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# INCLUSION THEOREMS ON MATRIX TRANSFORMATIONS OF SOME SEQUENCE SPACES OVER NON-ARCHIMEDIAN FIELDS IV

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Necessary and sufficient conditions for an infinite matrix defined over a field $K$ with non-trivial non-archimedian valuation to transform $\Gamma^* (K)$ into $C_0 (K)$ and $\Gamma (K)$ to $\Gamma^* (K)$ are investigated. The sequence spaces $C_0 (K)$, $\Gamma (K)$ and $\Gamma^* (K)$ are all defined over such a field $K$.

## 1. INTRODUCTION

The object of the present paper is to obtain some inclusion theorems of certain sequence spaces over non-archimedian fields which were not considered earlier[3-5]. In section 2 we shall describe the required preliminaries, whereas in section 4, we shall prove the main theorem of this paper.

## 2. ENTIRE FUNCTIONS OVER $K$

Let $K$ be a non-archimedian non-trivially valued field which is complete under the metric of real valuation. Let $N_k$ be defined as

$$N_k = \{ | x | : x \in K \}.$$

Let $\alpha (x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ be a power series with coefficients in $K$. As noted by Raghunathan[2] (p. 517), $\alpha (x)$ is an entire function over $K$ if and only if

$$| a_n |^{1/n} \to 0 \text{ as } n \to \infty.$$

If $\alpha (x)$ is an entire function, let

$$M (R) = \underset{| x | = R}{\text{Max}} | \alpha (x) | , (R \in N_k)$$

Then by a known result[1] (p. 85), we have

$$M (R) = \sup_n [ | a_n | R^n, n \geqslant 0] \qquad \ldots(2.1)$$

where

$$\alpha = \sum_{n=0}^{\infty} a_n x^n.$$

## 3. Definition of Different Sequence Spaces

In what follows, the notion of convergence and boundedness will be in relation to the metric of valuation of the field.

$C_0(K)$ : The set of all null sequences $x = (x_n)$

$\Gamma(K)$ : The set of all sequences $x = (x_n)$ with

$|x_n|^{1/n} \to 0$ as $n \to \infty$.

$\Gamma^*(K)$ : The set of all sequences $x = (x_n)$ with

$\{|x_n|^{1/n}\}$ bounded.

If $x = (x_n)$ is a sequence over $K$, let us define $\|x\| = \sup_{n \geqslant 1} |x_n|$. The norm is evidently non-archimedian in the sense that it satisfies stronger form of triangle inequality. With this as the norm $C(K)$ is a non-archimedian Banach space.

Let $x = (x_n) \in K$. Let $|x| = \text{Sup}\{|x_n|^{1/n}, n \geqslant 1\}$. Then $|x|$ satisfies the following conditions.

(i) $|x| > 0$, $|x| = 0$ if and only if $x = (0, 0, \ldots\ldots)$ where 0 is the zero element of the field.

(ii) $|x + y| < \text{Max}\{|x|, |y|\}$

(iii) $|tx| < A(t)|x|$, $t \in K$, $A(t) = \text{Max}\{1, |t|\}$.

Hence $|x - y|$ defines a metric in the set of all sequences $\Gamma(K)$. So $\Gamma(K)$ is a metric space defined over $K$, the topology being defined by the metric given above. Raghunathan[2] (p. 518) has proved that $\Gamma(K)$ is a complete linear metric space which is totally disconnected.

According to Raghunathan[2] every continuous linear functional $f(x)$ defined for $x \in \Gamma(K)$ is of the form $f(x) = \Sigma C_n x_n$, $x = (x_n)$ where $\{|C_n|^{1/n}\}$ is bounded. Hence $\Gamma^*(K)$ is identified as the dual space of $\Gamma(K)$. It was known from Raghunathan[2] (p. 524), $\Gamma^*(K)$ is a complete metric space which is not linear.

## 4. Matrix Transformation of $\Gamma^*(K)$ into $C_0(K)$

Let us consider the matrix transformation

$$y_n = \sum_{p=1}^{\infty} a_{np} x_p, \; n = 1, 2, 3, \ldots, \text{ and } a_{np} \in K. \qquad \ldots(4.1)$$

*Theorem* 1— A necessary and sufficient condition that $(y_n) \in C_0(K)$ whenever $(x_n) \in \Gamma^*(K)$ is that

$\{f_n(x)\}$ is bounded uniformly on every finite circle

$$| x | \leqslant R, \; R \in N_k \qquad \ldots(4.2)$$

where

$$f_n(x) = \sum_{p=1}^{\infty} a_{np} x_p, \; n = 1, 2, 3, \ldots \text{ and } a_{np} \in K$$

is a sequence of entire functions over $K$.

PROOF : The condition is sufficient.

Let

$$f_n(x) = \sum_{p=1}^{\infty} a_{np} x_p, \; n = 1, 2, 3, \ldots \text{ and } a_{np} \in K$$

be uniformly bounded on every finite circle $| x | \leqslant R$, $R \in N_k$. Then the condition (4.2) implies that there exists a constant $M(R)$ such that $| f_n(x) | \leqslant M(R)$ for every $x$ with the property that $| x | \leqslant R$, $R \in N_k$. By using (2.1) and the condition (4.2), we have

$$| a_{np} | R^p \leqslant M(R) \text{ for each fixed } p. \qquad \ldots(4.3)$$

Since $(x_p) \in \Gamma^*(K)$, there is a constant $c$ such that

$$| x_p |^{1/2} \leqslant c \text{ so that } | x_p | \leqslant c^p \text{ for all } p. \qquad \ldots(4.4)$$

From (4.1),

$$| y_n | \leqslant \sup_{1 \leqslant p \leqslant \infty} | a_{np} | \; | x_p | .$$

By using (4.3) and (4.4) in the above, we have

$$| y_n | \leqslant \frac{M(R)}{R^p} c^p.$$

Choose $R = \mu c$ where $\mu \in N_k$ and $\mu > 1$.

Then $| y_n | \leqslant \dfrac{M(\mu c)}{\mu^p c^p} c^p = \dfrac{M(\mu c)}{\mu^p} \leqslant \epsilon$ for large $p$ so that $y_n \to 0$ as $n \to \infty$.

The condition is necessary.

Since $f_n(x) = \Sigma a_{np} x^p$ is defined for every $n$ and $x = R$, the series is convergent for $| x | = R$.

$$| a_{np} z^p | = | a_{np} | R^p \to 0 \text{ as } p \to \infty. \qquad \ldots(4.5)$$

When we consider the special sequence (0, 0, ..., 0, 1, 0, ...) where 1 is in the $p$th place,

$$y_n = a_{np}, \; n = 1, 2, 3, \ldots .$$

Since $(y_n) \in C_0(K)$, we have for each fixed $p$

$$a_{np} \to 0 \text{ as } n \to \infty. \quad \text{..(4.6)}$$

If the condition (4.2) is not satisfied, there is some finite circle $|x| \leqslant R$, $R \in N_k$ such that $\{f_n(x)\}$ is not bounded uniformly on $|x| = R$. Since $|x| = R$ is bounded in the metric of valuation, we can find a sequence $(x_n)$ such that for $|x_n| = R$, $f_n(x_n) \to \infty$ as $n \to \infty$, through a subsequence of values of $n$.

Using this assumption, we shall construct a sequence $(x_p) \in \Gamma^*(R)$ with the additional condition $|x_p| = R^p$ for every $p$ such that the corresponding $(y_n)$ is not bounded. This proves the necessity of the condition.

Choosing $n_1$ such that

$$|f_{n_1}(x_{n_1})| > \epsilon \text{ for some } \epsilon > 0. \quad \text{..(4.7)}$$

Since $|x_{n_1}| = R$, (2.1) and (4.7) together imply,

$$\sup_{1 \leqslant p < \infty} |a_{n_1 p}| R^p > \epsilon. \quad \text{..(4.8)}$$

By using (4.5), we have

$$\sup_{p_{n_1}+1 \leqslant p < \infty} |a_{n_1 p}| R^p < \frac{\epsilon}{2} \text{ for } p \geqslant p_{n_1}. \quad \text{..(4.9)}$$

From (4.8) and (4.9), we get

$$\sup_{1 \leqslant p \leqslant p_{n_1}} |a_{n_1 p}| R^p > \epsilon.$$

Hence there is a $p_1$ in $1 \leqslant p \leqslant p_{n_1}$ such that

$$|a_{n_1 p_1}| R^p > \epsilon. \quad \text{...(4.10)}$$

Now choose $x_p$ in $1 \leqslant p \leqslant p_{n_1}$ as follows.

$$x_p = \begin{cases} x^p \text{ for } p = p_1 \text{ and } |x| = R \\ 0 \text{ for all } p \text{ in } 1 \leqslant p \leqslant p_{n_1}. \end{cases} \quad \text{...(4.11)}$$

Then

$$y_{n_1} = \sum_{p=1}^{p_{n_1}} a_{np}\, x_p + \sum_{p_{n_1}+1}^{\infty} a_{n_1}\, x_p. \quad \text{...(4.12)}$$

$$\left| \sum_{p_{n_1}+1}^{\infty} a_{n_1 p}\, x_p \right| \leqslant \sup_{p_{n_1}+1 \leqslant p < \infty} |a_{n_1 p}| \, |x_p| < \frac{\epsilon}{2} \text{ by (4.9).} \quad \text{...(4.13)}$$

Hence we shall have from (4.12)

$$| \sum_{p=1}^{p=p_{n_1}} a_{n_1 p} x_p | = | a_{n_1 p} x_{p_1} | \leqslant \text{Max} \{ | y_{n_1} | , | \sum_{p_{n_1}+1}^{\infty} a_{n_1 p} x_p | \}.$$

By (4.10), (4.11) and (4.13), we have

$$\epsilon < \text{Max} \{ | y_{n_1} | , \frac{\epsilon}{2} \}$$

Therefore from the above, we have

$$| y_{n_1} | > \epsilon.$$

Since $a_{n_p} \to 0$ as $n \to \infty$ for each fixed $p$. So $| a_{np} | R^p \to 0$ as $n \to \infty$ for each fixed $p$

$$| a_{np} | R^p < \epsilon \text{ for all } n > n_p.$$

Now choose $n_2 > n_1$ such that

$$\left. \begin{aligned} &\sup_{1 \leqslant p < \infty} | a_{n_2 p} | R^p > \epsilon \\ \text{and} \quad & \\ &\sup_{1 \leqslant p \leqslant p_{n_1}} | a_{n_2 p} | R^p < \frac{\epsilon}{2}. \end{aligned} \right\} \quad \text{..(4.14)}$$

This is possible if $n_2$ is large enough such that

$$n_2 > \text{Max } (n_p) \text{ where } 1 \leqslant p \leqslant p_{n_1}.$$

Then there exists by (4.5) $p_{n_2}$ greater than $p_{n_1}$ such that

$$\sup_{p_{n_1}+1 \leqslant p < \infty} | a_{n_2 p} | R^p < \frac{\epsilon}{2}. \quad \text{..(4.15)}$$

Therefore from (4.14) and (4.15), we have

$$\sup_{1 \leqslant p \leqslant p_{n_2}} | a_{n_2 p} | R^p > \epsilon. \quad \text{...(4.16)}$$

Hence there exists a $p_2$ in $1 \leqslant p \leqslant p_{m_2}$ such that

$$| a_{n_2 p_2} | R^{p_2} > \epsilon. \quad \text{...(4.17)}$$

So $p_2$ chosen in (4.17) exceeds $p_{n_1}$ by (4.14). Now choose $x_p$ as follows.

$$x_p = \begin{cases} x^p \text{ when } p = p_2 \text{ with } | x | = R \\ 0 \text{ for all } p \text{ in } p_{n_1} + 1 \leqslant p \leqslant p_{n_2} \end{cases} \quad \text{...(4.18)}$$

$$y_{n_2} = \sum_{p=1}^{p=p_{n_1}} a_{n_2 p} x_p + \sum_{p_{n_1}+1}^{p_{n_2}} a_{n_2 p} x_p + \sum_{p_{n_2}+1}^{\infty} a_{n_2 p} x^p$$

$$\sum_{p_{n_1}+1}^{p_{n_2}} a_{n_2 p} x_p = y_{n_2} - \sum_{p=1}^{p_{n_1}} a_{n_2 p} x_p - \sum_{p_{n_2}+1}^{\infty} a_{n_2 p} x_p .$$

By using (4.18) in the above, we get

$$| \sum_{p_{n_1}+1}^{p_{n_2}} a_{n_2 p} x_p | = | a_{n_2 p_2} x_{p_2} |$$

$$\leqslant \text{Max} \left\{ | y_{n_2} | , | \sum_{p=1}^{p_{n_1}} a_{n_2 p} x_p | , | \sum_{p_{n_2}+1}^{\infty} a_{n_2 p} x_p | \right\} \qquad \ldots(4.19)$$

By (4.15), we have

$$| \sum_{p_{n_2}+1}^{\infty} a_{n_2 p} x_p | \leqslant \sup_{p_{n_2}+1 \leqslant p < \infty} | a_{n_2 p} | R^p < \frac{\epsilon}{2}. \qquad \ldots(4.20)$$

By (4.14), we have

$$| \sum_{p=1}^{p_{n_1}} a_{n_2 p} x_p | \leqslant \sup_{1 \leqslant p \leqslant p_{n_1}} | a_{n_2 p} | R^p < \frac{\epsilon}{2}.$$

Using (4.17), (4.18), (4.20), (4.21) in (4.19), we get

$$\epsilon < \text{Max} \left\{ | y_{n_2} | , \frac{\epsilon}{2}, \frac{\epsilon}{2} \right\}$$

This implies $| y_{n_2} | > \epsilon$ .

Proceeding in the same manner, we can find $n_k$ such that $| y_{n_k} | > \epsilon$. That is $| y_{n_k} |$ does not tend to zero as $n \to \infty$ through a subsequence of values of $n$. This shows that $(y_n)$ does not belong to $C_0 (K)$, while the sequence $(x_p) \in \Gamma^* (K)$. This contradiction proves the necessity of the condition.

By using a method similar to the proof of the above Theorem 1, we can prove following theorem.

*Theorem* 2— A necessary and sufficient condition that $\{y_n\}$ should belong to $\Gamma (K)$ whenever $(x_p)$ belong to $\Gamma^* (K)$ is that $| f_n (x) |^{1/n} \to 0$ as $n \to \infty$ uniformly on every finite circle $|x| < R$, $R \in N_k$ where

$$f_n(x) = \sum_{p=1}^{\infty} a_{np} x^p, \; x \; n = 1, 2, 3, \ldots \text{ and } a_{np} \in K$$

is a sequence of entire functions over $K$.

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