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ON SOLVABILITY OF THE COMPLEX LINEAR COMPLEMENTARITY PROBLEM

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Generalizing earlier results in complex space, several existence theorems are established for a given complementarity problem over a polyhedral cone.

1. INTRODUCTION

Given a matrix $M \in C^{n \times n}$, a vector $q \in C^n$, and a polyhedral cone $S \subset C^n$, the complex linear complementarity problem, (q, M), considered here is the following :

Find a $z \in C^n$ such that

$$z \in S,\ Mz+q \in S^*,\ \text{Re} < z, Mz+q > = 0 \qquad \ldots(1)$$

where S^* is the polar cone of S.

For $S = \{z : |\arg z_j| \leqslant \beta_j, 0 \leqslant \beta_j \leqslant \pi/2, j = 1, \ldots, n\}$, the problem (q, M) was studied by McCallum (1972). The problem in the form (1) was studied by Mond (1973) and Berman (1974). In these works, the solvability of the complex linear complementarity problem was shown under various conditions on M, by considering a related complex quadratic programming problem and then invoking the complex Frank-Wolfe theorem (McCallum 1972) and the duality theory of complex quadratic programming. We present solvability results under some generalized set of conditions on M, using a perturbation technique that involves a result on variational inequalities. Our approach is simple and does not depend heavily on the theory of complex programming.

2. PRELIMINARIES

Let C^n (R^n) denote the n-dimensional complex (real) space with Hermitian (Euclidean) norm $\| \cdot \|$. R_+ and R^n_+ denote the set of nonnegative numbers and the nonnegative orthant of R^n, respectively. If A is a complex matrix or vector, then $\bar{A}$

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and A^H denote respectively the complex conjugate and conjugate transpose. For $x, y \in C^n$, $<x, y> = y^H x$ denotes the usual inner product of x and y. By e, we denote any vector with all components unity; its dimension follows from the context.

A nonempty set $S \subset C^n$ is a polyhedral cone if there is a positive integer k and a matrix $A \in C^{n\times k}$ such that $S = AR^k_+$. A cone is said to be pointed if $S \cap (-S) = \{0\}$. We shall assume, without loss at generality, that the columns of A are linearly independent.

The polar of $S = AR^k_+$ is the cone S^* defined by

$$S^* = \{y \in C^n : x \in S \Rightarrow \text{Re} <y, x> \geqslant 0\}$$

or, equivalently, by $S^* = \{y \in C^n : \text{Re}\ A^H y \geqslant 0\}$. The interior of S^* is given by

$$\text{int}\ S^* = \{y \in S^* : 0 \neq x \in S \Rightarrow \text{Re} <y, x> > 0\}$$

or, equivalently, by int $S^* = \{y \in C^n : \text{Re}\ A^H y > 0\}$ since A has linearly independent columns.

A matrix $M \in C^{n\times n}$ is said to be positive semidefinite if Re $z^H Mz \geqslant 0$ for all $z \in C^n$ and positive definite is Re $z^H Mz > 0$ unless $z = 0$.

A matrix $M \in C^{n\times n}$ is said to be S-copositive if Re $z^H Mz \geqslant 0$ for all $z \in S$ and strictly S-copositive if Re $z^H Mz > 0$ for all $0 \neq z \in S$.

A matrix $M \in C^{n\times n}$ is said to reverse the sign of $z \in C^n$ if Re $[\bar{z}_j (Mz)_j] \leqslant 0$, $j = 1, ..., n$.

A matrix $M \in C^{n\times n}$ is said to be regular if the problem (pt, M) for some $p \in$ int S^*, $t \geqslant 0$ has a unique solution, namely, $z = 0$.

As a natural extension of the result on variational inequalities in real space (Hartman and Stampacchia 1966) to complex space, we have the following result which will be needed in the sequel.

Theorem 1—If $G : C^n \to C^n$ is a continuous mapping on the nonempty, compact and convex set $D \subset C^n$, then there is a $z°$ in D with

$$\text{Re} <z-z°, G(z°) \geqslant 0 \text{ for all } z \in D.$$

Lemma 1 —Let S be a pointed polyhedral cone in C^n, and $p \in$ int S^*. Then the set

$$D_\alpha = \{z : z \in S,\ \text{Re} <z, p> \leqslant \alpha\}$$

is bounded for every fixed $\alpha > 0$.

PROOF : Assume D_α is unbounded. Then there is a sequnence $\{z^j\}$ in D_α such that $\| z^j \| \to \infty$ as $j \to \infty$. Define $y^j = z^j/\| z^j \|$ for each j. Since S is a cone, the sequence $\{y^j\}$ is also in S. Moreover, it is bounded and so it has a convergent subsequence. If we retain the same superscript to denote the subsequence and take y as its limit, we have $\| y \| = 1$ and Re $<y, p> > 0$. But

$$\text{Re} <y, p> = \lim_{j\to\infty} \text{Re} < y^j, p >$$
$$= \lim_{j\to\infty} \frac{\text{Re} <z^j, p>}{\| z^j \|} \leqslant \lim_{j\to\infty} \frac{\alpha}{\| z^j \|} \to 0$$

yields a contradiction. Hence the lemma follows.

3. Solvability of the Complementarity Problem

The next theorem will be the main tool for developing an existence theory for the problem (q, M).

Theorem 2—Let $M \in C^{n\times n}$, $q \in C^n$, $S \subset C^n$ be a pointed, polyhedral cone and let $p \in \text{int } S^*$. If (q, M) has no solution, then there exist vectors u and $\tilde{z}$ and a complex scalar $\tilde{\xi}$ such that

$$v = Mu + g(u)\, p \in S^*, \text{ Re} <u, v> = 0$$
$$g(u) = -\text{Re}(u^H Mu) \geqslant 0,\ 0 \neq u \in S; \qquad \ldots(2)$$
$$\tilde{W} = \tilde{M}z + q + (\text{Re } \tilde{\xi})\, p \in S^*, \text{ Re} <\tilde{z}, \tilde{W}> = 0$$
$$0 \neq \tilde{z} \in S, \text{ Re } \tilde{\xi} > 0; \qquad \ldots(3)$$
$$\text{Re} < \tilde{W}, u> = 0, \text{ Re} <v, \tilde{z} > = 0. \qquad \ldots(4)$$

Proof: Using Lemma 1, it is easy to show that the sets

$$D_\alpha = \{z : z \in S, \text{ Re} <z, p> \leqslant \alpha\}$$

are compact and convex for real $0 < \alpha < \infty$.

Hence, it follows from Theorem 1 that there exists $z^\alpha \in D_\alpha$ such that

$$\text{Re} <z - z^\alpha, Mz^\alpha + q > \geqslant 0 \text{ for all } z \in D_\alpha$$

and applying the duality theory of linear programming in complex space (Ben-Israel 1969) we get a $\xi_\alpha \in C^1$ such that

$$Mz^\alpha + q + \frac{1}{2}\xi_\alpha p + \frac{1}{2}\bar{\xi}_\alpha p \in S^*, z^\alpha \in S \qquad \ldots(5)$$
$$\text{Re} <Mz^\alpha + q + \frac{1}{2}\xi_\alpha p + \frac{1}{2}\bar{\xi}_\alpha p, z^\alpha > = 0 \qquad \ldots(6)$$
$$\left.\begin{array}{l}\text{Re} <z^\alpha, p> \leqslant \alpha,\ \xi_\alpha \in R_+ + i R^1 \\ \text{Re}[\bar{\xi}_\alpha (\alpha - \text{Re} <z^\alpha, p>)] = 0.\end{array}\right\} \qquad \ldots(7)$$

It is clear from (5) and (6) that if Re $\xi_\alpha = 0$ for some α, then z^α for that α solves the problem (q, M). Therefore, we can conclude that if (q, M) has no solution, then Re $\xi_\alpha > 0$ for every $0 < \alpha < \infty$. Now by (7), we have Re $<z^\alpha, p> = \alpha$ for all these α. Let $u^\alpha = z^\alpha/\alpha$. Then Re $<u^\alpha, p> = 1$, and $u^\alpha \in S$ since S is a cone. This shows that the set of points u^α lies in a compact set, and hence, there is a convergent sequence of u^α with $\alpha \to +\infty$. Let this sequence be one with $\alpha = \alpha_1, \alpha_2, \alpha_3, \ldots$, or, briefly, with $\alpha \in \Gamma$. Let u be the limit of the sequence. Thus we have

$$u = \lim_{\alpha \in \Gamma} u^\alpha = \lim_{\alpha \in \Gamma} (z^\alpha/\alpha)$$

where $u \in S$ and $\text{Re} <u, p> = 1$. Clearly, $u \neq 0$.

Further, from (6), we have

$$0 > -\text{Re}\,\xi_\alpha = \frac{1}{\alpha}\,\text{Re} <Mz^\alpha + q, z^\alpha>$$

$$= \alpha\,\text{Re} <Mu^\alpha, u^\alpha> + \text{Re} < q, u^\alpha >$$

for $\alpha \in \Gamma$, which implies that $\text{Re}\,(u^H Mu) \leqslant 0$. Now taking the limit in (5) and (6), we get (2).

Since $z^\alpha, u \in S$, there exists a matrix $A \in C^{n\times k}$ such that $z^\alpha = At^\alpha$, $u = At$, where $t^\alpha, t \in R^k_+$. This implies that the sequence

$$\{s^\alpha\}_{\alpha \in \Gamma} = \{(t^\alpha/\alpha)\}_{\alpha \in \Gamma} \to t.$$

Let $J_1 = \{j : t_j > 0\}$, $J_2 = \{j : t_j = 0$ and $s_j^\alpha > 0$ for all sufficiently large $\alpha \in \Gamma\}$, and let $J = J_1 \cup J_2$. Then choose α_0 so large that for all $\alpha \in \Gamma$ and $\alpha \geqslant \alpha_0$, $s_j^\alpha > 0$ if $j \in J$ and $s_j^\alpha = 0$ if $j \notin J$. Hence also, for $\alpha \in \Gamma$ and $\alpha \geqslant \alpha_0$, $t_j^\alpha > 0$ if $j \in J$ and $t_j^\alpha = 0$ if $j \notin J$. By taking $\tilde{z} = z^\alpha$, $\tilde{\xi} = \xi_\alpha$ for a fixed $\alpha = \tilde{\alpha} \geqslant \alpha_0$, (3) follows from (5) and (6). In order to show that $\text{Re} <\tilde{W}, u> = 0$, we find

$$\text{Re} <\tilde{W}, u> = \text{Re} <\tilde{W}, At>$$

$$= \text{Re}\,[t^T (A^H \tilde{W})] = \sum_{j=1}^{n} t_j\,\text{Re}\,(A^H \tilde{W})_j$$

and

$$0 = \text{Re} <\tilde{W}, \tilde{z}> = \sum_{j=1}^{n} t_j^\alpha\,\text{Re}\,(A^H \tilde{W})_j.$$

But $\text{Re}\,(A^H \tilde{W})_j \geqslant 0$, $j = 1, \ldots, n$, since, $\tilde{W} \in S^*$. Hence, it follows from the way α_0 is chosen that $\text{Re} <\tilde{W}, u> = 0$. Similarly, we can have $\text{Re} <v, \tilde{z}> = 0$. This completes the proof of the theorem.

Several existence results can be derived from Theorem 2.

Theorem 3—Let $M \in C^{n\times n}$, $S \subset C^n$ be a pointed, polyhedral cone and let $p \in \text{int}\, S^*$. Then (q, M) has a solution for every $q \in C^n$ if the system

$$\left.\begin{array}{l} Mu + g(u)\,p \in S^*,\ 0 \neq u \in S \\ \text{Re} <u, Mu + g(u)\,p> = 0,\ g(u) = -\text{Re}\,(u^H Mu) \geqslant 0 \end{array}\right\} \quad \ldots(8)$$

is inconsistent.

The following corollary is a consequence of Theorem 3 and the definitions of the matrices involved.

Corollary 1—Let $M \in C^{n \times n}$ and S be a pointed, polyhedral cone in C^n. Then (q, M) has a solution for every $q \in C^n$ if M is any of the following matrices : positive definite, strictly S-copositive and regular matrix (for regular matrix, take $g(u) = t$ in (8)).

Remark 1 : If we take $T = \left\{ z : |\arg z| \leqslant \beta, 0 \leqslant \beta \leqslant \frac{\pi}{2} e \right\}$, then the solvability result for a matrix M which reverses the sign of only the zero vector also follows from Theorem 3. Because, in this special case, we have

$$|\arg u_j| \leqslant \beta_j, \ |\arg p_j| < \frac{\pi}{2} - \beta_j$$

$$|\arg [(Mu)_j + g(u) p_j]| \leqslant \frac{\pi}{2} - \beta_j$$

which gives $\text{Re}\,[\bar{u}_j (Mu)_j + g(u)\bar{u}_j p_j] \geqslant 0$ and $\text{Re}\,\bar{u}_j p_j \geqslant 0$.
But since

$$0 = \text{Re} < Mu + g(u) p, u> \ = \sum_{j=1}^{n} \text{Re}\,[\bar{u}_j (Mu + g(u) p)_j]$$

we thus have $\text{Re}\,[\bar{u}_j (Mu)_j + g(u)\bar{u}_j p_j] = 0$, which implies that

$$\text{Re}\,[\bar{u}_j (Mu)_j] = -g(u)\,\text{Re}\,\bar{u}_j p_j \leqslant 0.$$

It is known (Berman 1974, Mond 1973) that if M is positive semidefinite and the set

$$Z(M, q) = \{z : z \in S, Mz + q \in S^*\}$$

is nonempty, then (q, M) has a solution. The following theorem is a generalization of this result.

Theorem 4—Let $M \in C^{n \times n}$ be such that if u satisfies the system

$$Mu + g(u) p \in S^*, \ 0 \neq u \in S$$

$$\text{Re} < Mu + g(u) p, u> \ = 0, \ g(u) = -\text{Re}\,(u^H Mu) \geqslant 0$$

then there exists a $0 \neq y \in S$ satisfying $u - y \in S$, $-M^H y \in S^*$ and $Mu + g(u) p + M^H y \in S^*$. Then (q, M) has a solution for that q for which $Z(M, q)$ is nonempty.

PROOF : Assume that $Z(M, q)$ is nonempty, but (q, M) has no solution. By Theorem 2 and the hypothesis of the present theorem, we can now have vector $u, \tilde{z}$ and y and a complex scalar $\tilde{\xi}$, which satisfy

$$0 = \text{Re} < M\tilde{z} + q + (\text{Re}\,\tilde{\xi}) p, u> \ > \text{Re} < M\tilde{z} + q + (\text{Re}\,\tilde{\xi}) p, y> \ \geqslant 0$$

$$0 = \text{Re} < Mu + g(u) p, \tilde{z} > \ \geqslant \text{Re} < -M^H y, \tilde{z} > \ > 0.$$

This implies that Re $<q, y> + (\text{Re}\,\tilde{\xi})\,\text{Re} < p, y> = 0$. Consequently, Re $<q, y> < 0$ since Re $\tilde{\xi} > 0$, and Re $< p, y> > 0$. Thus we have Re $<q, y> < 0$ for a $0 \neq y \in S$ with $-M^H y \in S^*$, implying that [see Mond 1973, Corollary, p. 251] $Z(M, q)$ is empty. This contradiction establishes the theorem.

As easy deductions from Theorem 2, we also obtain the following existence results.

Theorem 5—Let $M \in C^{n\times n}$, $q \in C^n$, $S \subset C^n$ a pointed polyhedral cone. (a) The problem (q, M) has a solution if Re $<z, Mz+q>$ is bounded below over the polyhedral cone S.

(b) Let M be S-copositive. There is a solution to (q, M) if the system $y \in S$, $q - M^H y \in S^*$ is feasible.

PROOF: (a) From (2), (3) and (4) in Theorem 2, we have for every real $\lambda > 0$

$$\text{Re} <\tilde{z}+\lambda u, M(\tilde{z}+\lambda u)+q> + (\text{Re}\,\tilde{\xi} + \lambda g(u))\,\text{Re} < \tilde{z} + \lambda u, p> = 0. \quad \ldots(9)$$

This is impossible since, in the left-hand side of (9), the first term is bounded below by the assumption, and the second term is strictly positive and increases with λ. Hence (a) follows from Theorem 2.

(b) It is also easy to see from (9) above that if M is S-copositive, then Re $<u, q> < 0$. Thus, if (q, M) has no solution, then we have a $0 \neq u \in S$ with $Mu \in S^*$, Re $<u, q> < 0$. But by Mond's theorem of the alternative [Mond 1973, Theorem 2] the system $y \in S$, $q - M^H y \in S^*$ has no solution. This leads to a contradiction which establishes the result.

By Theorem 5, part (a), there is a $z^\circ \in Z(M, q)$ with $f(z^\circ) = \text{Re} < z^\circ, Mz^\circ + q> = 0$ if $f(z)$ is bounded below over the polyhedral cone S. This assertion leads us to the following proposition.

Proposition 1—Let $f(z) = \text{Re}\,[z^H(Mz + q)]$ be the real part of a complex quadratic function which is bounded below over a nonempty, pointed polyhedral cone $S \subset C^n$. Then

$$\inf_{z \in S} f(z) \leqslant \min_{z \in Z(M,q)} f(z) = 0.$$

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