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# FIXED POINT THEOREMS ON COMPLETE AND COMPACT SPACES

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In this paper, a slight variant of a fixed point result of Fisher is proved using the techniques of Shih and Yeh. Also a result of Chatterjee is extended to a sequence of self maps on a complete metric space.

Recently Mukherjee (1981) proved the following :

*Theorem* 1 (Th. 2, Mukherjee (1981))—Let $f$ and $g$ be two self maps on a compact metric space $(X, d)$ such that $fg = gf$, $g(X) \subset f(X)$, $f$ is continuous and for $fx \neq fy$

$$d(gx, gy) < a_1 d(gx, fx) + a_2 d(gy, fy) + a_3 d(gx, fy) + a_4 d(gy, fx) + a_5 d(fx, fy)$$

where $a_i \geqslant 0$ with $a_1 + a_2 + a_3 + 2a_4 + a_5 = 1$.

Then $f$ and $g$ have a unique common fixed point.

Unfourtunately, it is not valid in view of the following :

*Example* 2 — Let $X = \{0, 1, 2\}$ with the usual metric. Let $f, g: X \to X$ be defined by $f = I$ (Identity map) and $g0 = g2 = 1, g1 = 0$.

Example 2 also shows that (Th. 4 of Mukherjee (1981)) is also not valid.

Fisher (1981) proved a theorem related to the above.

*Theorem* 3 (Fisher 1981, Th. 5)—Let $f$ and g be two self maps on a compact metric space $(X, d)$ such that $gf = fg$, $g(X) \subset f(X)$, $f, g$ are continuous and

$$d(gx, gy) < \text{Max}\{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$$

for all $x, y \in X$ for which the R. H. S. $> 0$.

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Then $S$ and $T$ have a unique common fixed point.

Now we prove a slight variant of the above by using the technique of Shih and Yeh (1982).

*Theorem* 4—Let $(X, \tau)$ be a nonempty compact Hausdorff space, $F : X \times X \to \mathbf{R}_+$ be continuous and $F(x, x) = 0$ for all $x \in X$.

Let $f, g$: $X \to X$ be such that $fg = gf$; for some positive integers $i, j, p, q$ we have $f^i g^j$ is continuous and

$$F(g^p x, g^q y) < \delta \{ hx, hy \mid h \in \mathscr{F} \} \text{ whenever } g^p x \neq g^q y \qquad ...(4.1)$$

where $\mathscr{F}$ is the semi group of self maps on $X$ generated by $f$ and $g$ and for any subset $A$ of $X$, $\delta(A) = \sup \{F(x, y) \mid x, y \in A\}$.

Then $f$ and $g$ have a unique common fixed point, say, $z \in X$ and for any $x \in X$, the sequences $\{F((fg)^n x, z)\}$ and $\{F(z, (fg)^n x)\}$ converge to zero.

PROOF : Let $H = \bigcap_{n=1}^{\infty} (f^i g^j)^n X$.

Clearly $\{(f^i g^j)^n X\}$ is a decreasing sequence of nonempty compact subsets of $X$ and hence $H$ is a nonempty compact set.

Now we will show that $f^i g^j H = H$.

Since $H \subseteq (f^i g^j)^n X$ for all $n$, we have $f^i g^j H \subseteq (f^i g^j)^{n+1} X$ for all $n$ so that $f^i g^j H \subseteq H$.

Let $x \in H$. Then $x \in (f^i g^j)^{n+1} X$ for all $n = 1, 2, ....$ Therefore, there exists $x_n \in (f^i g^j)^n X$ such that $f^i g^j x_n = x$ for $n = 1, 2, ...$

Since $\{x_n, x_{n+1}, ....\} \subseteq (f^i g^j)^n X$ and $(f^i g^j)^n X$ is compact, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with limit, say, $p$. Since $(f^i g^j)^n X$ is closed (being a compact subset of Hausdorff space), follows that $p \in (f^i g^j)^n X$ for each $n$ so that $p \in H$. Since $f^i g^j$ is continuous, we have $f^i g^j p = x$. Hence $H \subseteq f^i g^j H$. Thus $f^i g^j = H$. Clearly $fH \subseteq H$, $gH \subseteq H$. Also $H = f^i g^j H \subseteq fgH \subseteq gH \subseteq H$ so that $gH = H$. Similarly $fH = H$. Therefore $g^p H = H = g^q H$.

Now, there exist $z_1, z_2 \in H$ such that

$$F(z_1, z_2) = \delta(H) = \text{Sup}\{F(x, y) \mid x, y \in H\}.$$

Also there exist $x_1, x_2 \in H$ such that $z_1 = g^p x_1$, $z_2 = g^q x_2$.

Suppose $z_1 \neq z_2$. By (4.1),

$\delta(H) = F(z_1,z_2) = F(g^p x_1, g^q x_2) < \delta\{hx_1, hx_2 \mid h \in \mathcal{F}\} \leqslant \delta(H)$, a contradiction. Hence $z_1 = z_2$. Therefore $\delta(H) = 0$.

Clearly, as above, $x, y \in H$ with $F(x, y) = \delta(H)$ implies $x = y$.

Hence $H$ is a singleton, say, $z$.

Therefore $fz = z = gz$ since $fH = H = gH$.

Since every common fixed point of $f$ and $g$ is a point of $H$ and $H = \{z\}$,

it follows that $z$ is the only common fixed point of $f$ and $g$.

Now we will show that for any $x \in X$, $F((fg)^n x, z) \to 0$.

Suppose for some $x \in X$, $F((fg)^n x, z) \not\to 0$.

Then there exist an $\epsilon > 0$ and a subsequence $\{n_k\}$ of $\{n\}$ such that $F((fg)^{n_k} x, z) > \epsilon$ for $k = 1, 2, \ldots$.

Since $X$ is compact, the sequence $\{(fg)^{n_k} x\}$ has a convergent subsequence with the limit, say, $y$. Without loss of generality, we assume that

$(fg)^{n_k} x \to y$. Also assume $i \leqslant j$

For any positive integer $m$, $n_k \geqslant jm$ for sufficiently large $k$; hence we have $(fg)^{n_k} X \subseteq (f^i g^j)^n X$ for sufficiently large $k$.

Since $(f^i g^j)^n X$ is closed and the sequence $\{(fg)^{n_k} x\}$ converges to $y$, it follows that $y \in (f^i g^j)^n X$. This is true for each $n$. Hence $y \in H$. Further $F(y,z) \geqslant \epsilon$. Therefore $y \neq z$. This is a contradiction to the fact that $H = \{z\}$. The remaining follows similarly.

Chatterjee (1979) recently proved the following:

*Theorem* 5 (Chatterjee (1979) Th. 2)—Let $T$ be a self map on a complete metric space $(X, d)$ such that for any non-negative integer $p$,

$$d(T^{p+1} x, T^{p+2} y) \leqslant \alpha\, d(T^{p+1} y, T^{p+2} y) \frac{[1 + d(T^p x, T^{p+1} x)]}{[1 + d(T^p x, T^{p+1} y)]} + \beta d(T^p x, T^{p+1} y)$$

for all $x, y \in X$, where $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$.

Then $T$ has a unique fixed point.

Massa Silvio in his review on Chatterjee's paper (1981) (see Mathematical Reviews No. 80e, 54055) showed that the result is false if $p > 0$ in view of the following :

*Example* 6 —Let $X = [0, 1]$ with the usual metric, $Tx = x/2$ for $x \neq 0$,

$T0 = 1$. Here $p = 1$, $\alpha = 0$, $\beta = \frac{1}{2}$.

However we extend the above theorem for a sequence of self maps with a suitable modification.

*Theorem* 7— Let $\{T_i\}_{i=0}^{\infty}$ be a sequence of self maps on a complete metric space $(X, d)$ and $T_i T_j = T_j T_i$ for all non-negative integers $i$ and $j$. For $i = 1, 2, \ldots$

$$d(T_i^{p+1} x, T_0^{2p+2} y) \leqslant \alpha\, d(T_0^{p+1} y, T_0^{2p+2} y) \frac{[1 + d(x, T_1^{p+1}x)]}{[1 + d(x, T_0^{p+1} y)]} + \beta d(x, T_0^{p+1} y)] \quad \ldots(7.1)$$

for all $x, y \in X$, where $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$. Also assume that for any $x_0 \in X$, the sequence $\{x_n\}$ defined by

$x_{2n-1} = T_0^{p+1} x_{2n-2}$ and $x_{2n} = T_n^{p+1} x_{2n-1}$, $n = 1, 2, \ldots$ is decreasing i. e. $d(x_{n+1}, x_n) \leqslant d(x_n, x_{n-1})$ for all $n = 1, 2, \ldots$

Then $\{T_i\}_{i=0}^{\infty}$ has a unique common fixed point, say, $z$ and $z$ is the only fixed point of each $T_i$, $i = 0, 1, 2, \ldots$

Proof : We first prove the theorem for $p = 0$. For $p = 0$, we have

$$\text{(I)}\quad d(T_i x, T_0^2 y) \leqslant \alpha\, d(T_0 y, T_0^2 y) \frac{[1 + d(x, T_i x)]}{[1 + d(x, T_0 y)]} + \beta\, d(x, T_0 x)$$

for $i = 1, 2, \ldots$ and $x_{2n-1} = T_0 x_{2n-2}$, $x_{2n} = T_n x_{2n-1}$, $n = 1, 2, \ldots$ with

$$\text{(II)}\quad d(x_{n+1}. x_n) \leqslant d(x_n, x_{n-1}) \text{ for all } n = 1, 2, \ldots,$$

Now using (I) to $d(x_{2n+1}, x_{2n})$ we get that

$$d(x_{2n+1}, x_{2n}) \leqslant \frac{\beta}{1 - \alpha}\, d(x_{2n-1}, x_{2n}).$$

By (I) and (II), we have

$$d(x_{2n}, x_{2n-1}) = d(T_n x_{2n-1}, T_0^2 T_{n-1} X_{2n-4})$$

$$\leqslant \alpha\, d(x_{2n-2}, x_{2n-1}) \frac{[1 + d(x_{2n-1}, x_{2n})]}{[1 + d(x_{2n-1}, x_{2n-2})]} + \beta\, d(x_{2n-1}, x_{2n-2})$$

$$\leqslant (\alpha + \beta)\, d(x_{2n-1}, x_{2n-2}).$$

Thus $d(x_{n+1}, x_n) \leqslant k\, d(x_n\, x_{n-1})$ where $k = \max\left\{ \frac{\beta}{1-\alpha}, \alpha + \beta \right\} < 1.$

Therefore $\{x_n\}$ is Cauchy. Since $X$ is complete, there exists a $z \in x$ such that $x_n \to z$ as $n \to \infty$.

Applying (I) to $d(T_i\, z, x_{2n+1})$ for $i = 1, 2, \ldots.$ and letting $n \to \infty$.

We get that $T_i z = z$ for $i = 1, 2, \ldots.$ Consider.

Now

$$d(T_0\, z, T_0^2 z) = d(T_0\, T_i\, z, T_0^2) \text{ for } i = 1, 2, \ldots.$$

$$= d(T_i\, T_0 z, T_0^2\, z)$$

$$\leqslant \alpha\, d(T_0 z, T_0^2\, z) \frac{[1 + d(T_0\, z, T_i\, T_0\, z)]}{[1 + d(T_0\, z, T_0\, z)]} + \beta\, d(T_0\, z, T_0 z)$$

i. e. $d(T_0 z, T_0^2 z) \leqslant \alpha\, d(T_0\, z, T_0^2\, z)$ which implies that $T_0^2\, z = T_0\, z$.

Now applying (I) to $d(T_1\, z, T_0^2\, z)$ we get that $T_0\, z = z$.

Thus $z$ is a common fixed point of $\{T_i\}_{i=0}^{\infty}$ Uniqueness of common fixed point follows easily by (I).

Since every fixed point of $T_i$ $(i = 1, 2, \ldots.)$ is a fixed point of $T_0$ and vice versa, we conclude that $z$ is the only fixed point of each of $T_i$, $i = 0, 1, 2, \ldots.$

Suppose $p$ is a positive integer.

Then put $T_i^{p+1}\, x = S_i\, x$ for $i = 1, 2, \ldots.$ and $T_0^{p+1}\, x = S_0\, x$ for all $x$.

Then we have for $i = 1, 2, \ldots,$

$$d(S_i\, x, S_0^2\, y) \leqslant \alpha\, d(S_0\, y, S_0^2\, y) \frac{[1 + d(x, S_i\, x)]}{[1 + d(x, S_0\, y)]} + \beta\, d(x, S_0\, y)$$

for all $x, y \in X$ and $x_{2n-1} = S_0\, x_{2n-2}$, $x_{2n} = S_n\, x_{2n-1}$, $n = 1, 2, \ldots.$

By proceeding as above, we conclude that $\{S_i\}_{i=0}^{\infty}$ has a unique common fixed point, say $w$ and $w$ in the only fixed point of each $S_i$, $i = 0, 1, 2, \ldots$.

Now, $S_i T_i w = T_i^{p+1} T_i w = T_i T_i^{p+1} w = T_i S_i w = T_i w$ for $i = 0, 1, 2, \ldots$.

Thus $T_i w$ is a fixed point of $S_i$ for $i = 0, 1, 2, \ldots$. By the uniqueness of fixed point of $S_i$, we conclude that $T_i w = w$ for $i = 0, 1, 2, \ldots$.

Thus the theorem is proved.

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