Indian J. pure appl. Math., 20 (3) : 284-291, March 1989

ROCHE HARMONICS FOR STELLAR MODELS DISTORTED BY DIFFERENTIAL ROTATION

V. P. SINGH AND VANDANA GUPTA
Institute of Paper Technology, University of Roorkee, Saharanpur 247001

(*Received* 22 *March* 1988)

Kopal[1,2] obtained expliciting expressions for Roche harmonics associated with the stellar models of a star distorted by rotation (solid body) forces Following Kopal's approach we obtain explicit expressions for Roche harmonics associated with the stellar model of a star distorted by the differential rotation (the law of differential rotation has been assumed of the form $\omega = b_1 + b_2 s^2$, where ω is the angular velocity of rotation of a fluid element distant s from the axis of rotation and b_1, b_2 are certain constants).

1. INTRODUCTION

Kopal[2] introduced a family of new auxiliary functions, which he called the Roche harmonics, generated by the solution of the Laplace equation

$$\nabla^2 \phi = 0 \qquad \ldots(1)$$

derived in terms of the Roche curvilinear coordinates, which are associated with the Roche equipotential surfaces in the same way as the spherical harmonics are associated with a sphere. Roach[3] discussed some more theoretical aspects of Roche harmonics. In fact, as was pointed out by Kopal (*op. cit.*), spherical harmonics may be regarded as a limiting case of the more general Roche harmonics, and the latter may be more appropriate for the study of many problems arising in double star astronomy. Kopal obtained explicit expressions for Roche harmonics for rotationally or tidally distorted stellar models. In the case of Roche harmonics associated with equipotential surfaces which are being distorted by the rotational forces, Kopal obtained explicit expressions by taking into account the effects of solid body rotation only. Singh[4], Mohan, Singh[5] extended the analysis of Roche coordinates taking into account the effect of differential rotation. The law of differential rotation has been assumed to be of the form $\omega = b_1 + b_2 s^2$, where ω is the angular velocity of an element distant s from the axis of rotation, b_1 and b_2 are constants.

In this paper we present the analysis of Roche harmonics for stellar models distorted by differential rotation, using the above cited law of differential rotation as $\omega = b_1 + b_2 s^2$, to obtain the explicit expressions of Roche harmonics associated with Roche equipotential surfaces distorted by differential rotation. This is performed in Section 4.

For completeness the work of Kopal[1,2] is respected in Section 2 considering solid body rotation effects only. The system of Roche coordinates for the Roche-model distorted by differential rotation as obtaine dby Mohan and Singh[5] are presented in section 3. Concluding remarks are given in section 5.

2. Explicit Expressions of Roche Harmonics for Roche Equipotential Surfaces Distorted solid by Body Rotation Forces

Following Kopal[1] it can be shown that,

$$\xi = \frac{1}{r} + nr^2 (1 - \nu^2) = \text{constant} \qquad ...(2)$$

represents the equipotential surface of a star distorted by rotation (solid body) forces. Here $n = \omega^2/2$, ω being the angular velocity of rotation (solid body rotation), r is the dimensionless measurement of distance and $\nu = Z/r$ is the direction cosine with respect to Z-axis, the axis of rotation. The surfaces generated by setting ξ = constant in eqn. (2) are referred to as Roche equipotentials.

Choosing ξ as defined in eqn. (2) as one coordinate Kopal has shown that the second Roche coordinate η is given by

$$\eta = \phi. \qquad (3)$$

The expression for the third Roche coordinate ζ is not possible in closed analytic form. Kopal obtained expression for this third Roche coordinate ζ in ascending powers of the angular velocity of rotation as

$$\cos \zeta = \sum_{j=0}^{\infty} (2n)^j . r^{3j} X_j (\nu) \qquad ...(4)$$

where $X_0 (\nu) = 1$, $X_1 (\nu) = - \frac{1}{3} (1 - \nu^2)$, while for $j > 1$ all subsequent $X_j (\nu)$'s can be generated with the help of the recursion formula

$$3j X_j + (1 - \nu^2) [(\nu X_{j-1})' - 3 (j - 1) X_{j-1}] = 0.$$

Here prime denotes differentiation with respect to ν.

Kopal also obtained expressions for the metric coefficients h_1, h_2 and h_3 in terms of polar spherical coordinates upto second order terms in square of angular velocity of solid body rotation.

These are :

$$h_1 (\xi, \zeta) = r_0^2 [1 + 4n r_0^3 \sin^2\varepsilon - \frac{1}{3} n^2 r_0^6 \sin^2\varepsilon (22 - 85 \sin^2\varepsilon) + ...]$$

$$h_2 (\xi, \zeta) = r_0 \sin\zeta [1 - \frac{1}{3} nr_0^3 (2 - 5 \sin^2\varepsilon) + \frac{1}{9} n^2 r_0^6 (2 - 50 \sin^2\varepsilon + 7 \sin^4\varepsilon) + ...;$$

$$h_3(\xi,\zeta) = r_0[1-\tfrac{1}{3}nr_0^3(2-7\sin^2\zeta)+\tfrac{1}{9}n^2r_0^6(2-88\sin^2\zeta + 145\sin^4\zeta)+\ldots]. \qquad \ldots(5)$$

Here $r_0 = 1/\xi$ denotes the mean fractional radius of the equipotential surface ξ = constant.

Making use of h_1, h_2 and h_3 as given by (5), Kopal[2] has shown that the Laplace equation (1) becomes

$$\nabla^2\phi = [1 - nr_0^3(1-\nu_0^2)+\frac{2}{3}n^2r_0^6(1-\nu_0^2)(9+13\nu_0^2)] \frac{\partial}{\partial r_0}\left(r_0^2\frac{\partial\phi}{\partial r_0}\right) - 4n r_0^4\frac{\partial\phi}{\partial r_0} + \left[1-\frac{2}{3}nr_0^3(5-7\nu_0^2) - \frac{1}{9}n^2r_0^6(43-194\nu_0^2+143\nu_0^4)\right](1-\nu_0^2)\frac{\partial^2\phi}{\partial\nu_0^2} - \left[1+\frac{4}{3}nr_0^3\nu_0^2+\frac{1}{9}n^2r_0^6(75-34\nu_0^2-33\nu_0^4)\right] 2\nu_0\frac{\partial\phi}{\partial\nu_0} = 0 \qquad \ldots(6)$$

where

$$\nu = \cos\zeta,\ r_0 = 1/\xi.$$

Assuming a series of the form

$$\phi = \sum_j a_j r_0^j R_j \qquad \ldots(7)$$

where

$$R_j = P_j(\nu_0) + nr_0^3 X_2^{(j)}(\nu_0) + n^2r_0^6 X_4^{(j)}(\nu_0) + \ldots \qquad \ldots(8)$$

Kopal has shown that for $j > 1$, $X_2^{(j)}(\nu_0)$ and $X_4^{(j)}(\nu_0)$ are found to be of the form

$$X_2^{(2)}(\nu_0) = -(1-\nu_0^2)(1-5\nu_0^2) \qquad \ldots(9)$$

$$X_2^{(3)}(\nu_0) = -\tfrac{1}{2}(1-\nu_0^2)(11-25\nu_0^2)\nu_0 \qquad \ldots(10)$$

$$X_2^{(4)}(\nu_0) = \frac{1}{6}(1-\nu_0^2)(9-120\nu_0^2+175\nu_0^4) \qquad \ldots(11)$$

$$X_2^{(5)}(\nu_0) = \frac{5}{8}(1-\nu_0^2)(17-98\nu_0^2+105\nu_0^4)\nu_0 \qquad \ldots(12)$$

$$X_4^{(2)}(\nu_0) = -\frac{1}{6}(1-\nu_0^2)(21-172\nu_0^2+175\nu_0^4) \qquad \ldots(13)$$

etc.

Equation (8) with (9)-(13) constitute the explicit form of the Roche harmonics associated with Roche equipotential surfaces (2) distorted by solid body rotation forces.

3. Roche Coordinates for Roche Model Distorted by Differential Rotation

For the Roche model of mass M, rotating according to the law

$$\omega = b_1 + b_2 s^2 \qquad \dots(14)$$

the equation of hydrostatic equilibrium may be written in the form

$$d\Omega = dv + \tfrac{1}{2}\omega^2 d(s^2) \qquad \dots(15)$$

where Ω denotes potential at a point P distant r from the centre of the star. $v = GM/r$, is the gravitational potential, ω is the angular velocity of rotation of an element of the fluid distant s from the axis of rotation.

On substituting (14) in (15) we have

$$d\Omega = dv + \tfrac{1}{2}(b_1^2 + 2b_1 b_2 s^2 + b_2^2 s^4)\, d(s^2).$$

On integration and simplification this gives

$$\Omega = \frac{GM}{r} + \tfrac{1}{2}(x^2 + y^2)[b_1^2 + b_1 b_2 (x^2 + y^2) + \tfrac{1}{3} b_2^2 (x^2 + y^2)^2] \dots(16)$$

In spherical polar coordinates

$$\left.\begin{aligned} x &= r\cos\phi\sin\theta = r\lambda \\ y &= r\sin\phi\sin\theta = r\mu \\ z &= r\cos\theta \quad\;\;\; = r\nu \end{aligned}\right\} \qquad \dots(17)$$

Equation (16) may be expressed in non-dimensional form as

$$\xi = \frac{1}{r} + \tfrac{1}{2} r^2 (1 - \nu^2)[b_1^2 + b_1 b_2 (1 - \nu^2) + \tfrac{1}{3} b_2^2 r^4 (1 - \nu^2)^2] \qquad \dots(18)$$

where $\xi = R\Omega/GM$ is a non-dimensional parameter denoting potential and ω is now non-dimensional angular velocity in units of GM/R^3, R being the unit of distance.

The surfaces generated by setting ξ = constant in (18) are referred as the Roche equipotentials.

Now if we take $r_0 = 1/\xi$ as our first approximation to the distance of the equipotential surface from the centre, Mohan and Singh[5] have shown that

$$r = r_0 [1 + \tfrac{1}{6} r_0^3 (1 - \nu_0^2)(b_1^2 + b_1 \omega_0 + \omega_0^2)] \qquad \dots(19)$$

where

$$\nu_0 = \cos\zeta \text{ and } \omega_0 = b_1 + b_2 r_0^2 (1 - \nu_0^2).$$

In the system of Roche coordinates (ξ,η,ζ), the ξ-coordinate is defined by Roche equipotential surfaces of the form (18) while coordinates η and ζ are defined by the requirement that they are orthogonal to ξ as well as to each other. In this triple orthogonal system of Roche coordinates, Mohan and Singh[5] have shown that the second and third coordinates are given by

$$\eta = \frac{\lambda}{\sqrt{1-\nu^2}} \qquad \text{...(20)}$$

and

$$\cos\zeta = \nu\left[1 - \frac{(1-\nu^2)\, r^3}{105}\,(15\omega^2 + 12 b_1\,\omega + 8 b_1^2) + \ldots\right] \qquad \text{...(21)}$$

whereas the expression for η given in (20) is exact, the expression for ζ obtained in (21) is correct upto second order terms in angular velocity ω.

Mohan and Singh[5] also obtained the explicit expressions for the metric coefficients h_1, h_2 and h_3 correct upto sectond order terms in ω, which are found to be

$$h_1(\xi,\zeta) = r_0^2\left[1 + \frac{r_0^3 \sin^2\zeta}{3}(4\omega_0^2 + b_1\,\omega_0 + b_1^2) + \ldots\right]$$

$$h_2(\xi,\zeta) = r_0 \sin\zeta\left[1 - \frac{r_0^3}{210}(15\,\omega_0^2 + 12 b_1\,\omega_0 + 8 b_1^2) + \frac{r_0^3\sin^2\zeta}{210}(65\omega_0^2 + 9 b_1\,\omega_0 + 51 b_1^2) + \ldots\right]$$

$$h_3(\xi,\zeta) = r_0\left[1 - \frac{r_0^3}{210}(150\,\omega_0^2 - 48\, b_1\,\omega_0 - 32 b_1^2) - \sin^2\zeta\,(215\omega_0^2 + 11 b_1\,\omega_0 + 19 b_1^2) + \ldots\right] \qquad \text{...(22)}$$

4. Explicit Expressions of Roche Harmonics for Roche Equipotential Surfaces Distorted by Differential Rotation Forces

Using expressions of h_1, h_2 and h_3 as given by (22), the Laplacian

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial\xi}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial\zeta}\right)\right]$$

now becomes

$$\nabla^2 = \left[1 - \frac{r_0^3}{105}(1-\nu_0^2)(280\omega_0^2 + 70\, b_1\,\omega_0 + 70\, b_1^2)\right]\frac{\partial}{\partial r_0}\left(r_0^2\frac{\partial}{\partial r_0}\right)$$

(equation continued on p. 289)

$$- \left[\frac{3r_0^2}{105} (90\omega_0^2 - 12b_1\omega_0 - 8b_1^2) \right.$$

$$\left. + \frac{2r_0^4 b_2}{105} (1 - \nu_0^2)(180\omega_0 - 12b_1) \right] r_0^2 \frac{\partial}{\partial r_0}$$

$$- \left[2 + \frac{r_0^3}{210} (600\omega_0^2 - 192 b_1\omega_0 - 128 b_1^2) \nu_0^2 \right.$$

$$+ (1 - \nu_0^2) b_2 r_0^2 ((520 (1 - \nu_0^2)$$

$$\left. + 480) \omega_0 + (236 (1 - \nu_0^2) - 144) b_1) \right] \nu_0 \frac{\partial}{\partial \nu_0}$$

$$+ \left[1 - \frac{r_0^3}{210} - \nu_0^2 (430\omega_0^2 + 22 b_1\omega_0 + 38b_1^2) \right.$$

$$\left. + 130\omega_0^2 + 118b_1\omega_0 + 102 b_1^2 \right] (1 - \nu_0^2) \frac{\partial^2}{\partial \nu_0^2} .$$

Therefore the Laplace equation (1) takes the explicit form in this case as

$$\left[1 - \frac{r_0^3}{105} (1 - \nu_0^2)(280\omega_0^2 + 70b_1\omega_0 + 70b_1^2) \right] \frac{\partial}{\partial r_0} \left(r_0^2 \frac{\partial \phi}{\partial r_0} \right)$$

$$- \left[\frac{3r_0^2}{105} (90\omega_0^2 - 12b_1\omega_0 - 8b_1^2) + \frac{2r_0^4 b_2}{105} (1 - \nu_0^2) \right.$$

$$\left. (180\omega_0 - 12b_1) \right] r_0^2 \frac{\partial \phi}{\partial r_0}$$

$$- \left[2 + \frac{r_0^3}{210} (600\omega_0^2 - 192 b_1\omega_0 - 128b_1^2) \nu_0^2 \right.$$

$$+ (1 - \nu_0^2) b_2 r_0^2 ((520 (1 - \nu_0^2)$$

$$\left. + 480) \omega_0 + 236 (1 - \nu_0^2) - 144) b_1) \right] \nu_0 \frac{\partial \phi}{\partial \nu_0}$$

$$+ \left[1 - \frac{r_0^3}{210} - \nu_0^2 (430\omega_0^2 + 22b_1\omega_0 + 38b_1^2) \right.$$

$$\left. + 130\omega_0^2 + 118b_1\omega_0 + 102b_1^2 \right] (1 - \nu_0^2) \frac{\partial^2 \phi}{\partial \nu_0^2} = 0. \quad \text{.. (23)}$$

As in section 2, the solution of this equation can be obtained in series form where

$$\phi = \sum_j a_j r_0^j R_j$$

and setting

$$R_j = P_j(\nu_0) + \frac{\omega_0^2}{2} r_0^3 X_2^{(j)}(\nu_0) + \ldots \qquad \ldots(24)$$

It can be shown (consistent with the adopted scheme of approximation) that the functions $X_2^{(j)}(\nu_0)$ assume the forms because of the influence of the terms pertaining b_2^2 and $b_1 b_2$, as given by

$$X_2^{(2)}(\nu_0) = -1.8571\,\nu_0^4 + 2.1905\,\nu_0^2 - .3333 \qquad \ldots(25)$$

$$X_2^{(3)}(\nu_0) = -4.6428\,\nu_0^5 + 6.5714\,\nu_0^3 - 1.9286\,\nu_0 \qquad \ldots(26)$$

$$X_2^{(4)}(\nu_0) = -10.8333\,\nu_0^6 + 17.9762\,\nu_0^4 - 7.6428\,\nu_0^2 + .5 \qquad \ldots(27)$$

$$X_2^{(5)}(\nu_0) = -24.375\,\nu_0^7 + 46.458\,\nu_0^5 - 25.7440\,\nu_0^3 + 3.6607\,\nu_0; \qquad \ldots(28)$$

$$X_2^{(2)}(\nu_0) = -2.7\,\nu_0^4 + 3.2\,\nu_0^2 - .5 \qquad \ldots(29)$$

$$X_2^{(3)}(\nu_0) = -6.75\,\nu_0^5 + 9.6\,\nu_0^3 - 2.85\,\nu_0 \qquad \ldots(30)$$

$$X_2^{(4)}(\nu_0) = -15.75\,\nu_0^6 + 26.25\,\nu_0^4 - 11.25\,\nu_0^2 + .75 \qquad \ldots(31)$$

$$X_2^{(5)}(\nu_0) = -.33.4375\,\nu_0^7 + 67.8125\,\nu_0^5 - 37.8125\,\nu_0^3 + 5.4375\,\nu_0 \ldots(32)$$

etc.

Equation (24) with (25) - (32) constitutes the explicit form of the Roche harmonics associated with the Roche equipotentials surfaces (18) up to second order approximation of the differential rotation terms, which indeed, is the outcome of our analysis.

5. Concluding Remarks

The influence of b_1^2 terms on $X_2^{(j)}(\nu_0)$ is same as obtained by Kopal[7]. In the present analysis we could only present the results which are because of the influence of the terms pertaining b_2^2 and $b_1 b_2$. This is because of our approximation scheme for angular velocity ω upto second order. The consideration of higher order terms in ω, should lead to the appropriate formulation of this problem which we intend to investigate this problem in subsequent study.

It may be pointed out that although we have studied here the problem of Roche harmonics associated with equipotential surfaces by assuming the Roche model of the star, the present method of Roche coordinates can be used also when some more realistic structure is assumed for the interior of the model. We can still either approximate the distorted equipotentials of such stars by Roche model or use their more realistic forms by using the system of Clairaut's coordinates (cf. Kopal[6,7]), when the law of differential rotation may be assumed of the form $\omega = b_1 + b_2 s^2$.

Acknowledgement

The authors will like to express their gratitude to Professor N. J. Rao, Director, Institute of Paper Technology (University of Roorkee), Saharanpur, for providing necessary research facilities. They are also thankful to Professor C. Mohan, Department of Mathematics, University of Roorkee, for valuable discussions and suggestions. The research is supported in part by DRIL grant No. 106/99 (University of Roorkee).

References

1. Z. Kopal, ***Astrophys. Space Sci.*** **8** (1970), 149–171.
2. Z Kopal, ***Adv. Astr. Astrophys.*** **9** (1972), 1–9.
3. G. F. Roach, ***Astrophys. space, Space Sci.*** 36 (1975), 159–67.
4. V. P. Singh, Ph. D. thesis Ch. IV University of Roorkee, India, 1978.
5. C. Mohan and V. P. Singh, ***Proc. Nat. Aca. Sci., India,*** 50 A (1980), 25–31.
6. Z. Kopal, ***Astrophys. Space Sci.*** **70** (1980), 407.
7. Z. Kopal, ***Astrophys. Space Sci.*** **76** (1981), 187–212.