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ON THE AVERAGE NUMBER OF CROSSINGS OF AN ALGEBRAIC POLYNOMIAL

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The present paper provides an estimate of the expected number of crossings of a polynomial of degree n with the line $y = mx$ where the coefficients are independent normally distributed and m is a constant indepenpent of x. There are many known asymptotic estimates for the case of $m=0$. It is shown that the result is still valid even for $m \to \infty$ as long as $m = o(\sqrt{n})$.

1. INTRODUCTION

Let

$$P(x) = \sum_{i=0}^{n-1} a_i x^i \qquad \text{...(1.1)}$$

where $a_0, a_1, a_2, \ldots, a_{n-1}$ is a sequence of random variables, and $N_m\ (a, b) \equiv N\ (a, b)$ be the number of real roots of the algebraic equation $P(x) = mx$ in the interval (a, b), where $m_n \equiv m$ is a constant independent of x, and multiple roots are counted only once. Kac[5] found that in the case of $m = 0$, and when the coefficients of (1.1) are independent normally distributed with mean zero and variance one, the mathematical expectation of the number of real roots, $EN(-\infty, \infty)$, is asymptotic to $(2/\pi) \log n$. From later works[3,6,8], there is ground to believe that whatever class of distributions for the coefficients we choose it will not greatly affect the result as long as $E(a_i) = 0$ $(i = 0, 1, 2, \ldots, n-1)$. Further a reduction to the expected number of real roots appears to occur in the work of Ibragimov and Maslova[4] and Sambandham[7] when they consider the cases of the coefficients having non zero means or being dependent. An asymptotic formula for the expected number of real roots for $P(x) = m$ is obtained by the author[2], which is, indeed, the number of times that $P(x)$ crosses a line parrallel to the x-axis. Here we study the number of times that $P(x)$ crosses a line which is not necessarily parallel to the x-axis. We state our result in the following theorem.

Theorem—If the coefficients of (1.1) are independent, standard normal random variables, then for any constant m such that (m^2/n) tends to zero the mathematical expectation of the number of real roots of the equation $P(x) = mx$ satisfies

$$EN\,(-1,1) \sim (1/\pi)\log(n/m^2) \quad \text{if } m \to \infty \text{ as } n \to \infty$$

$$EN\,(-1,1) \sim (1/\pi)\log n \qquad \text{if } m \text{ is bounded}$$

$$EN\,(-\infty,-1) = EN\,(1,\infty) \sim (1/2\pi)\ \text{long}\ n.$$

From the theorem it is interesting to note that for sufficiently large n we still obtain a sizeable number of crossings even when the line tends to be perpendicular to the x-axis (i. e. $m \to \infty$).

2. A Formula for the Number of Crossings

Let $\phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-y^2/2)\,dy$ and $\phi(x) = \phi'(x) = (2\pi)^{-1/2} \exp(-x^2/2)$,

then by using the expected number of level crossings [(Cramer and Leadbetter[1], p. 285)] for our equation $P(x) - mx = 0$ we have

$$EN\,(a, b) = \int_a^b \beta^{1/2} \alpha^{-1/2} (1 - \mu^2)^{1/2} \phi\,(\omega/\alpha)^{1/2} [2\phi(\eta) + \eta\,\{2\phi(\eta) - 1\}]\, dx \ldots(2.1)$$

where

$$\alpha(x) \equiv \alpha = \text{Var}\,\{P(x) - mx\}, \quad \beta(x) \equiv \beta = \text{Var}\,[\{P(x) - mx\}']$$

$$\mu = \alpha^{-1/2} \beta^{-1/2}\, \text{Cov}\,[\{P(x) - mx\}, \{P(x) - mx\}'], \ \omega = E\,\{P(x) - mx\}$$

$$\eta = \beta^{-1/2} (1 - \mu^2)^{-1/2} \{\nu - \beta^{1/2} \mu\, \omega\, \alpha^{-1/2}\} \text{ and } \nu = E\,[\{P(x) - mx\}'].$$

Since the coefficients of $P(x)$ are independent random variables with mean zero and variance one we can easily find that

$$\omega = -mx, \qquad \alpha = \text{Var}\,\{P(x)\} = \sum_{i=1}^{n-1} x^{2i}$$

$$\beta = \text{Var}\,\{P'(x)\} = \sum_{i=1}^{n-1} i^2 x^{2i-2}, \ \nu = -m$$

$$\mu = \alpha^{-1/2} \beta^{-1/2} E\,\{P(x).\,P'(x)\} = \alpha^{-1/2} \beta^{-1/2} \sum_{i=1}^{n-1} i\, x^{2i-1} = \alpha^{-1/2} \beta^{-1/2} \gamma$$

(say)

and

$$\eta = m\,\alpha^{-1/2} (\alpha\beta - \gamma^2)^{-1/2} (\gamma x - \alpha).$$

Hence from (2.1) and since $\phi(x) = \frac{1}{2} + (2\pi)^{-1/2}\, \text{erf}\,(x)$ we have the extension of Kac-Rice formula[9].

$$EN\,(a, b) = \int_a^b (\Delta^{1/2}/\pi\alpha) \exp\{(-\alpha m^2 + 2m^2 \gamma x - \beta m^2 x^2)/2\Delta\}$$

(equation continued on p. 3)

$$+ \{ | m(\gamma x - \alpha) | / \pi\alpha^{3/2} \} \exp(-m^2 x^2/2\alpha) \operatorname{erf}$$
$$\{ | m(\gamma x - \alpha) | / \alpha^{1/2} \Delta^{1/2} \} \, dx$$
$$= \int_a^b I(x)\, dx = \int_a^b I_1(x)\, dx + \int_a^b I_2(x)\, dx \text{ (say)} \qquad \ldots(2.2)$$

where

$$\Delta(x) \equiv \Delta = \alpha\beta - \gamma^2.$$

3. Proof of the Theorem

First we consider the case when $m \to \infty$ as $n \to \infty$. For finding an upper limit of $EN(0, 1)$, let $0 \leq a \leqq 1$ and divide $(0, 1)$ into two sub-intervals $(0, 1 - 1/n^a)$ and $(1 - 1/n^a, 1)$. For $0 \leqq x \leqq 1 - 1/n^a$ and all sufficiently large n we have

$$\gamma = \{(n-1)x^{2n+1} - nx^{2n-1} + x\}(1 - x^2)^{-2}$$
$$= x(1 - x^{2n})(1 - x^2)^{-2} + O\{n^{1+a} \exp(-2n^{1-a})\} \qquad \ldots(3.1)$$

and

$$\beta = (1 + x^2)(1 - x^{2n})(1 - x^2)^{-3} + O\{n^{2+a} \exp(-2n^{1-a})\}. \qquad \ldots(3.2)$$

From (3.1) and (3.2) we have

$$\Delta = (1 - x^{2n})(1 - x^2)^{-4} + O\{n^{2+2a} \exp(-2n^{1-a})\}. \qquad \ldots(3.3)$$

Now we choose $a = 1 - \log\log(n^{10})/\log n$. Then since

$$\exp(-2n^{1-a}) \exp\{-2\log(n)^{10}\} = n^{-20}$$

for n sufficiently large, all the terms inside the $0\{\ \}$ will tend to zero and also $a \to 1$ which is necessary for obtaining our further result. Hence from (2.2), (3.1), and (3.3) for all sufficiently large n we have

$$\int_0^{1-1/n^a} I_1(x)\, dx = \int_0^{1-1/n^a} (\Delta^{1/2}\pi\alpha) \exp[\{-m^2(1 - x^2)/(1 - x^{2n})\} \{(1 - x^2)^2/2$$
$$- x^2(1 - x^2) + x^2(1 + x^2)/2\} \{1 + O(n^{2+a}$$
$$\times \exp(-2n^{1-a}))\}]\, dx. \qquad \ldots(3.4)$$

Now we note that for all x

$$(1 - x^2)^2/2 - x^2(1 - x^2) + x^2(1 + x^2)/2 = (1 - 3x^2 + 4x^4)/2$$
$$\geqq 7/32 > 1/5$$

and since from Kac[5] we have $\Delta^{1/2}/\alpha < (1 - x^2)^{-1}$, (3.4) gives

$$\int_0^{1-1/n^a} I_1(x)\, dx < (1/\pi) \int_0^{1-1/n^a} (1 - x^2)^{-1} \exp\{-m^2(1 - x^2)/5(1 - x^{2n})\}\, dx$$

(*equation continued on p.* 4)

$$< (1/\pi) \int_0^{1-1/n^a} (1 - x^2)^{-1} \exp\{-m^2 (1 - x^2)/5\} \, dx. \qquad \ldots(3.5)$$

Let $\lambda = m^2/5$, then from (3.5) and since $\exp\{-\lambda (1 - x)^2)\} < \{1 - \lambda (1 - x^2)^{-1}\}$ we have

$$\int_0^{1-1/n^a} I_1(x) \, dx < (1/\pi) \int_0^{1-1/n^a} (1 - x^2)^{-1} \{1 + \lambda (1 - x^2)\}^{-1} dx$$

$$= (1/\pi) \int_0^{1-1/n^a} [(1 - x^2)^{-1} - \lambda/\{1 + \lambda (1 - x^2)\}] \, dx$$

$$= (1/2\pi) \log \frac{2 - 1/n^a}{1/n^a} - (1/2\pi) (1 - (1/\lambda)^{-1/2}$$

$$\log \frac{\sqrt{1 + (1/\lambda)} + 1 - (1/n^a)}{\sqrt{1 + (1/\lambda)} - 1 + (1/n^a)} . \qquad \ldots(3.6)$$

Now since $(1 + 1/\lambda)^{1/2} = 1 + 1/2\lambda + O(1/\lambda^2)$ for all $|\lambda| < 1$ and also by noting that $a \to 1$, $\lambda \to \infty$ and $\lambda/n \to 0$ as $n \to \infty$, from (3.6) for all sufficiently large n we have

$$\int_0^{1-1/n^a} I_1(x) \, dx = (1/2\pi) (\log 2 + \log n)$$

$$- (1/2\pi) (1 + 1/\lambda)^{-1/2} \log \frac{2 + 1/2\lambda + O(1/\lambda^2) - 1/n}{1/2\lambda + O\{1/\lambda^2) + 1/n}$$

$$< (1/2\pi) (\log 2 + \log n) - (1/2\pi) (1 + 1/\lambda)^{-1/2}$$

$$\{\log \lambda + \log (4 - 1/n)\} < (1/2\pi) \log n/m^2 + 1. \qquad ..(3\ 7)$$

Also for $1 - 1/n^a\, 1 \leqq x \leqq 1$, since from Ibragimov and Maslova[4] $\Delta^{1/2}/\alpha < (2n - 1)^{1/2} (1 - x)^{-1/2}$, we have

$$\int_{1-1/na}^{1} I_1(x) \, dx < \int_{1-1/na}^{1} (2n - 1)^{1/2} (1 - x)^{-1/2} \, dx$$

$$= 2 (2n^{1-a} - n^{-a})^{1/2} = O(\log n)^{1/2}. \qquad \ldots(3.8)$$

Now we find an upper limit for $\int_0^1 I_2(x) \, dx$. We have

$$mx\gamma\alpha^{-3/2} = mx (1 - x^2)^{-1/2} \{(n - 1) x^{2n+1} - n x^{2n-1} + x\} (1 - x^{2n})^{-3/2} \qquad \ldots(3.9)$$

which gives

$$| m (\gamma x - \alpha) \alpha^{-3/2} | = | m | (1-x^2)^{-1/2} (1-x^{2n})^{-3/2} | (2x^2 - 1)(1-x^{2n}) - n x^{2n} (1 - x^2) | . \quad \ldots(3.10)$$

Now for $0 \leqq x \leqq \sqrt{3}/2$ given any ϵ positive exists an integer n_0 such that for all $n \geqq n_0$

$$n x^{2n} (1 - x^2) < n (3/4)^n < \epsilon.$$

We note that $(2x^2 - 1)(1 - x^{2n}) < (2x^2 - 1) < \epsilon$ only for a small interval of $\frac{1}{2}(1 - \epsilon) < x^2 < \frac{1}{2}(1 + \epsilon)$. Hence from (3.10) for all sufficiently large n we have

$$\int_{\sqrt{(1-\epsilon)/2}}^{\sqrt{(1+\epsilon)/2}} I_2 (x)\, dx < | m | (\sqrt{2/\pi}) \{1 - (3/4)^n\} \{\epsilon^2 + O(\epsilon^3)\}. \quad \ldots(3.11)$$

Using the fact that ϵ can be made arbitrarily small we can see that the above integral tends to zero as $n \to \infty$ (for example choosing $\epsilon = 1/m$). On the other hand, let $\int_0^{1-1/n} \ldots dx$ indicate the integral over $0 < x \leqq 1 - 1/n$ excluding $\sqrt{(1-\epsilon/2)} \leqq x \leqq \sqrt{(1+\epsilon)/2}$, then from (3.10) and since for $0 \leqq x \leqq 1 - 1/n$ and n sufficiently large $x^{2n} < (1 - 1/n)^{2n} \to e^{-2}$, we have

$$\int_0^{1-1/n} I_2 (x)\, dx \leqq | m | (\sqrt{2/\pi}) (1 - e^{-2})^{-1/2} \int_0^{1-1/n} (1 - 2x^2)(1 - x^2)^{-1/2}$$

$$\exp\{- m^2 x^2 (1 - x^2)\} \leqq | m | (\sqrt{2/\pi}) (1 - e^{-2})^{-1/5} \int_0^{1-1/n}$$

$$(1- \; -2x^2)(1 - x^2)^{-1/2} \exp\{- m^2 x^2 (1 - x^2)\}\, dx$$

$$< (2\pi)^{-1/2} (1 - e^{-2})^{-1/2} \int_0^k u^{-1/2} e^{-u}\, du$$

where $u = m^2 x^2 (1 - x^2)$ and $k = m^2 (1 - 1/n)^2 (2/n - 1/n^2)$. Now by integrating by parts and since $u^{1/2} e^{-u} < 1$ for all sufficiently large n, we have

$$\int_0^{1-1/n} I_2 (x)\, dx < (2\pi)^{-1/2} (1 - e^{-2})^{-1/2} [\{2 (m^2/n)(2 - 1/n)(1 - 1/n)^2\}^{1/2}$$

$$\exp\{-(m^2/n)(2 - 1/n)(1 - 1/n)^2\} + 2 (m^2/n)(1 - 1/n)^2$$

$$(2 - 1/n)\}] \quad \ldots(3.12)$$

which also tends to zero as $n \to \infty$. In order to estimate $\int_{1+1/n}^{1} I_2 (x)\, dx$ we note that always $\gamma > \alpha/x$, then

$$| \gamma x/\alpha^{3/2} - 1/\alpha | < 2 | \gamma x/\alpha^{3/2} | . \qquad ...(3.13)$$

Since

$$\gamma + \sum_{i=1}^{n-1} i\, x^{2n-1} < (n/x) \sum_{i=1}^{n-1} x^{2i} = (n/x)\, \alpha$$

for $1 \leqq x \leqq 1 - 1/n$ we can obtain

$$\gamma x/\alpha^{3/2} \leqq n\, \alpha^{-1/2} \leqq \sqrt{n}\, (1 - 1/n)^{-(n-1)}.$$

This and (3.13) gives

$$\int_{1-1/n}^{1} I_2(x)\, dx < (2 | m | /\pi) \int_{1-1/n}^{1} (\gamma x/\alpha^{3/2})\, dx$$

$$< (2 | m | /\pi \sqrt{n})\, (1 - 1/n)^{-(n-1)} \qquad ...(3.14)$$

which also tends to zero as $n \to \infty$. From (3.7), (3.8), (3.11), (3.12) and (3.14) we have

$$EN(0, 1) < (1/2\pi) \log (n/m^2) + O \{\log n\}^{1/2}. \qquad ...(3.15)$$

In order to obtain a lower estimate of $EN(0, 1)$ without loss of the generality we can assume $2m^2 > 1$. Then from (3.4) and since for all $0 \leqq x \leqq 1$

$$(1 - x^2)^2/2 - x^2 (1 - x^2) + x^2 (1 + x^2)/2 \leqq 1$$

we have

$$\int_0^{1-1/n^a} I_1(x)\, dx > (1/\pi) \int_0^{1-1/n^a} (\Delta^{1/2}/\alpha) \exp \{-m^2 (1 - x^2)/(1 - x^{2n})\}\, dx$$

$$\geqq (1/2\pi) \int_0^{1-1/n^a} (1 - x)^{-1} \exp \{- 2m^2 (1 - x)\}\, dx$$

$$= (1/2\pi) \int_{2m^2/n^a}^{2m^2} u^{-1} e^{-u}\, du = (1/2\pi) \{ \int_{2m^2/n^a}^{2m^2} u^{-1}\, du$$

$$- \int_{2m^2/n^u}^{1} u^{-1} (1 - e^{-u})\, du - \int_1^{2m^2} u^{-1}\, du + \int_1^{2m^2} u^{-1} e^{-u}\, du\} \qquad ...(3.16)$$

where $u = 2m^2 (1 - x)$. Now by noticing that $u^{-1} (1 - e^{-u}) < 1$ and $\int_1^{2m^2} u^{-1} e^{-u}\, du$ is always positive, from (3.19) for all sufficiently large n we have

$$\int_0^{1-1/n^a} I_1(x)\, dx > (1/2\pi) \{a \log n - \log m^2 - 1 - \log 2\}$$

$$= (1/2\pi) \log (n/m^2) + O (\log \log n). \qquad ...(3.17)$$

Finally from (3.15) and (3.17) we obtain the asymptotic formula

$$EN(0,\ 1) \sim \log (n/m^2).$$

On the other hand for m bounded from (3.5) and (3.8) we have

$$\int_0^1 I_1(x)\, dx < (1/\pi) \int_0^{1-1/n^a} (1-x^2)^{-1}\, dx + (1/\pi) \int_{1-1/n^a}^1 (2n-1)^{1/2}(1-x)^{-1/2}$$

$$\times\, dx < (1/2\pi) \log n + O\{\log n\}^{1/2}.$$

From this and (3.11), (3.12) and (3.14) we have

$$EN(0,\ 1) < (1/2\pi) \log n + O\{\log n\}^{1/2}. \quad \ldots(3.18)$$

Also in order to obtain a lower estimate, from (3.16) for all sufficiently large n we have

$$\int_0^{1-1/n^a} I_1(x)\, dx > (a/2\pi) \log n - 2m^2$$

$$= (1/2\pi) \log n + O\{\log \log n\}. \quad \ldots(3.19)$$

Hence from (3.18) and (3.19) for m bounded we have

$$EN(0,\ 1) \sim (1/2\pi) \log n.$$

Now we shall find the asymptotic relation for $EN(1, \infty)$. Let $y = 1/x$ then we have

$$\int_1^\infty I(x)\, dx = \int_0^1 I(1/y)\, y^{-2}\, dy. \quad \ldots(3.20)$$

Since from Farahmand[2] (p. 707) we have $\Delta(1/y) = y^{-2(2n-4)} \Delta(y)$ and $\alpha(1/y) = y^{-(2n-2)}$ $\alpha(y)$, from (2.2) and (3.20) we have

$$\int_1^\infty I_1(x)\, dx < (1/\pi) \int_0^1 y^2 \{\Delta(1/y)\}^{1/2}/\alpha(1/y)\}\, dy$$

$$= (1/\pi) \int_0^1 \{\Delta(y)\}^{1/2}/\alpha(y)\, dy$$

$$< (1/\pi) \int_0^{1-1/n} (1-y^2)^{-1}\, dy + (1/\pi) \int_{1-1/n}^1 (2n-1)^{1/2}(1-y)^{-1/2}\, dy$$

$$< (1/2\pi) \log n + 3.5. \quad \ldots(3.21)$$

Now we find an upper estimate for $I_2(x)$. We have

$$\{(m/y)\, \gamma(1/y) - m(\alpha(1/y)\}\, \{\alpha(1/y)\}^{-3/2}$$

$$= m\, y^{n-1} (1-y^2)^{-1/2} (1-y^n)^{-3/2} \{n(1-y^2) - (2-y^2)(1-y^{2n})\}. \quad \ldots(3.22)$$

Let a be the same constant as before. From (2.2), (3.13) and (3.22) and for all sufficiently large n we have

$$\int_0^{1-1/n^a} y^{-2} I_2(1/y)\,dy \leqq (2\,|\,m\,|\,n/\pi) \int_0^{1-1/n^a} y^{n-3}(1-y^2)^{1/2}(1-y^{2n})^{-3/2}\,dy$$

$$\leqq (2\,|\,m\,|\,n/\pi)\exp(-n^{1-a})\{1-\exp(2n^{1-a})\}/(n-2) \quad ..(3.23)$$

which, since $\exp(-n^{1-a}) = n^{-10}$, tends to zero as $n \to \infty$. On the other hand for $1-(1/n^a) \leqq y \leqq 1$

$$x\gamma(x)\{\alpha(x)\}^{-3/2} < n\{\alpha(x)\}^{-1/2} = n\,y^{n-1}(1-y^2)^{1/2}(1-y^{2n})^{-1/2}$$

$$< n\,y^{n-1}\{2n^{-a} - n^{-2a}\}^{1/2}\{1-(1-(1/n^a)^{2n}\}^{-1/2}.$$

Hence from this, (2.2) and (3.13) we have

$$\int_{1-1/n^a}^{1} y^{-2} I_2(1/y)\,dy \leqq (2\,|\,m\,|\,n\sqrt{2/\pi})\,n^{-a/2}\{1-(1/n^a)^{2n}\}^{-1/2}/n-2$$

$$= O\{\log n\}^{1/2}. \quad ...(3.24)$$

Since $n^{-a/2} = (n^{-1}\log n^{10})^{1/2}$. Hence from (3.21), (3.23) and (3.24) we obtain

$$EN(1,\infty) < (1/2\pi)\log n + O\{\log n\}^{1/2}. \quad ...(3.25)$$

To get the lower estimate for $EN(1,\infty)$, for $0 \leqq y \leqq 1 - 1/n^a$ we have

$$\alpha = y^{-(2n-2)}(1-y^{2n})/(1-y^2),\quad \gamma = y^{-(2n-3)}\{1-(1-y^{2n})/(1-y^2)\}/(1-y^2),$$

$$\Delta = y^{-(4n-8)}\{(1-y^{2n})^2(1-y^2)^{-4} - n^2 y^{2n-2}(1-y^2)^{-2}\}$$

$$= y^{-(4n-8)}(1-y^{2n})^2(1-y^2)^{-4} + O\{n^{2+2a}\exp(-2n^{1-a})\}$$

and

$$\beta/\Delta = y^{2n-4}(1-y^2)(1-y^{2n})^{-1}\{1+y^2+n^2(1-y^2)^2(1-y^{2n})^{-1}$$

$$-2n(1-y^2)(1-y^{2n})\} + O\{n^{2+2a}\exp(-2n^{1-a})\},$$

Hence, from these, and simple algebra, we find

$$(-\alpha + 2\gamma/y - \beta/y^2)/\Delta = y^{2n-6}(1-y^2)(1-y^{2n})^{-1}[(1-y^2)^2$$

$$-2(1-y^2)^2(1-y^{2n})^{-1}\{1-(1-y^{2n})/(1-y^2\} + 1 + y^2 + n^2$$

$$(1-y^2)^2/(1-y^{2n}) - 2n(1-y^2)/(1-y^{2n})]$$

$$+ O\{n^{2+2a}\exp(-2n^{1-a})\} < n^2\exp(2n^{1-a}) = n^{-18} \quad ...(3.26)$$

for all sufficiently large n. Now, since from (3.26), $\exp\{m^2(-\alpha + 2\gamma/y - \beta/y^2)/2\Delta\}$ tends to one as $n \to \infty$, from (3.20), (3.26) and for all sufficiently large n

$$EN(1, \infty) > \int_1^\infty I_1(x)\,dx > (1/\pi) \int_0^{1-1/n^a} (1-y^2)^{-1}\,dy$$

$$= (a/2\pi) \log n - (1/2\pi) \log (2 - 1/n^a) = (1/2\pi) \log n + O(\log \log n). \quad \ldots(3.27)$$

Finally from (3.25) and (3.27) we have asymptotic formula

$$EN(1, \infty) \sim (1/2\pi) \log n.$$

Sinnce a_j and $-a_j$ ($j = 0, 1, \ldots, n-1$) both have the standard normal distribution

$EN(-1, 0) = EN(0, 1)$ and $EN(-\infty, -1) = EN(1, \infty)$, we have proof of the theorem.

If we add this result to asymptotic formula in Farahmand[2] for k constant such that (k^2/n) tends to zero we can obtain the result for the case in which $P(x) = mx + k$. For this case

$$\omega = -mx - k \text{ and } \eta = \{\gamma(mx+k) - \alpha m\}/\alpha^{1/2}\Delta^{1/2}$$

while α, β, γ and μ will remain the same. The Kac-Rice formula will be

$$EN(a, b) = \int_a^b (\Delta^{1/2}/\pi\alpha) [\exp [\{-\alpha m^2 + 2m\gamma(mx+k) - \beta(mx+k)^2\}/2\Delta]$$

$$+ [\{-\alpha m + \gamma(mx+k)\}/\pi\alpha^{3/2}] \exp\{-(mx+k)^2/2\alpha\}$$

$$\times \operatorname{erf}[\{-\alpha m + \gamma(mx+k)\}/\alpha^{1/2}\Delta^{1/2}]\}\,dx$$

and it is interesting to know for this case

$EN(-1, 1) \sim (1/\pi) \log(n/(m^2 + k^2))$ if either m or k or both tend to infinity as $n \to \infty$

$EN(-1, 1) \sim (1/\pi) \log n$ if m and k are bounded

$EN(-\infty, 1) = EN(1, \infty) \sim (1/2\pi) \log n$.

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