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MATRIX TRANSFORMATIONS OF ORTHONORMAL SERIES

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Let D denote the set of Hausdorff matrices for which $\max_k h_{nk} = O(n^{-1/2})$, E, the set of lower triangular matrices with row sums one satisfying $\sum_{j=0}^{k-1} a_{nj} = O(k/n)$, uniformly in k. In this paper we establish number of theorems involving the summability of orthonormal series by matrices in either class D or E. These results significantly extend some of the corresponding theorems established by Meder[4] and Patel[5-7] for the Euler matrix of order 1.

Meder[4] and Patel[5-7] established several results involving the Euler summability of order one of orthonormal series. In this paper we generalize some of these theorems to Euler summability of order p. For the others we replace the Euler matrix with any matrix from a large class of matrices whose entries satisfy certain growth conditions.

A Hausdorff matrix $H \equiv (h_{nk})$ is a lower triangular matrix whose nonzero entries are of the form $h_{nk} = \binom{n}{k} \Delta^{n-k} \mu_k$, where $\binom{n}{k}$ is the ordinary binomial coefficient, $\{\mu_n\}$ is a real or complex sequence, an Δ is the forward difference operator defined by $\Delta^0 \mu_k = \mu_k$, $\Delta \mu_k = \mu_k - \mu_{k+1}$, $\Delta^{n+1} \mu_k = \Delta (\Delta^n \mu_k)$. Examples of Hausdorff matrices are the Cesaro matrices of order α, obtained by setting $\mu_k = 1/\binom{k+\alpha}{\alpha}$, and the Euler matrices, (E, p), obtained by setting $\mu_k = (1+p)^{-k}$.

Let $\mu(x)$ be a positive, bounded, monotone increasing function defined over an interval $[a, b]$, whose derivative is nonnegative and nonexistent at most on a set of measure zero. $\{\phi_n(x)\}$ will denote a system of functions orthonormal to the distribu- $d\mu(x)$ over $[a, b]$, with partial sums

$$S_n(x) = \sum_{i=0}^{n} a_i \phi_i(x) \qquad \text{...(1)}$$

where the a_i are real and satisfy

$$\sum_{n=0}^{n} a_n^2 < \infty. \qquad \text{...(2)}$$

Let $l = \{x = \{x_k\} : \Sigma_x \mid x_k \mid < \infty\}$. Then l is called the set of absolute convergent sequences, where the convergence of $\Sigma \mid x_k \mid$ means the absolute convergence of the series Σx_k rather than that of the sequence $\{x_k\}$. $B(l)$ will denote the set of bounded linear operators on l; i. e., a matrix $A \equiv (a_{nk}) \in B(l)$ if, for each $x \in l$, $Ax = \{\Sigma_k a_{nk} x_k\} \in l$. A Hausdroff matrix $H \in B(l)$ iff $\int_0^1 t^{-1} \mid d\beta(t) \mid < \infty$, where $\beta(t) \in BV[0, 1]$ and $\mu_n = \int_0^1 t^n d\beta(t)$ (see Rhoades[8]). Let D denote the set of Hausdorff matrices for which $\max_k (h_{nk}) = O(n^{-1/2})$.

For H a Hausdorff matrix define $\tau_n(x) = \Sigma_{k=0}^{n} h_{nk} s_k(x)$.

Theorem 1—The series

$$\sum_{n=1}^{\infty} \int_a^b n(\tau_n(x) - \tau_{n-1}(x))^2 \, d\mu(x) \qquad \ldots(3)$$

is convergent if $H \in B(l) \cap D$ and

$$\sum_{n=1}^{\infty} \sqrt{n}\, a_n^2 < \infty. \qquad \ldots(4)$$

PROOF : It has been shown[3] that, if a series Σa_k has partial sums τ_n, then

$$\tau_n(x) - \tau_{n-1}(x) = \frac{1}{n} \sum_{j=0}^{k} j h_{nj} a_j.$$

Using the orthonormaly of $\{\phi_n(x)\}$,

$$\int_a^b n[\tau_n(x) - \tau_{n-1}(x)]^2 \, d\mu(x)$$

$$= \int_a^b \frac{1}{n} [\sum_{j=0}^{n} h_{nj} \, j a_j \, \phi_j(x)] \, [\sum_{k=0}^{n} h_{nk} \, k a_k \, \phi_k(x)] \, d\mu(x)$$

$$= \frac{1}{n} \sum_{k=0}^{n} k^2 \, h_{nk}^2 \, a_k^2$$

$$\leqslant \frac{1}{n} n^{3/2} \max_k \mid h_{nk} \mid \sum_{k=0}^{n} \mid h_{nk} \mid \sqrt{k} a_k^2 .$$

$$= O(1) \sum_{k=0}^{n} \mid h_{nk} \mid \sqrt{k} a_k^2 . \qquad \ldots(5)$$

Since $H \in B(l)$, $\sum_{n=1}^{\infty} \sum_{k=1}^{n} | h_{nk} | \sqrt{k} a_k^2$ is finite, and Theorem 1 is proved.

There are many Hausdorff matrices that belong to D. We shall list here two examples.

For the gamma methods, $\mu_n = a/(n + a)$, a real, and

$$h_{nk} = \frac{\Gamma(n+1)\,\Gamma(k+a)}{\Gamma(k+1)\,\Gamma(n+a+1)}.$$

$$\frac{h_{nk}}{h_{n,k+1}} = \frac{k+1}{k+a} \begin{cases} \geqslant 1 \text{ for } a \leqslant 1 \\ < 1 \text{ for } a > 1 \end{cases}.$$

Therefore

$$\max_k h_{nk} = \begin{cases} h_{n0}, & 0 < a \leqslant 1 \\ \mu_n, & a > 1. \end{cases}$$

$$= \begin{cases} \dfrac{\Gamma(a)\,\Gamma(n+1)}{\Gamma(n+a+1)}, & 0 < a \leqslant 1 \\[2ex] \dfrac{a}{n+a}, & a > 1. \end{cases}$$

The mass functions for the gamma methods are $\theta(t) = t^a$. Therefore each operator method belongs to D for $a \geqslant 1/2$, and belongs to $B(l)$ for $a > 1$.

For the Euler methods, using either Theorem 138 of Hardy[2] or Lemma 1 of Ziza[11], $\max_k h_{nk} = O(n^{-1/2})$. That each Euler method is in $B(l)$ follows from Theorem 1 of Rhoades[8].

Meder[4] has shown that (3) implies (4) for $(E, 1)$. On the other hand, Ziza[11] has shown that all Euler matrices are equivalent a. e. for every orthonormal series with coefficients satisfying (2). Therefore a reasonable conjecture is that Meder's result can be extended to (E, p) for $p > 0$.

Corollary 1—Let (E, p) be an Euler matrix of order $p > 0$. Then (3) converges a. e. iff (4) converges.

That (4) implies (3) follows from Theorem 1, since every Euler matrix belongs to D.

To prove the converse, note that one can replace the convergence of (3) by that of (5). For N sufficiently large, and for $n/(p+1) - \sqrt{n} \geqslant 1$, $n/(p+1) + \sqrt{n} \leqslant n$, for all $n \geqslant N$, condition (5) implies that

$$\sum_{n=N}^{\infty} \frac{1}{n} \sum_{\frac{n}{p+1} - \sqrt{n} \leqslant k \leqslant \frac{n}{p+1} \sqrt{k}} k^2 h_{nk}^2 a_k^2 < \infty.$$

But, from Theorem 138 of Hardy[2], $h_{nk} > c/\sqrt{k}$ over the range of the inner sum, where c is some positive constant. Therefore we have

$$\sum_{n=N}^{\infty} \frac{1}{n} \sum_{\frac{n}{p+1} - \sqrt{n} \leq k \leq \frac{n}{p+1} + \sqrt{k}} k a_k^2 < \infty$$

which in turn implies that

$$\sum_{n=N}^{\infty} \sum_{\frac{n}{p+1} - \sqrt{n} \leq k \leq \frac{n}{p+1} + \sqrt{k}} a_k^2 < \infty.$$

The above sum can be rewritten in the form

$$\sum_{k} N(k)\, a_k^2$$

where $N(k)$ denotes the number of integers $n \geq N$ satisfying $n/(p+1) - \sqrt{n} \leq k \leq n/(p+1) + \sqrt{k}$. It can be shown that $N(k)$ is asymptotic to $\sqrt{k}$, and the result is proved.

Let E denote the set of regular lower triangular matrices, with row sums one, satisfying

$$\sum_{j=0}^{k-1} a_{nj} = O(k/n)$$

uniformly in k. We shall show that E is a rather large class.

First, observe that, since each A in E is regular, each a_{nk} is bounded. Therefore, if η is any real number satisfying $0 < \eta < 1$, then, automatically,

$$\sum_{j=\eta n}^{n} a_{nj} = O(k/n)$$

$k \geq \eta n$. Thus, it is enough to verify the equality for $k < \eta n$.

Let (N, p) be any regular Norlund method satisfying $np_n = O(|P_n|)$. The entries of a regular Norlund matrix are $c_{nk} = p_{n-k}/P_n$, $P_n = \sum_{k=0}^{n} p_k$.

$$\sum_{j=0}^{k-1} c_{nj} = 1 - P_{n-k}/P_n.$$

Claim : $P_{n-k}/P_n = 1 + O(k/n)$.

$$P_{n-1}/P_n = 1 - p_n/P_n = 1 + O(1/n).$$

Then $\log|P_{n-1}/P_n| = O(1/n)$ and $\log|P_{n-k}/P_n| = O(k/n)$. Therefore $|P_{n-k}/P_n| = \exp(O(k/n)) = 1 + O(k/n)$

For a regular weighted mean method $(\bar{N}, p)$, the entries are $c_{nk} = p_k/P_n$, and

$$\sum_{j=0}^{k-1} c_{nj} = \sum_{j=0}^{k-1} p_j/P_n = P_{k-1}/P_n = O(k/n).$$

uniformly for $1 \leqslant k \leqslant n$ for $P_n \sim n^\alpha$, $\alpha \geqslant 1$.

For any regular Hausdorff matrix H, $\{\mu_n\}$ has the representation

$$\mu_n = \int_0^1 t^n \, d\beta(t)$$

where $\beta \in BV[0, 1]$. Suppose that H also satisfies

$$\int_0^t |d\beta(t)| = O(t), \text{ as } t \to 0+. \qquad ..(6)$$

Since $\int_0^1 |d\beta(t)| < \infty$, it follows that, if (5) is true for small values of t, then it continues to remain true, possibly with a different constant, uniformly over $0 \leqslant t \leqslant 1$. It is sufficient to show that

$$\sum_{j=0}^{k-1} h_{nj} = \sum_{j=0}^{k-1} \int_0^1 \binom{n}{j} t^j (1-t)^{n-j} \, d\beta(t)$$

$$= \int_0^1 \left[\sum_{j=0}^{k-1} \binom{n}{j} t^j (1-t)^{n-j}\right] d\beta(t) = O(k/n) \qquad ...(7)$$

uniformly for $1 \leqslant k \leqslant n/4$.

Since the expression in brackets in (7) is bounded above by one; it follows from (6) that

$$\int_0^{2k/n} \left[\sum_{j=0}^{k-1} \binom{n}{j} t^j (1-t)^{n-j}\right] d\beta(t) = O(k/n).$$

It remains to estimate the contribution of the integral in (7) over the interval $[2k/n, 1]$.

Since $[2j/n, 1]$ includes $[2k/n, 1]$ for $0 \leqslant j \leqslant k-1$, it is sufficient to show that, uniformly for $0 \leqslant j \leqslant n/4$,

$$\binom{n}{j}\int_{2j/n}^{1} t^j (1-t)^{n-j}\,|\,d\beta(t)\,| = O(1/n). \qquad \ldots(8)$$

With $\chi(t) = \int_0^t |\,d\beta(u)\,|$, $\chi(1) = O(1)$, and the left hand side of (8) can be written in the form

$$-\binom{n}{j}\int_{2j/n}^{1} [\chi(t) - \chi(2j/n)]\ \frac{d}{dt}\ (t^j (1-t)^{n-j})\,dt.$$

Since $d\,(t^j(1-t)^{n-j})/dt < 0$ in the interval of integration, the above expression is dominated by a constant times

$$-\binom{n}{j}\int_{2j/n}^{1} t\ \frac{d}{dt}\ (t^j (1-t)^{n-j})\,dt$$

$$= \binom{n}{j}\{[t^{j+1}(1-t)^{n-j}]_{t=2j/n} + \int_{2j/n}^{1} t^j (1-t)^{n-j}\,dt\}$$

$$= I_1, + I_2, \text{ say.}$$

$$I_2 \leqslant \binom{n}{j}\int_0^1 t^j(1-t)^{n-j}\,dt = 1/(n+1).$$

For $j = 0$, $I_1 = 1$. For $j \geqslant 1$,

$$I_1 = \binom{n}{j}(2j/n)^{j+1}(1-2j/n)^{n-j}$$

$$= \binom{n}{j}\frac{(2j)^{j+1}(n-2j)^{n-j}}{n^{j+1}\,n^{n-j}}$$

$$\sim \frac{n^{n+1/2}}{j^{j+1/2}(n-j)^{n-j+1/2}}\cdot\frac{2^{j+1}\,j^{j+1}(n-2j)^{n-j}}{n^{n+1}}$$

$$= \frac{2^{j+1}\sqrt{j}}{\sqrt{n}\sqrt{n-j}}\left[\frac{n-2j}{n-j}\right]^{n-j} = O\left[\left[\frac{j^{1/2}}{u}\right]\left[\frac{2}{e}\right]^j\right].$$

The Hausdorff matrices with $\beta(t) = t^\alpha$, $0 < \alpha < 1$ do not satisfy (6) and do not belong to E. Therefore condition (6) cannot be weakened.

Let $\sigma_n(x)$ denote the nth term of the $(C, 1)$ transform of (1); i. e.,

$$\sigma_n(x) = (n+1)^{-1}\sum_{k=0}^{n} s_k(x)\ .$$

The next two theorems require the following lemmas.

Lemma 1—If the orthonormal series (1) satis fies (2), then, for any $A \in E$ with

$$t_n(x) = \sum_{k=0}^{n} a_{nk} s_k(x)$$

$$\sum_{n=1}^{\infty} [\sigma_n(x) - t_n(x)]^2/n \qquad \ldots(9)$$

converges a.e.

PROOF : $\sigma_n(x) - t_n(x) = \sum_{k=0}^{n} \left[\frac{1}{n+1} - a_{nk} \right] s_k(x)$

$$= \sum_{k=0}^{n} \left[\frac{1}{n+1} - a_{nk} \right] \sum_{j=0}^{k} a_j \phi_j(x)$$

$$= \sum_{j=0}^{n} a_j \phi_j(x) \sum_{k=j}^{n} \left[\frac{1}{n+1} - a_{nk} \right]$$

$$= \sum_{j=0}^{n} a_j \phi_j(x) \left[[(n-j+1)/(n+1) - \sum_{k=j}^{n} a_{nk} \right]$$

$$= \sum_{j=0}^{n} a_j \phi_j(x) \left[1 - \frac{j}{n+1} - \sum_{k=j}^{n} a_{nk} \right]$$

$$= \sum_{j=0}^{n} a_j \phi_j(x) \left[\sum_{k=0}^{j-1} a_{nk} - j/(n+1) \right]. \qquad \ldots(10)$$

Since $A \in E$,

$$\sum_{n=1}^{\infty} \int_a^b ([\sigma_n(x) - t_n(x)]^2/n)\, d\mu(x) = O(1) \sum_{n=1}^{\infty} n^{-1} \sum_{k=0}^{n} a_k^2 (k/n)^2$$

$$= O(1) \sum_{k=1}^{\infty} k^2 a_k^2 \sum_{n=k}^{\infty} n^{-3} = O(1) \sum_{k=1}^{\infty} a_k^2 .$$

Using the theorem of Levi (see, e.g. Alexits[1], p. 11) (9) converges a. e.

Lemma 2—If the orthonormal series (1) satisfies (2) and is summable (A) a.e. to $s(x)$, for $A \in E$, then

$$\lim \frac{1}{n} \sum_{k=1}^{n} [S_k(x) - S(x)]^2 = 0.$$

PROOF : $s_n(x) - \sigma_n(x) = \dfrac{1}{n+1} \displaystyle\sum_{k=1}^{n} k\, a_k\, \phi_k(x).$

Therefore

$$\sum_{n=1}^{\infty} \int_a^b \frac{1}{n} [s_n(x) - \sigma_n(x)]^2\, d\mu(x) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} \sum_{k=1}^{n} k^2 a_k^2$$

$$< \sum_{k=1}^{\infty} a_k^2 k^2 \sum_{n=k}^{\infty} n^{-3} = O(1) \sum_{k=1}^{\infty} a_k^2 .$$

From the theorem of Levi,

$$\sum_{n=1}^{\infty} [s_n(x) - \sigma_n(x)]^2/n$$

converges a.e. From Kronecker's Theorem

$$\frac{1}{n} \sum_{k=1}^{n} [s_k(x) - \sigma_k(x)]^2 \to 0 \text{ as } n \to \infty.$$

Using Minkowski's inequality,

$$\left\{\frac{1}{n} \sum_{k=1}^{n} [s_k(x) - s(x)]^2\right\}^{1/2} \leqslant \left\{\frac{1}{n} \sum_{k=1}^{n} [s_k(x) - \sigma_k(x)]^2\right\}^{1/2}$$

$$+ \left\{\frac{1}{n} \sum_{k=1}^{n} [\sigma_k(x) - t_k(x)]^2\right\}^{1/2}$$

$$+ \left\{\frac{1}{n} \sum_{k=1}^{n} [t_k(x) - s(x)]^2\right\}^{1/2}.$$

Since $t_k(x) \to s(x)$, the third series on the right converges to zero. Applying Kronecker's Theorem to (9) yields

$$\frac{1}{n}\sum_{k=1}^{n}[\sigma_k(x) - t_k(x)]^2 \to 0 \text{ as } n \to \infty.$$

Theorem 2—If the orthonormal series (1) satisfies (2) and is summable (A) a.e. to $s(x)$, $A \in E$, then it is summable $(C, 1)$ a. e. to the same sum.

PROOF :

$$[\sigma_n(x) - s(x)]^2 \leqslant \left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x) - s(x)|\right\}^2$$

$$\leqslant \frac{1}{n+1}\sum_{k=0}^{n}[s_k(x) - s(x)]^2$$

and the result follows by Lemma 2.

Theorem 3—If the orthonormal series (1) satisfies

$$\sum_{n=2}^{\infty} a_n^2 (\log\log n)^2 < \infty$$

and $A \in E$, then $\lim_n t_{2n}(x)$ exists a.e.

PROOF : From the proof of Lemma 1,

$$\sum_{n=0}^{\infty}\int_a^b [\sigma_{2^n}(x) - t_{2^n}(x)]^2\, d\mu(x)$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{2n} a_k^2 \left[\sum_{j=1}^{k-1} a_{2^n,j} - \frac{k}{2^n+1}\right]^2$$

$$= O(1)\sum_{n=1}^{\infty} 4^{-n}\sum_{k=1}^{2^n} a_k^2 k^2 = O(1)\sum_{k=1}^{\infty} a_k^2$$

by the argument of Alexits[1] (p. 145).

The remainder of the proof is the same as that in Meder[4].

Theorem 4—Let A be a nonnegative Hausdorff matrix, $A \in D \cap E$. If the orthonormal series (1) is (A) summable a.e. to $s(x)$ and if (4) is satisfied, then it is strongly A-summable, with index 2, a.e. to $s(x)$.

PROOF: $\sum_{k=0}^{n} a_{nk} [s_k(x) - s(x)]^2 \leqslant 2 \sum_{k=0}^{n} a_{nk} [s_k(x) - t_k(x)]^2$

$$+ 2 \sum_{k=0}^{n} a_{nk} [t_k(x) - s(x)]^2 = S_1 + S_2, \text{ say.}$$

By hypothesis $S_2 \to 0$.

$$S_1 = 2 \sum_{k=0}^{n} a_{nk} [s_k(x) - t_k(x)]^2 = \frac{O(1)}{\sqrt{n}} \sum_{k=0}^{n} [s_k(x) - t_k(x)]^2$$

$$s_k(x) - t_k(x) = S_k(x) - \sum_{j=0}^{k} a_{kj} s_j(x)$$

$$= \sum_{j=0}^{k} a_j \phi_j(x) - \sum_{j=0}^{k} a_{jk} \sum_{i=0}^{j} a_i \phi_i(x)$$

$$= \sum_{j=0}^{k} a_j \phi_j(x) - \sum_{i=0}^{k} a_i \phi_i(x) \sum_{j=1}^{k} a_{kj}$$

$$= \sum_{i=0}^{k} a_i \phi_i(x) [1 - \sum_{j=i}^{k} a_{kj}]$$

$$= \sum_{i=0}^{k} a_j \phi_j(x) \sum_{j=0}^{i-1} a_{kj}.$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_a^b [s_n(x) - t_n(x)]^2 \, d\mu(x) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{i=0}^{n} a_i^2 \left[\sum_{j=0}^{i-1} a_{nj} \right]^2$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_i^2 \, (i^2/n^2)$$

$$= O(1) \sum_{i=1}^{\infty} i^2 a_i^2 \sum_{n=i}^{\infty} n^{-5/2}$$

$$= O(1) \sum_{i=1}^{\infty} \sqrt{i} \; a_i^2 \,.$$

By Levi's Theorem

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [s_n(x) - t_n(x)]^2$$

converges a.e. By Kronecker's Theorem,

$$\sum_{k=1}^{\infty} [s_k(x) - t_k(x)]^2 = O(\sqrt{n})$$

and $S_1 \to 0$.

Theorem 5—Let the orthonormal series (1) satisfy

$$\sum_{m=0}^{\infty} A_m < \infty \qquad \ldots(11)$$

where

$$A_m = \Big[\sum_{i=2^m+1}^{2^{m+1}} a_i^2\Big]^{1/2}.$$

Then

$$\sum_{n=1}^{\infty} | s_n(x) - t_n(x) | /n < \infty$$

for each A in E

PROOF :

$$s_n(x) - t_n(x) = \sum_{k=0}^{n} a_k \phi_k(x) \sum_{j=0}^{k-1} a_{nj}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b | s_n(x) - t_n(x) | \, d\mu(x) = O(1) \sum_{n=1}^{\infty} n^{-2} \Big[\sum_{k=1}^{n} k^2 a_k^2\Big]^{1/2}.$$

Without loss of generality we may assume $a_1 = 0$. It is well known that

$$\sum_{k=2}^{n} k^2 a_k^2 < \sum_{r=0}^{[\log n]} 2^{2(r+1)} A_r^2 .$$

Thus

$$\sum_{n=2}^{\infty} \frac{1}{n} \int_a^b | s_n(x) - t_n(x) | \, d\mu(x) < O(1) \sum_{n=2}^{n} n^{-2} \sum_{r=0}^{[\log n]} 2^{r+1} A_r$$

(*equation continued on p.* 161)

$$= O(1) \sum_{r=2}^{\infty} 2^{r+1} A_r \sum_{\log[n] \geqslant r} n^{-2}$$

$$= O(1) \sum_{r=2}^{\infty} A_r.$$

Theorem 6—If $\{A_m\}$ satisfies (11), then, for each H in D.

$$\sum_{k=0}^{n} |\tau_k(x) - \tau_{k-1}(x)| = o(\sqrt{n}).$$

PROOF : We may assume $a_0 = a_1 = 0$. Using (5),

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_a^b |\tau_n(x) - \tau_{n-1}(x)| \, d\mu(x)$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[\int_a^b |\tau_n(x) - \tau_{n-1}(x)|^2 \, dx \right]^{1/2}$$

$$= \sum_{n=1}^{\infty} n^{-3/2} \left\{ \sum_{j=2}^{n} j^2 h_{nj}^2 \, a_j^2 \right\}^{1/2}$$

$$= O(1) \sum_{n=1}^{\infty} n^{-2} \left\{ \sum_{j=2}^{n} j^2 a_j^2 \right\}^{1/2}$$

$$= O(1) \sum_{j=2}^{\infty} 2^{j+1} A_j \sum_{\log[n+1] \geqslant j} n^{-2}$$

$$= O(1) \sum_{j=2}^{\infty} A_j < \infty.$$

The proof is completed by using the theorems of Levi and Kronecker.

We now prove two theorems dealing with lacunary series.

Theorem 7—If $\{a_n\}$ satisfies (2) and $\{n_k\}$ is an increasing sequence of indices satisfying

$$1 < q \leqslant n_{k+1}/n_k, \; k = 0, 1, 2, \ldots \qquad \ldots(12)$$

then, for each $A \in E$,

$$\sum_{k=0}^{\infty} [s_{n_k}(x) - t_{n_k}(x)]^2 \qquad \ldots(13)$$

converges a.e. in $[a, b]$.

PROOF : $s_n(x) - t_n(x) = \sum_{k=0}^{n} a_k \phi_k(x) \sum_{j=0}^{n} a_{nj}$.

$$\left\{\int_a^b | s_n(x) - \tau_n(x) | \, d\mu(x)\right\}^2 = O(1) \sum_{k=1}^{n} a_k^2 \, (k/n)^2$$

since $A \in E$.

Therefore

$$\sum_{k=1}^{\infty} \int_a^b | s_{n_k} - t_{n_k}(x) |^2 d\mu(x) = O(1) \sum_{k=1}^{\infty} \frac{1}{n_k^2} \sum_{j=1}^{n_k} j^2 a_j^2$$

$$= O(1) \sum_{j=1}^{\infty} j^2 a_j^2 \sum_{n_k \geqslant j} \frac{1}{n_k^2}$$

$$= O(1) \sum_{j=1}^{\infty} j^2 a_j^2 \frac{1}{j^2} \sum_{m=0}^{\infty} q^{-2m}$$

$$= O(1) \sum_{j=1}^{\infty} a_j^2 < \infty.$$

Now apply Levi's Theorem.

Theorem 8—If $\{a_n\}$ satisfies (4) and $\{n_k\}$ is an increasing sequence of indices satisfying

$$1 < q \leqslant n_{k+1}/n_k \leqslant r, \; k = 0, 1, 2, \ldots \qquad \ldots(14)$$

then (1) is (E, p) summable a.e. in $[a, b]$ iff $\{s_{n_k}(x)\}$ converges a.e. in $[a, b]$.

PROOF : Suppose (1) is summable (E, p). Then (14) implies (12) and (13) is satisfied. Since $\{\tau_n(x)\}$ converges, $\{s_{n_k}(x)\}$ converges a.e.

Suppose $\{s_{n_k}(x)\}$ converges a.e. for $\{n_k\}$ satisfying (12). Let s satisfy $n_k \leqslant s < n_{k+1}$.

$$(\tau_s(x) - \tau_{n_k}(x))^2 = \left[\sum_{n=n_k+1}^{s} (\tau_n(x) - \tau_{n-1}(x))\right]^2$$

(equation continued on p. 163)

$$< \sum_{n=n_k+1}^{n_{k+1}} n\,(\tau_n(x) - \tau_{n-1}(x))^2 \sum_{n=n_k+1}^{n_{k+1}} 1/n. \qquad \ldots(15)$$

But

$$\sum_{n=n_k+1}^{n_{k+1}} 1/n < \frac{1}{n_k}(n_{k+1} - n_k) = n_{k+1}/n_k - 1 < r - 1.$$

From Theorem 1, (5) implies (2). By Levi's Theorem

$$\sum_{n=1}^{\infty} n\,[\tau_n(x) - \tau_{n-1}(x)]^2$$

converges a.e. Therefore the right-hand side of (15) tends to zero a.e. From Theorem 7 $\{\tau_{n_k}(x)\}$ converges a.e. Therefore $\{\tau_n(x)\}$ converges from (15).

The author is indebeted to Brian Kuttner for the idea of extending the results of Meder[4] and Patel[5-7] to the classes D and E, and for the proof that those Hausdorff matrices satisfying (6) belong to E.

Remarks

1. Theorems 1—3 are generalizations of Theorems 1,2, and 4, respectively, of Meder[4].

2. Lemmas 1 and 2 are generalizations of Lemmas 1 and 2 of Meder[4].

3. Theorem 4 generalizes the Theorem in Patel[7].

4. Theorem 5 generalizes the corresponding result in Patel[6].

5. Theorem 6 generalizes corresponding Theorem of Patel[5]:

6. Theorems 7 and 8 are generalizations of Theorems 1 and 2 of Ziza[10]

7. In Sharma[9] the result of Patel[6] is extended to double Euler summability $(E, 1, 1)$. However, the results in that paper are incorrect since the author assumes that

$$\sum_{j=0}^{m} \sum_{k=0}^{n} - \sum_{j=i}^{m} \sum_{k=l}^{m} = \sum_{j=0}^{i-1} \sum_{k=0}^{l-1}.$$

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