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ON CONTROLLABILITY OF NONLINEAR SYSTEMS WITH DISTRIBUTED DELAYS IN THE CONTROL

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It is known that most natural applications give rise to mechanisms of indirect actions, where the decisions in the control function u are shifted, twisted or combined before affecting the evolution. An example of this delayed action is the model defined by

$$\dot{x}(t) = g(t, x(t), u(t)) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s).$$

We give sufficient conditions for the null controllability of such systems when the controls are limited in the sense that they lie in a closed unit ball with zero in its interior. These extend known results.

1. INTRODUCTION

Consider the nonlinear control system with distributed delays given by

$$\dot{x}(t) = A(t, x(t), u(t))\, x(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) + f(t, x(t), u(t)) \qquad \ldots(1.1)$$

or

$$\dot{x}(t) = g(t, x(t), u(t)) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) \qquad \ldots(1.2)$$

where $x \in E^n$, $u \in IB \equiv L_\infty([a, b], E^m)$, the space of functions measurable and essentially bounded on finite intervals. It is assumed that the system (1.1) (= (1.2)) has a unique solution $x = x(t) = x(t, t_0, x_0, u)$, for each admissible control $u \in IB$, and each initial state $x(t_0) = x_0$.

The controllability of system (1.1) has been studied by Klamka[4,5], where he showed that under certain conditions on the matrix functions $A(t, x, u)$, $f(t, x, u)$, $B(t, s)$, system (1.1) is globally relatively controllable if

$$\inf_{z\, u \in C_{nm}[t_0, t_1]} \det W(t_0, t_1; z, v) \geq c.$$

Here

$$W(t_0, t_1; z, v)$$

is the controllability matrix (to be defined later). In this method, fixing the variable arguments of the matrix A and the function f by $(z, v) \in C_{nm}[t_0, t_1]$, he arrived at the linear dynamical system

$$\dot{x}(t) = A(t, z(t), v(t))\, x(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s)$$

$$+ f(t, z(t), v(t)).$$

He then applied Schauder's fixed point theorem to this linear system to obtain conditions for global controllability of (1.1).

In our own case we shall treat the equivalent system of (1.1) given by (1.2), where $g(t, x(t), u(t))$ is nonlinear. We shall employ the Alek seev-type variation of constant formula given by Khanh[2] to obtain an integral equation for system (I.2). We shall then use this integral equation to derive necessary and sufficient conditions for the null controllability with constraints for the linear variational system of (1.2). With this result, we shall then provide sufficient criteria for the null controllability of (1.2) with constraints. Indeed, we shall show that if

(i) $g(t, 0, 0) = 0$;

(ii) the system

$$\dot{x}(t) = A(t, x(t)) + B(t)\, u(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) \qquad \text{...(1.3)}$$

is controllable,

where $A(t) = D_2\, g(t, 0, 0)$; $B(t) = D_3\, g(t, 0, 0)$;

(iii) the system

$$\dot{x}(t) = g(t, x(t), 0) \qquad \text{...(1.4)}$$

is uniformly asymptotically stable, so that the solution of (1.4) satisfies $| x(t) | \leq k \, | x_0 | \, ^{-\alpha(t-t_0)}$, $\alpha > 0$, $t > 0$, then (1.2) is null controllable with constraints provided g satisfies all smoothness conditions for the existence and uniqueness of solutions.

We shall further apply our results specifically to the systems

$$\dot{x}(t) = g(x(t), u(t)) + \sum_{i=1}^{p} C_i\, u(t - h_i) \qquad \text{...(1.5)}$$

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{p} C_i\, u(t - h_i) \qquad \text{...(1.6)}$$

where

$$A = D_1 g(0, 0);\ B = D_2 g(0, 0). \qquad \ldots(1.7)$$

Our results will show that if

(i) $g(0, 0) = 0$

(ii) rank $[B, AB, \ldots\ldots, A^{k-1} B, \ldots, \ldots\ldots, A^{k-1} C_k] = n$

and

(iii) the system

$$\dot{x}(t) = g(x(t), 0) \qquad \ldots(1.8)$$

is uniformly asymptotically stable then (1.5) is null controllable with constraints.

Sebakhy and Bayounmi[7] considered the system

$$\dot{x} = A(t)\, x(t) + B(t)\, u(t) + C(t)\, u(t - h).$$

This system is totally linear. Their results show that this system is null controllable if and only if

$$\text{rank}\ \Gamma(t_0\, t_1) = n$$

where $\Gamma(t_0, t_1)$ is the controllability Grammian. This result cannot be applied to systems of the form

$$\dot{x}(t) = g(t, x(t)) + B(t)\, u(t) + g(t)\, X(t)$$

where g and f are nonlinear. Thus, our results complements and extends those of Sebakhy and Bayounmi[7].

Khanh[2] considered the nonlinear system

$$\dot{x}(t) = g(t, x, u(t)) + B(t, x(t))\, u(t) + f(t, x(t), u(t)).$$

nonlinear though, but without control delays. Thus, our results extend those of Khanh[2] to include systems with delayed controls. However, our results are of a special nature in the sense that it considers controllability to the origin of E^n, a results that is basic in the study of optimal control problems. Klamka[3,4] considered the following systems

$$\dot{x} = A(t)\, x(t) + \int_{-h}^{0} [dH(t, s)]\, u(t + s)$$

a linear system, and

$$\dot{x}(t) = A(t, x(t))\, x(t) + \int_{-h}^{0} (d_s H(t, s)\, x(s))\, u(t + s)$$

nonlinear. He gave sufficient conditions for controllability of the above systems when the controls are unlimited. Our result specialises the results of the above to controllability to the origin of E^n with limited controls i.e. with constrained controls. Yamatoto[8] and Dauer[1] gave results for the complete controllability of quasi-linear systems of the form

$$\dot{x} = A(x, u, t) X + B(x, u, t) u + f(x, u, t).$$

when the controls are unlimited. Our result strengthens and extends the above results to include systems with control delays.

In section 2, we give the basic notations and define the system equations. Sections 3 gives the controllability and stability theorems for both the linear variational system and the nonlinear system (1.2).

Chapter 4 deals with the applications of the results to system (1.5).

Throughout the paper, we exploit the properties of the 'Reachable' sets in deriving our results.

2. Notations and Preliminaries

Let E denote the real line and $J = [t_0, t_1]$, an interval in E. For a positive integer n, we denote by E^n the space of real n-tuples with the Euclidean norm denoted by $|\cdot|$. If J is any interval of E, the usual Lebesgue space of functions, measurable and essentially bounded from J to E^n will be denoted by $L_\infty(J, E^n)$. M_{nm} will be used for the collection of all real $n \times m$ matrices with a suitable norm. Let $h > 0$ be given. For functions $u : [t_0 - h, t_0] \to E^n$, $t \in [t_0, t_1]$, we use u_t to denote the function on $[-h, 0]$ defined by $u_t(s) = u(t + s)$, for $s \in [-h, 0]$.

We shall consider the nonlinear system

$$\dot{x}(t) = A(t, x(t), u(t))\, x(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) + f(t, x(t), u(t)) \quad \text{...(2.1)}$$

or

$$\dot{x}(t) = g(t, x(t), u(t)) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) \quad \text{...(2.2)}$$

satisfied almost everywhere on $[t_0, t_1]$, where the integral is in Lebesgue-Steiltjes sense with respect to s, $x(t) \in E^n$, $u \in L_\infty([t_0, t_1], E^m)$, $B(t, s)$ is an $n \times m$ matrix-valued function absolutely continuous in s for each fixed t, and of bounded variation in s on $[-h, 0]$ for each $t \in [t_0, t_1]$. We shall assume that $g(t, x(t), u(t))$ (nonlinear in general) is continuous with respect to its arguments and continuously differentiable with respect to x. We shall also assume toat the functions g, g_x, B, are continuous regarding the arguments.

Throughout the sequel, the controls of interest are $IB = L_\infty([t_0, t_1], E^m)$ $IU \subseteq L_\infty([t_0, t_1], E^m)$, a closed and bounded subset of $1B$ with zero in its interior relative to $1B$.

Definition 2.1— The pair $z_t = \{x(t), u_t\}$ is said to be the complete state of the system (2.1) or (2.2).

Definition 2.2— System (2.1) is relatively controllable (or controllable) on $[t_0, t_1] = J$, if for every complete state z_t and every vector x_0, x_1 in E^n there exists a control $u(t)$ defined on $[t_0, t_1]$, such that the corresponding trajectory of system (2.1) satisfies the condition $x(t_1) = x_1$, with $x(t_0) = x_0$. Following Sebakhy and Bayounmi[7], we give the following definition of null controllability.

Definition 2.3— System (2.1) is said to be null controllable at $t = t_1$, if for any initial state $\{x_0, u_{t_0}\}$ on $[t_0 - h, t_0]$, there exists and admissible control $u(t) \in IB$, defined on $[t_0, t_1 - h]$ such that the response of the system is brought so the origin of E^n at $t = t_1$ using the control effort

$$u(t) = \{u(t) \text{ on}, [t_0, t_1 - h]\}$$
$$= 0 \text{ on}, [t_1 - h, t_1].$$

That is if given the initial state $\{x_0, u_{t_0}\}$ on $[t_0 - h, t_0]$ there exists a control $u(t) \in IB$ defined on $[t_0, t_1 - h]$ such that $x(t_1) = 0$.

System (2.1) is null controllable with constraints at $t = t_1$ if for any initial state x_0, u_{t_0} on $[t_0 - h, t_0]$, there exists an admissible control $u(t) \in IU$, defined on $[t_0, t_1 - h]$ such that the response $x(t)$ satisfies $x(t_1) = 0$, using the control

$$u(t) = \begin{Bmatrix} u(t) \in IU \text{ on } [t_0, t_1 - h], \\ 0 \in IU, \text{ on } [t_1 - h, t_1]. \end{Bmatrix}$$

Let the system

$$\dot{x}(t) = g(t, x(t), u(t)) \qquad \ldots(2.3)$$

have a unique solution $x = G = G(t, t_0, u, y)$, for each admissible control $u \in IB$ and each initial state $G(t_0, t_0, y, u) = y \in E^n$. If we now set

$$F(t, t_0, y, u) = G_y(t, t_0, y, u) \qquad \ldots(2.4)$$

and

$$K(t, x, u) = g_y(t, x, u) \qquad \ldots(2.5)$$

then it is not difficult to see that the following relations hold for the $n \times n$ matrix functions

$$F_t(t, s, y, u) = K(t, G(t, s, y, u), u(t)) F(t, s, y, u); \; F(t_0, t_0, u, y) = I_{nn}$$

$$F(t, t_0, u, y(t_0)) = I_{nn} + \int_{t_0}^{t} K(s, G(s, t_0, u, y(t_0), u(s))$$
$$F(s, t_0, y(t_0), u(s))\, ds. \qquad \ldots(2.6)$$

Thus, Alekseev-type variation of parameter formula for (2.2) (= (2.1)) is given via the result of Khanh[2] by

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{t_0}^{t} F(t, s, x(s), u(s))$$
$$\left[\int_{-h}^{0} (d_s B(t, s)\, u(t + s))\, ds\right]. \qquad \ldots(2.7)$$

When we consider the solution $x(t)$ of (2.2) given in (2.7), we note that the values of the control $u(t)$ for $t \in [t_0 - h, t_0]$ enter into the definition of the complete state $z(t_0)$. Thus, the last term of (2.7) must be transformed to take care of this by interchanging the order of the integration. Now, using the unsymmetric Fubini theorem (Klamka[4]), we have the following :

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{-h}^{0} dB_s \left(\int_{t_0+s}^{t_0} F(t, l - s, x(l - s),\right.$$
$$u(l - s))\, B(l - s, s)\, u_{t_0}\, dl\Big) + \int_{-h}^{0} d_{B_s}$$
$$\left(\int_{t_0}^{t+s} F(t, l - s, x(l - s), u(l - s)\, B(l - s, s)\, u(l)\, dl\right. \qquad \ldots(2.8)$$

where the symbol d_{B_s} denotes that the integration is in the Lebesque-Steiltges sense with respect to the variable s in the function $B(t, s)$. If we denote by

$$B_t(l, s) = \begin{Bmatrix} B(l, s) \text{ for } l \leqq t, s \in E \\ 0 \quad\quad \text{for } l > t, s \in E \end{Bmatrix} \qquad \ldots(2.9)$$

then (2.8) yields

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{-h}^{0} d_{B_s} \left(\int_{t_0+s}^{t_0} F(t, l-s, x(l-s), u(l-s)).\right.$$
$$B(l - s, s)\, u_{t_0}\, dl\Big) + \int_{-h}^{0} d_{B_s} \left(\int_{t_0}^{t} F(t, l - s, x, u)\, B_t(l - s; s)\, u(l)\, d(l)\right. \qquad \ldots(2.10)$$

Using the unsymmetric Fubini Theorem, (2.10) can be written in a more convinient form as follows :

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{-h}^{0} d_{B_s} \Big(\int_{t_0+s}^{t_0} F(t, l-s, x, u)$$

$$(B(l-s, s)\, u_{t_0}\, dl) + \int_{t_0}^{t} \int_{-h}^{0} F(t, l-s, x, u)$$

$$d_s B_l (l-s, s))\, u(l)\, dl. \qquad \ldots(2.11)$$

We now define the $n \times n$ controllability matrix of (2.2) (= (2.1)) by

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Big(\int_{-h}^{0} F(t_1, l-s, x, u)\, d_s B_{t_1} (l-s, s)\Big)$$

$$\Big(\int_{-h}^{0} F(t_1, l-s, x, u)\, d_s B_{t_1} (l-s, s)\Big)^T dl. \qquad \ldots(2.12)$$

Where the symbol T denotes the matrix transpose. Note that $W(t_0, t_1)$ is symmetric and non-negative-definite.

Definition 2.4— The reachable set of (2.2) at time t_1 using L_∞ control is the subset of E^n given by

$$IP(t_1, t_0) = \Big[\int_{t_0}^{t_1} F(t, s, x(s), u(s)) \int_{-h}^{0} (d_s B(t, s)\, u(t+s))\, ds : u \in IB\Big].$$

In a similar manner, we define the constraint reachable set of (2.2) as

$$IR(t_1, t_0) = \Big[\int_{t_0}^{t_1} F(t, s, x(s), u(s)) \Big[\int_{-h}^{0} (d_s B(t, s)\, u(t+s)\, ds : u \in IU\Big].$$

3. Controllability Theorems

Consider the system

$$\dot{x}(t) = A(t, x(t), u(t))\, x(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t+s) \qquad \ldots(3.1)$$

or

$$\dot{x}(t) = g(t, x(t), u(t)) + \int_{-h}^{0} (d_s B(t, s))\, u(t+s). \qquad \ldots(3.2)$$

Lemma 3.1— For each fixed $(t_0, s_0, x_0, u_0) \in E^+ \times E^+ \times E^n \times L_\infty$, let $\bar{D}x(t_0, s_0, x_0, u_0)$ denote the partial derivatives of x with respect to its last two arguments at the point (t_0, s_0, x_0, u_0). Then, for every $(h, u) \in E^n \, x \, L_\infty$, we have

$$\bar{D}(t_0, s_0, x_0, u_0)(h, u) = \lambda(t_0, s_0, x_0, u_0, h, u),$$

where the mapping $t \to \lambda(t, s_0, x_0, u_0, h, u)$ of E into E^n is the unique absolutely continuous solution of the linear differential equation

$$\dot{z}(t) = D_2 g(t, x(t, s_0, x_0, u_0), u_0(t)) z(t) + D_3 g(t, x(t, s_0, x_0, u_0), u_0) v(t) + \int_{-h}^{0} (d_s B(t, s)) v(t + s). \qquad \ldots(3.3)$$

Also, for every $W \in L_\infty$, we have

$$D_u x(t_0, s_0, x_0, u_0) w = \psi(t_0, s_0, x_0, u_0, w)$$

where the mapping $t \to \psi(t, s_0, x_0, u_0, w)$ is the unique absolutely continuous solution of (3.3) satisfying the initial condition $(s_0, 0)$ i.e. $z(s_0) = 0$.

PROOF : Let D_u be the partial derivative of $x(t, s_0, x_0, u)$ with respect to u. Then,

$$D_u x(s_0, s_0, x_0, u) = 0 \text{ in } [-h, 0]$$

$$D_u x(t, s_0, x_0, u) = \int_{s_0}^{t} D_2 g(s, x(s, s_0, x_0, u), u(s)). D_u x(s, s_0, x_0, u) ds + \int_{s_0}^{t} D_3 g(s, x(s, s_0, x_0, u), u(s)) ds + \int_{s_0}^{t} \int_{-h}^{0} (d_s B(t, s)) ds. \qquad \ldots(3.4)$$

Differentiating with respect to t, we obtain

$$\frac{d}{dt} D_u x(t, s_0, x_0, u) v = D_2 g(s, x(s, s_0, x_0, u), u). D_u x(s, s_0, x_0, u) v + D_3 g(s, x(s, s_0, x_0, u), v + \int_{-h}^{0} (d_s B(t, s)) v. \qquad \ldots(3.5)$$

Substituting the limits of integration in (3.4), we obtain

$$D_u x(s_0, s_0, x_0, u) v = D_u x_0 v = 0$$

proving the second assertion. To prove the first part, we note that

$$\bar{D}x(t, s_0, x_0, u)(h, v) = D_3 x(t, s_0, x_0, u) + D_4 x(t, s_0, x_0, u) v.$$

But,

$$D_3 x(t, s_0, x_0, u) = I + \int_{s_0}^{t} D_2 g(s, x(s, s_0, x_0, u), u(s)) \times D_3 x(s, s_0, x_0, u), u(s)) ds$$

$$D_4\, x\,(t, s_0, x_0, u) = \int_{s_0}^{t} D_3\, g\,(s, x\,(s, s_0, x_0, u), u\,(s))\, ds$$

$$+ \int_{s_0}^{t} [\int_{-h}^{0} (d_s\, B\,(t, s))]\, ds.$$

Therefore,

$$\bar{D}x\,(t, s_0, x_0, u)\ \ (h, v) = h + \int_{s_0}^{t} D_2\, g\,(s, x\,(s), u\,(s)).\ D_3\, x\,(s, s_0, x_0, u)\, hds$$

$$+ \int_{s_0}^{t} D_3\, g\,(s, x\,(s, s_0, x_0, u), u\,(s))\, vds$$

$$+ \int_{s_0}^{t} \int_{-h}^{0} (d_s\, B\,(t, s))\, vds. \qquad \ldots(3.6)$$

Taking the t derivative, we obtain :

$$\frac{d}{dt}\, \bar{D}x\,(t, s_0, x_0, u)\,(h, v) = 0 + D_2\, g\,(s, x\,(s, s_0, x_0, u), u\,(s))$$

$$\times\ D_3\, x\,(S))\, h + D_3 g\,(s, x\,(s, s_0, x_0, u), u\,(s))\, v$$

$$+ \int_{-h}^{0} (d_s\, B\,(t, s))\, v.$$

Moreover, from (3.6), we have

$$\bar{D}x\,(s_0, s_0, x_0, u)\,(h, v) = h + 0 + 0 + 0.$$

Hence, the Proposition.

Definition 3.1— Let X, Y be Banach Manifolds of class C^1 and let $h : X \rightarrow Y$ be a C^1 mapping. A point y_0 in $h\,(X)$ is called a normal value of h if there exists at least one $x_0 \in h^{-1}\,(y_0)$ such that the differential dh_{x_0} is a split-surjective linear mapping.

The next Lemma gives a necessary and sufficient condition for the derivative $D_u\, x\,(t, s_0, x_0, u_0)$ to be a surjective linear mapping, in which case the point $x\,(t, s_0, x_0, u_0)$ is a normal value.

Lemma 3.2— Given $x_0 \in E^n$, $u \in L_\infty$. For $t_1 > s_0 + h$, let $t \in [s_0 \cdot\ t_1]$. Let $u \rightarrow x\,(t_1, s_0, x_0, u)$ be the mapping $T : L_\infty \rightarrow E^n$, defined by

$$T\,(u) = x\,(t_1, s_0, x_0, u)$$

where $x\,(t_1, s_0, x_0, u)$ is a solution of (3.2). Then $DT\,(u) = \frac{d}{du}\, T\,(u) : L_\infty \rightarrow E^n$ has a continuous local right inverse (is a surjective linear mapping) if and only if the linear variational control system (3.3) of (3.2) along the response $t \to x(t_1, s_0, x_0, u)$ is controllable on $[s_0, t_1]$, $t_1 > s_0 + h$.

PROOF : For $t \in [s_0, t_1]$, $t_1 > s_0 + h$, $u \in L_\infty$, let $t \to z(t, t_0, x_0, u)$ denote the solution of (3.3). Clearly, the linear variational control system of (3.2) along the solution $t \to x(t, s_0, x_0, u)$ is controllable on $[s_0, t_1]$ if and only if the mapping $u \to z(t, s_0, x_0, u)$ is surjective, $t \in [s_0, t_1]$. But by Lemma 3.1,

$$z(t_1, s_0, x_0, u, v) = D_u x(t_1, s_0, x_0, u) = DT(u)\, v.$$

Therefore, the mapping $v \to z(t_1, s_0, x_0, u, v)$ is surjective if and only if $DT(u)$ is surjective. The following proposition will be made use of in the next theorem.

Proposition 3.1 (Surjective mapping theorem) (Graves)— Let U be open in a Banach space X. Let $f: U \to Y$ be a C^1 map into a Banach space Y. Let $x_0 \in X$. If $f'(x_0)$ is surjective, then f is locally open in a neighbourhood of x_0. More precisely, there exists an open neighbourhood V of x_0 contained in U having the following Lang[6] property :

For each $x \in V$, and open ball B_x centered at x, contained in V, the image $f(B_x)$ contains an open neighbourhood of $f(x)$.

Remarks 3.1: This is another form of the implicit function theorem given by (p. 193).

Theorem 3.1— The system (3.2) is controllable on $[s_0, t]$, $t > s_0 + h$, whenever the system

$$\dot{z}(t) = D_2 g(t, 0, 0)\, z(t) + D_3 g(t, 0, 0)\, v(t) + \int_{-h}^{0} (d_s B(t, s))\, v(t+s)\, ds \qquad \ldots(3.7)$$

is controllable on $[s_0, t]$, $t > s_0 + h$.

PROOF : For system (3.2), let $u \in L_\infty$, and $x(s_0, x_0, u)$ be its response. Let $T: L_\infty \to E^n$ be the map defined by $Tu = x(t, s_0, 0, u)$.

It follows from the conditions on g and (2.11) that $T(IU) = IR(t, s_0)$. Suppose system (3.7) is controllable, then by Lemmas 3.1 and 3.2,

$$DT(0) = \frac{d}{du}[T(u)]_{u=0} = D_4 x(t, s_0, 0, u) \mid u = 0$$

is a surjective linear mapping of $L_\infty \to E^n$. Therefore, by Lang[6] (p. 193), T is locally open. Hence, there is an open ball $B_p \subseteq L_\infty$ containing zero and an open ball $B_r \subseteq E^n$ containing zero such that $B_r \subseteq T(B_p)$.

Since IU contains an open ball containing zero, $r > 0$, $p > 0$ can be chosen such that $B_r \subseteq T(B_p \cap IU)$.

Therefore,

$$B_r \subseteq T(IU) = IR(t, s_0)$$

so that $0 \in \text{Int } IR(t, s_0)$.

Remarks 3.1 : The above theorem assures us that if the linear variational control system of (3.2) corresponding to the zero solution is controllable, then $0 \in$ Int $IR(t, t_0)$.

Definition 3.2— The domain **D** of null controllability of (3.2) is the set of all initial points $x_0 \in E^n$, for which the solution of (3.2) with $x(t_0) = x_0$, satisfies $x(t_1) = 0 \in E^n$, at some t_1 using $U \in IU$.

Theorem 3.2— In system (3.2), assume

(i) $g(t, 0, 0) = 0, t > s_0$

(ii) system (3.7) is controllable on $[s_0, t]$, $t > s_0 + h$

and

(iii) the smoothness assumptions hold for g

then

the domain of null controllability of (3.2) contains zero in its interior.

PROOF : The variation of parameter formula for system (3.2) is given from (2.7) as

$$x(t, t_0, x_0, u) = G(t, t_0, x_0, u) + \int_{t_0}^{t} F(t, s, x(s), u(s))$$

$$\int_{-h}^{0} (d_s B(t, s))\, u(t+s)\, ds. \qquad \ldots(3.8)$$

We note that if there exists a $u \in IU$ such that the solution of (3.2) satisfies

$$x(t_0, t_0, x_0, u) = x_0, x(t_1, t_0, x_0, u) = 0,$$

then from (3.8) we have

$$0 = G(t, t_0, x_0, u) + \int_{t_0}^{t} F(t, s, u, x(s)) \int_{-h}^{0} (d_s B(t, s))\, u(t+s)\, ds.$$

By (iii), (i) $0 \in \mathbf{D}$, the domain of null controllability of (3.2). Assume that $0 \notin$ Int **D**. Then there exists a sequence $\{x_i\}^{\infty}{}_{i=1} \subseteq E^n$ such that $x_i \to 0$ as $i \to \infty$ and no x_i is in **D**, hence, $x_i \neq 0$ for any i. Since $x_i \notin \mathbf{D}$ $x(t, t_0, x_i, u) \neq 0$, for any $t > t_0 + h$ and any $u \in IU$. Hence, from (3.8),

$$-G(t, t_0, x_i, u) \equiv \zeta_i \neq \int_{t_0}^{t} F(t, s, u, x(s)) \left[\int_{-h}^{0} (d_s B(t, s,))\, u(t+s)\right] ds$$

for any i, any $t > t_0 + h$ and $u \in IU$. Therefore, ζ_i is not contained in $IR(t, t_0)$ for any $t > t_0 + h$. Hence, we have a countable sequence $\{\zeta\}_i \subseteq E^n$, such that

$$\zeta_i \to 0 \text{ as } i \to \infty$$

$$\zeta_i \notin IR(t, t_0), \text{ for any } t > t_0 + h$$

$$\zeta_i \neq 0 \text{ for any } i$$

hence $0 \notin \text{Int } IR(t, t_0)$, for any $t > t_0 + h$.

But by (ii), the linear variational system of (3.2) about the zero solution is controllable, hence, by Theorem 3.1, $0 \in \text{Int } IR(t, t_0)$ $t > t_0 + h$ which is a contradiction. This contradiction shows that $0 \in \text{Int } \mathbf{D}$.

Remarks 3.3 : The above theorem says that, if the linear variational system is controllable, then (3.2) is locally null controllable with constraints.

Theorem 3.3— For system (3.2), assume

(i) g satisfies all smoothness conditions for the existence and uniqueness of solutions;

(ii) $g(t, 0, 0) = 0$;

(iii) system (3.7) is controllable on $[t_0, t_1]$, $t_1 > t_0 + h$;

(iv) the system

$$\dot{x}(t) = g(t, x(t), 0) \qquad \ldots(3.9)$$

is uniformly asymptotically stable so that every solution of (3.9) satisfies

$$\| x(t, t_0, x_0, 0) \| \leq k \mid e^{-\alpha(t-t_0)}, \; t > t_0, \; k > 0, \; \alpha > 0 \qquad \ldots(3.9a)$$

then

(3.2) is null controllable with constraints.

PROOF : By (i), (ii), (iii) the domain $\mathbf{D}$ of null controllability of (3.2) contains zero in its interior. Therefore, there exists a ball $B_1 \in \text{Int } \mathbf{D}$.

By (iv) every solution of (3.9) which is a solution of (3.2) (with $u = 0$) satisfies $x(t, t_0, x_0, 0) \to 0$ as $t \to \infty$, hence, at some $t_1 < \infty$, we have $x(t_1, t_0, x_0, 0) \equiv x_2 \in B_1 \subseteq \text{Int } \mathbf{D}$. Therefore, with t_1, x_2 as initial data there exists $u \notin IU$ and some $t_2 > t_1$, such that the solution $x(t)$ satisfies $x(t_2, t_1, x_2, u) = 0$; proving the theorem.

We now give necessary and sufficient conditions for the controllability of the linear variational system (3.7).

Now, consider the linear system

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) + \int_{-h}^{0} (d_s H(t, s))\, u(t + s). \qquad \ldots(3.10)$$

Following Klamka[4], the unique solution of (3.10) is given by

$$x(t) = F(t, t_0)\, x(t_0) + \int_{t_0}^{t} F(t, \tau) \left[\int_{-h}^{0} (d_s H(\tau, s))\, u(\tau + s) + B(\tau) \right] d\tau \qquad \ldots(3.11)$$

where $F(t, s)$ is the fundamental matrix for the homogeneous system

$$\dot{x}(t) = A(t)\, x(t). \qquad \ldots(3.12)$$

Following Klamka[4] and using the unsymmetric Fubini theorem, we can transform eqn. (3.11) to the following convenient form :

$$x(t) = F(t, t_0)\, x(t_0) + \int_{-h}^{0} d_H \left[\int_{t_0+s}^{t_0} F\, t\;\; l - s)\, H(l - s, s)\; u_{t_0}\, dl \right]$$

$$+ \int_{t_0}^{t} \left[\int_{-h}^{0} F(t, l - s)\, d\bar{H}(l - s, s) + F(t, l)\, B(l)\, u(l)\, dl \right]. \qquad \ldots(3.13)$$

We now define the $n \times n$ controllability matrix of (3.10) as

$$W_1(t_0, t_1) = \int_{t_0}^{t_1} \left[\int_{-h}^{0} F(t_1, t - s)\, d\bar{H}(t - s, s) \right] \left[\int_{-h}^{0} F(t_1, t - s)\, dH(t - s, s) \right]^T dt$$

$$+ \int_{t_0}^{t_1} [F(t_1, t)\, B(t)]\, [F(t_1, t)\, B(t)]^T\, dt. \qquad \ldots(3.14)$$

The proof of the next theorem follows exactly the same proof given in Theorem 1 of Klamka[3] and therefore will be omitted.

Proposition 3.1— System (3.10) is controllable on $[t_0, t_1]$ if and only if

$$\text{rank } W_1(t_0, t_1) = n;\ t_1 > t_0. \qquad \ldots(3.15)$$

Remark 3.4 : Here, we extend and complement this result of Klamka[3] to system (1.1) in the following manner.

The next theorem is a consequence of Theorem 3.3 and Proposition 3.1. Let

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) + \int_{-h}^{0} (d_s B(t, s))\, u(t + s) \qquad \ldots(3.16)$$

where

$$A(t) = D_2\, g(t, 0, 0),\ B(t) = D_3\, g(t, 0, 0) \qquad \ldots(3.17)$$

and let

$$W_1(t_0, t_1) = \int_{t_0}^{t_1} \left[\int_{-h}^{0} F(t_1, t - s)\, d\bar{B}(t - s, s) \right] \left[\int_{-h}^{0} F(t, t - s)\, d\bar{B}(t - s, s) \right]^T$$

(equation continued on p. 226)

$$+ \int_{t_0}^{t_1} [F(t_1, t) B(t)] [F(t_1, t)]^T \, dt \qquad \ldots(3.18)$$

where $F(t, s)$ is the fundamental matrix for the homogeneous system

$$\dot{x}(t) = A(t) x(t). \qquad \ldots(3.18)$$

Theorem 3.4— For system (3.2), assume

(i) g satisfies all smoothness conditions for the existence and uniqueness of solutions;

(ii) $g(t, 0, 0) = 0$;

(iii) rank $W_1(t_0, t_1) = n$; $t_1 > t_0$

where $W_1(t_0, t_1)$ is given by (3.14);

and

(iv) the system

$$\dot{x}(t) = g(t, x(t), 0) \qquad (3.19)$$

is uniformly asymptotically stable then

(3.2) is null controllable with constraints.

PROOF : Immediate from Theorem 3.3 and Proposition 3.1.

4. APPLICATIONS

If we now specialize to the constant system with multiple delays in the control defined by

$$\dot{x}(t) = g(x(t), u(t)) + \sum_{i=1}^{k} C_i u(t - h_i) \qquad (4.1)$$

then the following results follow :

It is known[1] that the system

$$\dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{k} C_i u(t - h_i) \qquad \ldots(4.2)$$

is controllable on $[0, t_1]$, $t_1 > \max\{h_1, \ldots\ldots, h_m\}$ if and only if

$$\text{rank}\,[B, \ldots, A^{n-1} B, C_1, \ldots, A^{n-1} C_1 \ldots, C_k, \ldots A^{n-1} C_k] = \Lambda. \qquad \ldots(4.3)$$

Remarks 4.1 : The above results of Sebakhy and Bayoummi[7] cannot be applied directly to the nonlinear system (4.1), without some modifications as will be shown by

Proposition 4.2. The linear variational system of (4.1) corresponding to the zero solution is given by

$$\dot{z}(t) = D_1 g(0, 0) z(t) + D_2 g(0, 0) u(t) + \sum_{i=1}^{k} C_i u(t - h_i). \qquad \ldots(4.4)$$

If we set

$$A = D_1 g(0, 0); \; B = D_2 g(0, 0) \qquad \ldots(4.5)$$

and suppose that A, B are autonomous, then the next Proposition follows from the reasoning above.

Proposition 4.1— The autonomous linear system (4.4) corresponding to the zero solution of (4.1) is controllable if and only if

$$\text{rank}\, [B, \ldots, A^{n-1} B, \ldots, C_k \ldots A^{n-1} C_k] = n$$

where A and B are given by (4.5).

The next Proposition follows from Theorem 3.4 and Proposition 4.1.

Proposition 4.2— For system (4.1), assume

(i) (4.2) is autonomous;

(ii) g satisfies all smoothness conditions for the existence and uniqueness of solutions;

(iii) $g(0, 0) = 0$;

(iv) $\text{rank}\, [B, \ldots, A^{n-1} B \ldots\ldots, C_k \ldots\ldots A^{n-1} C_k] = n$,

where A and B are defined by (4.5); and

(v) the system

$$\dot{x}(t) = g(x(t), 0) \qquad \ldots(4.6)$$

is uniformly assymptotically stable then

(4.1) is null controllable with constraints.

PROOF : Immediate.

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