$

$$F_F\,[a+b-1,\ a+b-1,\ a+b-1;\ c,\ d,\ c;\ a,\ b,\ b;\ x,\ y,\ z]$$

$$= (1-x-z)^{-c}\,(1-y)^{-d}\,G_A\left[1-a,\ c,\ d;\ b;\ \frac{x}{x+z-1},\frac{y}{y-1},\frac{z}{x+z-1}\right]. \qquad \ldots(1.8)$$

$$F_G\,[a+b-1,\ a+b-1,\ a+b-1;\ c,\ d,\ e;\ a,\ b,\ b;\ x,\ y,\ z]$$

$$= (1-x)^{-c}\,(1-y)^{-d}\,(1-z)^{-e}\,G_B\left[1-a,\ c,\ d,\ e;\ b;\ \frac{x}{x-1},\frac{y}{y-1},\frac{z}{z-1}\right]. \qquad \ldots(1.9)$$

The results (1.8) and (1.9) have already been obtained by Pandey (1963) by using the method of Pochhammer's double-loop contour integrals.

We shall give the proofs of (1.6) to (1.9) in section 2 with the help of elementary series manipulation formula [Rainville 1960, p. 56, Lemma 10 (1)]

$$\sum_{m+n=0}^{\infty} A(m, n) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} A(m-n, n) \qquad \ldots(1.10)$$

a conversion of double series into a single series [see Srivastava 1971, p. 4 (15)]

$$\sum_{m+n=0}^{\infty} f(m+n) \frac{x^m y^n}{m!\, n!} = \sum_{N=0}^{\infty} \frac{f(N)(x+y)^N}{N!} \qquad \ldots(1.11)$$

some relationships about factorial function [Rainville 1960, p. 32 (8, 9)]

$$\left.\begin{aligned} (a)_{n-k} &= \frac{(-1)^k (a)_n}{(1-a-n)_k}, \quad (0 \leqslant k \leqslant n) \\ (n-k)! &= \frac{(-1)^k n!}{(-n)_k} \\ (b)_{-n} &= \frac{(-1)^n}{(1-b)_n}, \quad (b \neq \text{an integer}). \end{aligned}\right\} \qquad \ldots(1.12)$$

Euler's first linear transformation [Rainville 1960 p. 60., Theorem 20 (4)] for Gauss's ordinary hypergeometric function ${}_2F_1$ [Rainville 1960, p. 45 (1)]

$${}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} x\right] = (1-x)^{-a}\, {}_2F_1\left[\begin{matrix} a, c-b; \\ c; \end{matrix} \frac{-x}{1-x}\right], \left(|x| < 1, \left|\frac{x}{1-x}\right| < 1 \right) \qquad \ldots(1.13)$$

and Gauss's summation theorem [Rainville 1960, p. 49, Th. 18]

$${}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \quad \mathrm{Re}\,(c-a-b) > 0. \qquad \ldots(1.14)$$

2. Proofs of (1.6) to (1.9)

Let us put

$$S = G_1\left[1-a, e-a, 1-e; \frac{-x}{1+x}, \frac{-y}{1+x}\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{(1-a)_{m+n}\,(e-a)_{n-m}\,(1-e)_{m-n}}{m!\, n!} \left(\frac{-x}{1+x}\right)^m \left(\frac{-y}{1+x}\right)^n$$

(*equation continued on p.* 1242)

$$= \sum_{n=0}^{\infty} \frac{(1-a)_n\,(e-a)_n}{(e)_n\,n!}\left(\frac{y}{1+x}\right)^n {}_2F_1\left[\begin{matrix}1-a+n,\ 1-e-n;\\ 1-e-n+a\end{matrix}\ ;\frac{x}{1+x}\right].$$

Now using (1.13), we get

$$S = (1+x)^{1-a}\sum_{n=0}^{\infty}\frac{(1-a)_n\,(e-a)_n\,y^n}{(e)_n\,n!}\ {}_2F_1\left[\begin{matrix}1-a+n,\ a;\\ 1-e-n+a;\end{matrix}\ -x\right]$$

$$= (1+x)^{1-a}\sum_{m,n=0}^{\infty}\frac{(1-a)_{m+n}\,(e-a)_{n-m}\,(a)_m\,x^m\,y^n}{(e)_n\,m!\,n!}.$$

Then by using the definition of another Horn's function H_1 [Erdélyi *et al.* 1953, p. 225 (13)], we arrive at (1.6) which is the correct form of the result (1.2). The transformation (1.7) is followed directly from a known transformation of Erdélyi [1950, p. 150 (22)] and is the correct form of the result (1.3).

Next we consider (1.8). Replace G_A by its defining series, expand $(1-y)^{-n-d}$ and $(1-x-z)^{-c-m-p}$ in binomial series and use (1.11) to rewrite the expression as :

$$T = \sum_{m,n,p,i,j,s=0}^{\infty}\frac{(1-a)_{n+p-m}\,(c)_{m+p+i+j}\,(d)_{n+s}\,(-1)^{m+n+p}}{(b)_{n+p-m}}$$

$$\times\frac{x^{m+i}\,y^{n+s}\,z^{p+j}}{m!\,n!\,p!\,i!\,j!\,s!}.$$

Then the application of (1.10), (1.12) and successive applications of Gauss's summation formula (1.14) will give us the result in terms of F_F.

Similarly we can derive (1.9) which is the correct form of the transformation (1.5).

3. Special Cases

When $y = 0$ or $d = 0$ in (1.8), we get a known transformation of Srivastava [1972, p. 102 (1.6)]. For $x = 0$ in (1.8) and $x = 0$ or $c = 0$ in (1.9), we have a known transformation [Erdélyi *et al.* 1953, p. 239 (1)]. Similarly setting $z = 0$ in (1.8) and $y = 0$ or $d = 0$ or $z = 0$ or $e = 0$ in (1.9), we get another known transformation [Erdélyi 1950, p. 149 (21)].

Lastly, it may be remarked that a result of Lal [1975, p. 624 (first equation)] written in its correct form, may be read as

$$F_3\,[c-a,\ a;\ -m,\ -n;\ c;\ 1,\ 1] = \frac{(a)_m\,(c-a)_n}{(c)_{m+n}} \qquad \ldots(3.1)$$

where F_3 is Appell's polynomial of third kind [Erdélyi *et al.* 1953, p. 224 (8)].

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