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FIXED POINT ITERATIONS FOR NONLINEAR HAMMERSTEIN EQUATION INVOLVING NONEXPANSIVE AND ACCRETIVE MAPPINGS

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The solution of the nonlinear Hammerstein operator equation $x + KNx = f$, where K and N are 'nonexpansive' and 'accretive' mappings, and K also satisfies a monotonicity condition is approximated in a Hilbert space by means of fixed point iteration processes.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real normed linear space. A mapping T with domain $D(T)$ and range $R(T)$ in X is called monotone[14] if for each x, y in $D(T)$ and some real number $t > 0$, the following inequality is satisfied :

$$\|x - y\| \leq \|x - y + t(Tx - Ty)\|. \qquad \text{..(1)}$$

Mappings satisfying (1) for all $t > 0$ are sometimes referred to as accretive[1]. If X is a Hilbert space, the accretive condition (1) reduces to

$$\text{Re} < Tx - Ty, x - y > \geq 0 \qquad \text{...(2)}$$

for all x, y in X. The accretive operators were introduced by Browder[1] and Kato[14]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \; u(0) = u_0 \qquad \text{...(3)}$$

is solvable if T is locally Lipschitzian and accretive on X. Browder also proved that if $T : X \to X$ is locally Lipschitzian and accretive then T is m-accretive, i.e., the map $(I + T)$, where I denotes the identity map of X, is surjective. This result was subsequently generalized by Martin[18] to continuous accretive operators. If H is a Hilbert space, Zarantonello[26] proved that the operator equation

$$x + Tx = h \qquad \text{...(4)}$$

for each $h \in H$, has a unique solution provided T is monotone and Lipschitzian.

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For a Banach space X we shall denote by J the normalized duality map from X to 2^{X^*} given by

$$Jx = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle\}$$

where X^* denotes the dual space of X and $\langle \, , \, \rangle$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex, then J is single-valued, and if X^* is uniformly convex, then J is uniformly continuous on bounded sets[2].

A mapping $A : H \to 2^H$ with domain $D(A)$ in a Hilbert space H is called hemicontinuous at $x_0 \in D(A)$, if, for any $x \in H$ such that $x_0 + tx \in D(A)$ for $0 \leqslant t \leqslant \alpha_x$ with $\alpha_x > 0$, and for any sequence $t_n \to 0$ with $0 < t_n \leqslant \alpha_x$, we have $A(x_0 + t_n x) \overset{w}{\to} Ax_0$, where $\overset{w}{\to}$ denotes weak convergence. A is called hemicontinuous if it is hemicontinuous at every $x_0 \in D(A)$. It is easily seen that linear maps as well as continuous maps are hemicontinuous.

In the sequel we shall be concerned with operators of the Hammerstein type, i. e., operators of the form $I + AB$. These operators play a crucial role in the study of feedback systems (see e. g., Dolezal[3], Chapter 4) and have been studied by several authors[16,25,26]. In Sh-Chepanovich[25] the following result appears :

Theorem Sh-Chepanovich[25]—Let X be a separable reflexive Banach space and let X^* denote it dual space. Let.

(a) $N : X^* \to X$ be a hemicontinuous monotone map ;

(b) $K : X \to X^*$ be a linear monotone map such that for some $\mu > 0$ and each $u \in X^*$,

$$\langle Ku, u \rangle \geqslant \mu \|Ku\|^2 \qquad \ldots(5)$$

where $\langle \, , \, \rangle$ denotes the duality pairing. Then the operator equation

$$u + KNu = f \qquad \ldots(6)$$

has a unique solution for each $f \in X^*$.

In section 2 we examine two fixed point iteration schemes and apply them (in section 3) to the iterative approximation of the solution of eqn. (6). In particular, we shall prove that both iteration schemes converge weakly to the solution of eqn. (6). We conclude with an open question.

2. Two Fixed Point Iteration Methods

In this section we describe two fixed point iteration methods given by the following :

(a) *The Ishikawa Iteration Process*[12,21] defined as follows : For K a convex subset of a Banach space X, and T a mapping of K into itself, the sequence $\{x_n\}_{n=1}^{\infty}$ in K

is defined by

$$x_0 \in K \qquad \text{...(7)}$$

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Ty_n \qquad \text{...(8)}$$

$$y_n = (1 - \beta_n)\, x_n + \beta_n\, Tx_n,\ n \geqslant 0 \qquad \text{...(9)}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ $\{\beta_n\}_{n=0}^{\infty}$, satisfy $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$ for all n,

$$\lim_n \beta_n = 0; \text{ and } \sum_n \alpha_n\, \beta_n = \infty.$$

(b) *The Mann Iteration Process*[17,24] which is similar to the Ishikawa iteration process above but with $\beta_n \equiv 0$ and different conditions placed on α_n. More precisely, with X, K and x_0 as in part (a), the Mann iteration process is defined by

$$x_0 \in K \qquad \text{...(10)}$$

$$x_{n+1} = (1 - C_n)\, x_n + C_n\, Tx_n,\ n \geqslant 0 \qquad \text{...(11)}$$

where $\{C_n\}_{n=0}^{\infty}$ is a real sequence satisfying $C_0 = 1$; $0 \leqslant C_n < 1$ for all $n \geqslant 1$, and $\sum_n C_n = \infty$. The condition $\sum_n C_n = \infty$ is, in some applications, replaced by $\sum_n C_n (1 - C_n) = \infty$.

The iteration processes described in (a) and (b) above have been studied extensively by several authors and have been successfully employed to approximate the fixed points of several nonlinear mappings in Banach spaces (when these mappings are already known to have fixed points) and to approximate solutions of several nonlinear operator equations in Banach spaces[3,5,8-13,16,19,20-24]. It is worth mentioning here that even though the iteration scheme (b) is similar to (a), the two schemes may exhibit different behaviours for different classes of nonlinear mappings[24].

3. Weak Convergence of the Hammerstein Operator Equation in Hilbret Space

We shall need the following results :

Lemma[21]—Let H be a Hilbert space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H such that $\{x_n\}_{n=1}^{\infty}$ converges weakly to x^* in H. Then the inequality

$$\liminf_{n\to\infty} \|x_n - y\| \geqslant \liminf_{n\to\infty} \|x_n - x^*\|$$

holds for all $y \neq x^*$.

Let K be a nonempty convex closed subset of a Hilbert space H and let T map K

into K. T is called demiclosed at 0 in K if $\{x_n\}_{n=1}^{\infty}$ is a sequence in K which converges weakly to x^* in K, and if $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to zero, then $Tx^* = 0$.

A mapping $T : H \to H$ of a Hilbert space H into itself is called nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for each x, y in H. It is well known[3] that if $T : H \to H$ is nonexpansive then $(I - T)$ is demiclosed at 0.

Theorem — Let H be a separable Hilbert space and let C be a nonempty bounded closed convex subset of H. Suppose

(a) $N : C \to C$ is a nonlinear nonexpansive monotone map;

(b) $K : C \to C$ is a nonexpansive monotone map;

such that for some $\mu > 0$ and each $x \in H$.

$$<Kx, x> \geqslant \mu \|Kx\|^2.$$

Define $S : C \to C$ by $Sx = f - KNx$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in C$,

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Sy_n \quad \ldots(12)$$

$$y_n = (1 - \beta_n) x_n + \beta_n Sx_n, \; n \geqslant 0 \quad \ldots(13)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying the following conditions :

(i) $0 \leqslant \alpha_n, \beta_n < 1$ for all n;

(ii) $\lim\sup \beta_n < 1$

(iii) $\sum_n \alpha_n \beta_n = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ converges weakly to the unique solution of

$$x + KNx = f. \quad \ldots(14)$$

Proof : We shall make use of the following inequality which is valid in every Hilbert space, H. For each x, y, z in H, and each real number $\lambda \in (0, 1)$,

$$\|\lambda x + (1 - \lambda) y - z\|^2 = \lambda \|x - z\|^2 + (1 - \lambda) \|y - z\|^2 - \lambda (1 - \lambda) \|x - y\|^2. \quad \ldots(15)$$

Observe that the nonexpansiveness of N implies its hemicontinuity. So, the existence of a unique solution to (14) follows from Theorem Sh. Let q denote this solution. Observe that q is a fixed point of S. Furthermore, for arbitrary $u, v \in H$, $\|Su - Sv\| \leqslant \|u - v\|$. From (12) and (13), using (15),

$$\|x_{n+1} - q\|^2 = \|(1 - \alpha_n) x_n + \alpha_n Sy_n - q\|^2$$
$$= (1 - \alpha_n) \|x_n - q\| + \alpha_n \|Sy_n - q\| - \alpha_n [1 - \alpha_n]$$
$$\times \|x_n - Sy_n\|^2 \leqslant (1 - \alpha_n) \|x_n - q\|^2 + \alpha_n \|y_n - q\|^2 \quad ..(16)$$

and

$$\|y_n - q\|^2 = \|(1 - \beta_n) x_n + \beta_n Sx_n - q\|^2$$
$$= (1 - \beta_n) \|x_n - q\|^2 + \beta_n \|Sx_n - q\|^2 - \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2$$
$$\leqslant (1 - \beta_n) \|x_n - q\|^2 + \beta_n \|x_n - q\|^2 - \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2$$
$$= \|x_n - q\|^2 - \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2. \quad ...(17)$$

Substitution of (17) in (16) yields

$$\|x_{n+1} - q\|^2 \leqslant \|x_n - q\|^2 - \alpha_n \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2.$$

Hence, $\|x_{n+1} - q\| \leqslant \|x_n - q\|$ and $\{\|x_n - q\|\}$ converges. Moreover,

$$\alpha_n \beta_n [1 - \beta_n] \|x_n - Sx_n\|^2 \leqslant \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

Summation of this inequality from 1 to N yields

$$\sum_{i=1}^{N} \alpha_j \beta_j [1 - \beta_j] \|x_j - Sx_j\|^2 \leqslant \|x_1 - q\|^2 - \|x_{N+1} - 1\|^2 < \infty. \quad ...(18)$$

Now, lim sup $\beta_n < 1$ implies for N sufficiently large, $1 - \beta_l \geqslant a > 0$ for all $j \geqslant N$ and some fixed real number a, so that the condition $\sum_n \alpha_n \beta_n = \infty$ and inequality (18) now yield, $\liminf_{n\to\infty} \|x_n - Sx_n\| = 0$. The boundedness of $\{x_n\}_{n=1}^{\infty}$ implies there exists a subsequnce $\{x_{n_k}\}_{k=0}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$ such that $\{x_{n_k}\}_{k=0}^{\infty}$ converges weakly to some $x^* \in H$. Moreover, $\{x^n\}_{k=0}^{\infty}$ is in C and C is weakly closed (since it is closed and convex), so it follows that $x^* \in C$. Also,

$$\lim_{n\to\infty} \|x_{n_k} - Sx_{n_k}\| = \lim_{k\to\infty} \|(I - S) x_{n_k}\| = 0.$$

Nonexpansiveness of S implies $(I - S)$ is demiclosed at 0, so it follows that $(I - S)x^* = 0$, i. e., x^* is a fixed point of S. By uniqueness of the fixed point, $x^* = q$. Thus any weak cluster point of $\{x_n\}_{n=1}^{\infty}$ is a fixed point of S. A standard argument[11,21] using the Lemma above now shows that $\{x_n\}_{n=0}^{\infty}$ has a unique weak cluster point so that $\{x_n\}_{n=1}^{\infty}$ converges weakly to q, completing the proof of the Theorem.

Corollary—Let H, N, C, f and S be as in the above Theorem. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by

$$x_0 \in C,$$

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Sx_n,\ n \geqslant 0 \qquad \ldots(19)$$

where $\{\alpha_n\}$ is a real sequence satisfying

(i) $0 \leqslant \alpha_n < 1$ for all n,

(ii) $\sum_n \alpha_n (1 - \alpha_n) = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ converges weakly to the unique solution of

$$x + KNx = f.$$

PROOF : Set $\beta_n = 0$ for all n, in equations (12) and (13); and replace the condition (iii) $\sum_n \alpha_n \beta_n = \infty$ by (iii) $\sum_n \alpha_n (1 - \alpha_n) = \infty$ in the above Theorem. Then the Corollary follows immediately from the Theorem.

Comments—If $K = I$ (the identity map of C) the Hammertsein equation $x + KNx = f$ reduces to the equation

$$x + Nx = f. \qquad \ldots(20)$$

Equation (20) has been studied by several authors[5,7,9], Dotson[9] showed that if $N : H \to H$ is nonexpansive and monotone, the Mann iteration process converges strongly to the unique solution of (20). This result was generalized by the author[6] to operators with Lipschitz constant $L \geqslant 1$ and to operators which need not be defined on the whole of H. Bruck[5], considered equation (20) when $T : H \to H$ is a multivalued nonlinear monotone map and proved, without any continuity assumption on N, that the Mann iteration process converges strongly to a solution of equation (20), if the initial guess is taken in a certain neighbourhood of the solution. This result has also been extended by the author to L_p spaces for $p \geqslant 2$. The methods used earlier[5,7,9] to establish the strong convergence of the Mann iteration process the unique solution of eqn. (20) seem not to be applicable to the Hammerstein equation (14) with nonexpansive, monotone maps N and K. This leads naturally to the following problem :

Problem—Does any of the Mann or Ishikawa iteration process converge strongly to the solution of the Hammerstein equation (14) when K and N are nonexpansive and monotone?

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References

1. F. E. Browder, *Bull. Am. Math. Soc.* 73 (1967), 875-82.
2. F. E. Browder, *Proc. Symp. Pure Math.* 18 (Part 2) *Amer. Math. Soc.* Providence, 1976.
3. F. E. Browder and W. V. Petryshyn, *Bull. Am. Math Soc.* 72 (1968), 571-75.
4. F. E. Browder and W. V. Petryshyn, *J. Math. Anal. Appl.* 29 (1967), 197-228.
5. R. E. Bruck (Jr.), *Bull. Am. Math. Soc.* 79 (1973), 1258-62.
6. C. E. Chidume *J. Austral. Math. Soc.* (*Series A*) 41 (1986), 59-63.
7. C. E. Chidume, *J. Math. Anal. Appl.*, 116 (1986), 531-37.
8. V. Dolezal, *Monotone operators and applications in control and network theory, studies in Automation and Control* 3, 32 Elsevier Sci. Publ. Company, N. Y. 1979.
9. W. G. Dotson, *Math. Comp.* 32 151 (1978), 223-25.
10. J. C. Dunn, *J. Funct. Anal.* 27 (1978), 38-50.
11. M. Edelstein and R. C. O'Brian, *J. Lond. Math. Soc.* (2), 17 (1978), 547-54.
12. S. Ishikawa, *Proc. Am. Math. Soc.* 149 (1974), 147-50.
13. S. Ishikawa, *Proc. Am. Math. Soc.* 73 (1976), 65-71.
14. T. Kato, *J. Math. Soc. Japan* 19 (1967), 508-20.
15. W. A. Kirk and C. Morales, *Canad. Math. Bull.* 24 (1981), 441-45.
16. I. M. Lavrent'ev and R. Sh. Chepanovich, *Publ. Inst. Math* (*Beograd*) *N. S.* 35 (49) (1984), 125-29.
17. W. R. Mann, *Proc. Am. Math. Soc.* 4 (1953), 506-10.
18. R. H. Martin (Jr.), *Proc. Am. Math. Soc.* 26 (1970), 307-14.
19. S. Maruster, *Proc. Am. Math. Soc.* 63 (1977), 69-73.
20. R. N. Mukerjee, *Indian. J. pure Appl. Math.* 15 (1984), 1183-89.
21. Z. Opial, *Bull. Am. Math. Soc.* 73 (1967), 591-97.
22. W. V. Petryshyn, *J. Math Anal. Appl.* 14 (1966), 276-84.
23. B. E. Rhoades, *Trans. Am. Math. Soc.* 196 (1974), 161-76.
24. B. E. Rhoades, *J. Math. Anal. Appl.* 56 (1976), 741-50.
25. R. Sh-Chepanovich, *Publ. Inst. Math.* (*Beograd*), *N. S.* 35 (49), 1984). 119-23. *MR.* # 86h: 47093a.
26. E. H. Zarantonello, *Technical Report No.* 160, U. S. Army Math. Res. Centre, Mai Madison, Wiscosin, 1960.