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AN OSCILLATION CRITERION FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

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A new oscillation criterion is given for the second order nonlinear differential equation

$$\ddot{x}(t) + q(t) f(x(t)) = 0$$

where the coefficient $q(t)$ is not assumed to be nonnegative for all large values of t. Condition on f of the form $\int^{\pm\infty} \frac{du}{f(u)} < \infty$, used by Onose, Philos and Wong is discarded.

1. INTRODUCTION

Consider the second order nonlinear differential equation

$$\ddot{x}(t) + q(t) f(x(t)) = 0, \left(\cdot = \frac{d}{dt} \right) \qquad ...(1)$$

where $q : [t_0, \infty) \to R = (-\infty, \infty)$, $f : R \to R$ are continuous and $x f(x) > 0$ for $x \neq 0$.

We consider only those solutions of eqn. (1) which exist on $[t_0, \infty)$. A solution of eqn. (1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all such solutions are oscillatory.

Recently, Philos[2] considered the strongly superlinear equation of the form (1) i.e., the function f in eqn. (1) is such that

$$\int^{+\infty} \frac{du}{f(u)} < \infty \text{ and } \int^{-\infty} \frac{du}{f(u)} < \infty \qquad ...(2)$$

and proved the following oscillation criterion.

Theorem A—Suppose that condition (2) holds,

$$f'(x) \geqslant k > 0 \text{ for } x \neq 0, \left(' = \frac{d}{dx} \right) \qquad ...(3)$$

and let ρ be a positive continuously differentiable function on the interval $[t_0, \infty)$ such that $\dot{\rho}$ is nonnegative and decreasing on $[t_0, \infty)$. Equation (1) is oscillatory if

$$\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)\, q(s)\, ds > -\infty \qquad \ldots(4)$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \rho(u)\, q(u)\, du\, ds = \infty. \qquad \ldots(5)$$

His criterion extended and improved some of the results due to Onose[1] and Wong[3-5]. Theorem A is only concerned with oscillatory behaviour of strongly superlinear equations of the form of eqn. (1). Therefore, the purpose of this paper is to establish a new oscillation criterion for eqn. (1), where condition (2) is discarded. We also discuss the asymptotic behaviour of the forced equation

$$\ddot{x}(t) + q(t) f(x(t)) = e(t) \qquad \ldots(6)$$

where $e : [t_0, \infty) \to R$ is continuous.

Our results extend and improve some of the results of Onose[1], Philos[2] and Wong[3-5].

2. Main Results

Theorem 1—Let condition (3) hold and suppose that there is a differentiable function

$$\rho : [t_0, \infty) \to (0, \infty)$$

such that

$$\dot{\rho}(t) \geqslant 0 \text{ for } t \geqslant t_0 \text{ and } \limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} \rho(s)\, ds < \infty \qquad \ldots(7)$$

$$\int^{\infty} \frac{1}{\rho(s)}\, ds = \infty \qquad \ldots(8)$$

$$\liminf_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)\, q(s) - \frac{\dot{\rho}^2(u)}{4k\, \rho(s)}\right] ds > -\infty \qquad \ldots(9)$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \left[\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4\, k\rho(u)}\right] du\, ds = \infty. \qquad \ldots(10)$$

Then every solution of eqn. (1) is oscillatory.

PROOF : Let $x(t)$ be a nonoscillatory solution of eqn. (1). Without loss of generality, we assume that $x(t) \neq 0$ for all $t \geqslant t_0$. Furthermore, we assume that $x(t) > 0$ for $t \geqslant t_0$, since the substitution $u = -x$ transforms eqn. (1) into an equation of the same form subject to the assumptions of the theorem.
Now we define

$$w(t) = \rho(t) \frac{\dot{x}(t)}{f(x(t))} \text{ for every } t \geqslant t_0.$$

Then for all $t \geqslant t_0$ we obtain

$$\dot{w}(t) = -\rho(t) q(t) + \frac{\dot{\rho}(t)}{\rho(t)} w(t) - \frac{1}{\rho(t)} f'(x(t)) w^2(t)$$

$$\leqslant -[\rho(t) q(t) - \frac{\dot{\rho}^2(t)}{4k\rho(t)}] - \frac{1}{\rho(t)}[\sqrt{k}\, w(t) - \frac{\dot{\rho}(t)}{2\sqrt{k}}]^2. \quad \text{...(11)}$$

Hence, for all $t \geqslant t_0$ we have

$$w(t) \leqslant w(t_0) - \int_{t_0}^{t} [\rho(s) q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}] ds - \int_{t_0}^{t} \frac{1}{\rho(s)}[\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2 ds. \quad \text{...(12)}$$

Next, we consider the following three cases for the behaviour of $\dot{x}$:

Case 1—$\dot{x}$ is oscillatory. Then there exists a sequence $\{t_m\}_{m=1,2,\ldots}$ in $[t_0, \infty)$ with $\lim_{m\to\infty} t_m = \infty$ and such that $\dot{x}(t_m) = 0$ $(m = 1,2,\ldots)$. Thus, (12) gives

$$\int_{t_0}^{t_m} \frac{1}{\rho(s)} [\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2 ds \leqslant w(t_0) - \int_{t_0}^{t_m} [\rho(s) q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}]^2 ds, \quad m = 1, 2, \ldots$$

and hence, by (g) we conclude that

$$\int_{t_0}^{\infty} \frac{1}{\rho(s)}[\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2 ds < \infty.$$

So, for some positive K we have

$$\int_{t_0}^{\infty} \frac{1}{\rho(s)} [k\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2 ds \leqslant K \text{ for every } t \geqslant t_0.$$

By the Schwarz inequality, for $t \geqslant t_0$ we get

$$\left| -\int_{t_0}^{t} [\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]\, ds \right|^2 = \left| \int_{t_0}^{t} \sqrt{\rho(s)}\,[\frac{1}{\sqrt{\rho(s)}}(\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}})]\, ds \right|^2 \leqslant [\int_{t_0}^{t} \rho(s)\, ds] \int_{t_0}^{t} \frac{1}{\rho(s)} [\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]^2\, ds$$

$$\leqslant K \int_{t_0}^{t} \rho(s)\, ds.$$

From (7), there exists a positive constant L such that

$$\int_{t_0}^{t} \rho(s)\, ds < Lt^2 \text{ for every } t \geqslant t_0$$

and hence for all $t \geqslant t_0$ we obtain

$$-\int_{t_0}^{t} [\sqrt{k}\, w(s) - \frac{\dot{\rho}(s)}{2\sqrt{k}}]\, ds \leqslant \sqrt{KL}\; t.$$

Since $\dot{\rho}(t) \geqslant 0$ for $t \geqslant t_0$ we have

$$-\int_{t_0}^{t} w(s)\, ds < \sqrt{\frac{KL}{k}}\; t.$$

Furthermore, (12) gives

$$\int_{t_0}^{t} [\rho(s)\, q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}]\, ds \leqslant -w(t) + w(t_0)$$

and therefore, for all $t \geqslant t_0$

$$\int_{t_0}^{t}\int_{t_0}^{s} [\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4k\rho(u)}]\, du\, ds \leqslant -\int_{t_0}^{t} w(s)\, ds + (t - t_0)\, w(t_0)$$

$$\leqslant \sqrt{\frac{KL}{k}}\; t + (t - t_0)\, w(t_0)$$

and hence

$$\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s} [\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4k\rho(u)}]\, du\, ds < \sqrt{\frac{KL}{k}} + \left(1 - \frac{t_0}{t}\right) w(t_0).$$

This contradicts condition (10). Thus $\dot{x}(t)$ is of constant sign for all $t \geqslant t_0$.

Case 2—$\dot{x} > 0$ on $[t_1, \infty)$ for some $t_1 \geqslant t_0$. In this case, from (12) it follows that for $t \geqslant t_1$

$$\int_{t_0}^{t} [\rho(s)\, q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}]\, ds < w(t_0)$$

and consequently

$$\frac{1}{t}\int_{t_1}^{t}\int_{t_0}^{s} [\rho(u)\, q(u) - \frac{\dot{\rho}^2(u)}{4k\rho(u)}]\, du\, ds \leqslant (1 - \frac{t_1}{t})\, w(t_0)$$

which again contradicts condition (10).

Case 3—$\dot{x} < 0$ on $[t_1, \infty)$ for some $t_1 \geqslant t_0$. From (11) and the fact that ρ is nondecreasing on $[t_0, \infty)$ it follows that

$$- w(t) \geqslant - w(t_1) + \int_{t_1}^{t} [\rho(s)\, q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}]\, ds + \int_{t_1}^{t} \frac{1}{\rho(s)} f'(x(s))\, w^2(s)\, ds. \qquad \text{...(13)}$$

If

$$\int_{t_1}^{\infty} \frac{1}{\rho(s)} f'(x(s))\, w^2(s)\, ds < \infty$$

then condition (3) ensures that

$$\int_{t_1}^{\infty} \frac{1}{\rho(s)} w^2(s)\, ds < \infty$$

and hence we can arrive at a contradiction by the procedure of Case 1. So, we suppose that the above improper integral is infinite. From (13) and condition (9) we derive

$$- w(t) \geqslant C + \int_{t_1}^{t} \frac{1}{\rho(s)} w^2(s)\, f'(x(s))\, ds$$

where C is a constant. Furthermore we choose a $t_2 \geqslant t_1$ so that

$$C + \int_{t_1}^{t_2} \frac{1}{\rho(s)} w^2(s)\, f'(x(s))\, ds = C_1 > 0$$

and then for every $t \geqslant t_2$ we get

$$\frac{1}{\rho(t)} w^2(t) f'(x(t)) [C + \int_{t_1}^{t} \frac{1}{\rho(s)} w^2(s) f'(x)(s)) ds]^{-1}$$
$$\geqslant - \frac{\dot{x}(t) f'(x(t))}{f(x(t))}$$

and hence, by integrating over $[t_2, t]$, we obtain

$$\log \frac{1}{C_1} [C + \int_{t_1}^{t} \frac{1}{\rho(s)} w^2(s) f'(x)) ds] \geqslant \log \frac{f(x(t_2))}{f(x(t))}.$$

Thus

$$C + \int_{t_1}^{t} \frac{1}{\rho(s)} w^2(s) f'(x(s)) ds \geqslant C_2 \frac{1}{f(x(t))}, \text{ for all } t \geqslant t_2$$

where $C_2 = C_1 f(x(t_2)) > 0$. So (13) yields

$$\dot{x}(t) \leqslant - C_2 \frac{1}{\rho(t)} \text{ for every } t \geqslant t_2$$

and consequently we have

$$x(t) \leqslant x(t_2) - C_2 \int_{t_2}^{t} \frac{1}{\rho(s)} ds \to -\infty \text{ as } t \to \infty$$

a contradiction to the fact that $x(t) > 0$ for $t \geqslant t_0$. This completes the proof.

Theorem 2—Let conditions (7) and (8) in Theorem 1 be replaced by

$$\dot{\rho}(t) \geqslant 0 \text{ and } \ddot{\rho}(t) \leqslant 0 \text{ for } t \geqslant t_0. \quad ...(14)$$

Then the conclusion of Theorem 1 holds.

PROOF : The proof is similar to that of Theorem 1 and hence is omitted.

From the proof of Theorem 1 we see the following results hold :

Corollary 1—Let the conditions (3), (7) and (8) hold and

$$\int^{\infty} \frac{\dot{\rho}^2(s)}{\rho(s)} ds < \infty. \quad ...(15)$$

Equation (1) is oscillatory if condition (4) and (5) hold.

Corollary 2—Let conditions (3), (4), (5), (14) and (15) hold, then equation (1) is oscillatory.

Remark 1 : Theorem 1 includes Theorem 3 in Philos[2] as a special case, and since condition (2) is disregarded in our criterion, Corollary 1 extends and improves Theorem A when condition (15) holds. On the other hand, Theorem 1 can be applied in some cases in which Theorem A is not applicable. Such a case is described in the following example.

Example 1—Consider the differential equation

$$\ddot{x}(t) + \left(-t^{-1/2}\sin t + \tfrac{1}{2}t^{-3/2}(2+\cos t)\right) f(x(t)) = 0,\ t \geqslant t_0 = \frac{\pi}{2} \qquad \ldots(16)$$

where f is any one of the following functions :

(i) $f(x) = cx,\ x \in R$ for $c > 0$.

(ii) $f(x) = |x|^{\gamma}\,\text{sgn}\,x + kx,\ x \in R$ for $\gamma > 0$ and $k > 0$.

(iii) $f(x) = x\log^2(\mu + |x|),\ x \in R$ for $\mu > 1$.

(iv) $f(x) = xe^{\lambda|x|},\ x \in R,\ \lambda \geqslant 0$.

We let $\rho(t) = t$. Conditions (3), (7) and (8) are easy to verify and

$$\int_{t_0}^{t}\left[\rho(s)q(s) - \frac{\dot{\rho}^2(s)}{4k\rho(s)}\right]ds = \int_{\pi/2}^{t}\left[\frac{2+\cos s}{2\sqrt{s}} - \sqrt{s}\sin s - \frac{1}{4ks}\right]ds$$

$$= \sqrt{t}(2+\cos t) - 2\left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{4k}\ln\frac{2t}{\pi} > \sqrt{t} - 2\left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{4k}\ln\frac{2t}{\pi}$$

where k is as in condition (3), and

$$\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[\rho(u)q(u) - \frac{\dot{\rho}^2(u)}{4k\rho(u)}\right]du\,ds > \frac{1}{t}\int_{\pi/2}^{t}\left[\sqrt{s} - 2\left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{4k}\ln\frac{2s}{\pi}\right]ds = \frac{2}{3}\sqrt{t} + \frac{4}{3}\left(\frac{\pi}{2}\right)^{3/2}$$

$$- 2\left(\frac{\pi}{2}\right)^{1/2} - \frac{1}{4k}\ln\frac{2t}{\pi} + \frac{1}{4k}\left(1 - \frac{\pi}{2t}\right).$$

Conditions (9) and (10) of Theorem 1 are satisfied, and hence every solution of eqn. (16) is oscillatory. We may note that Theorem A can be applied to eqn. (16) when f is strongly superlinear i. e. for the case (ii) with $\gamma > 1$, case (iii) and case (iv) with $\lambda > 0$. Also, Theorem 3 in Philos[2] is not applicable here since

$$\limsup_{t\to\infty} \frac{1}{t} \int_{\pi/2}^{t} \int_{\pi/2}^{s} \left[\frac{2+\cos u}{2u\sqrt{u}} - \frac{\sin u}{\sqrt{u}}\right] du\, ds < \infty.$$

We conclude that the class of the function f described is larger than that in Philos[2].

The following theorem is concerned with the asymptotic behaviour of all solutions of the forced equation (6)

Theorem 3—In addition to the hypotheses of Theorem 1, we let

$$\int^{\infty} \rho(s) \mid e(s) \mid ds < \infty \qquad \ldots(17)$$

then $\liminf_{t\to\infty} \mid x(t) \mid = 0$ for all solutions x of eqn. (6).

PROOF : Let x be a solution of eqn. (6) on $[t_0, \infty)$ with $\liminf_{\to\infty} \mid x(t) \mid > 0$. Clearly x is nonoscillatory. Without loss of generality, we assume that $x(t) > 0$ for $t \geqslant t_0$. Furthermore, we consider the function W defined as in the proof of Theorem1 and then for $t \geqslant t_0$ we get

$$\dot{w}(t) = \dot{\rho}(t) \frac{\dot{x}(t)}{f(x((t)))} \rho -(t)\, q(t) + \frac{1}{f(x)(t))} \rho(t)\, e(t) - \frac{1}{\rho(t)}$$

$$w^2(t) f'(x(t)) \leqslant -\rho(t)\, q(t) + \frac{1}{f(c)} \rho(t) \mid e(t) \mid$$

$$+ \dot{\rho}(t) \frac{\dot{x}(t)}{f(x(t))} - \frac{1}{\rho(t)} w^2(t) f'(x(t))$$

where $c = \inf_{t \geqslant t_0} x(t) > 0$. Thus

$$\int_{t_0}^{t} \rho(s)\, q(s)\, ds \leqslant -w(t) + w(t_0) + \int_{t_0}^{t} \frac{\dot{\rho}(s)}{\rho(s)} w(s)\, ds$$

$$+ \frac{1}{f(c)} \int_{t_0}^{t} \rho(s) \mid e(s) \mid ds - \int_{t_0}^{t} \frac{1}{\rho(s)} w^2(s) f'(x(s))\, ds.$$

$$\leqslant M - w(t) + \int_{t_0}^{t} \frac{\dot{\rho}(s)}{\rho(s)} w(s)\, ds - \int_{t_0}^{t} \frac{1}{\rho(s)} w^2(s) f'(x(s))\, ds.$$

where $M = w(t_0) + \frac{1}{f(c)} \int_{t_0}^{\infty} \rho(s) \mid e(s) \mid ds$. Now, we can complete the proof by procedure of the proof of Theorem 1 and hence we omit the detail.

The following examples are illustrative.

Example 2—Consider the second order forced equation

$$\ddot{x}(t) + \left[\frac{2 + \cos t}{2t\sqrt{t}} - \frac{\sin t}{\sqrt{t}} \right] x(t) = \frac{15}{4t^{7/2}} - \frac{1}{t_2} \sin t + \frac{2 + \cos t}{2t^3}, \; t \geqslant t_0 = \frac{\pi}{2}. \qquad \text{...(18)}$$

All conditions of Theorem 3 are satisfied with $\rho(t) = t$. Hence, all solutions x of equation (18) satisfy $\liminf_{t \to \infty} \mid x(t) \mid = 0$. One such solution is $x(t) = 1/t\sqrt{t}$. We may note that Proposition 1 in Philos[2] is not applicable to equation (18).

Example 3—The equation

$$\ddot{x}(t) + x(t) = \frac{2 \sin t}{t_3} - \frac{2}{t^2} \cos t, \; t > 0 \qquad \text{...(19)}$$

has the oscillatory solution $x(t) = \frac{\sin t}{t}$. All conditions of Theorem 3 are satisfied with $\rho(t) = 1, t \geqslant t_0 > 0$.

Remark 2 : From Example 2, we see that eqn. (18) without forcing term is oscillatory by Theorem 1 while the forced equation (18) has a nonoscillatory solution $x(t) = \frac{1}{t} \to 0$ as $t \to \infty$. On the other hand, from Example 3, we find that the unforced equation $\ddot{x} + x = 0$ is oscillatory and has solution $\sin t$ and $\cos t \not\to 0$ as $t \to \infty$ and eqn. (19) has oscillatory solution $x(t) = \sin t/t \to 0$ as $t \to \infty$. Therefore, it seems that the size of the forcing term is responsible for generating such behaviours. Now, it remains an open problem to find conditions on the forcing term "e" in eqn. (6) so that the oscillatory character of the unforced equation is either to be changed or maintained.

Remark 3 : The results of this paper can be extended to more general equations of the form

$$(a(t)\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t) f(x(t)) = 0 \qquad \text{...(20)}$$

and

$$(a(t)\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t) f(x(t)) = e(t) \qquad \text{...(21)}$$

where $a, e, p, q : [t_0, \infty) \to R$ and $f : R \to R$ are continuous, $a(t) > 0$ for $t \geqslant t_0$ and $x f(x) > 0$ for $x \neq 0$.

References

1. H. Onose, *Proc. Am. Math. Soc.* **51** (1975), 67–73.
2. Ch. G. Philos, *Math. Nachr.* **120** (1985), 127–38.
3. J.S.W. Wong, *Proc. Am. Math. Soc.* **40** (1973), 487–91.
4. J.S.W. Wong, *Funcial-Ekvac.* **11** (1968), 207–34.
5. J.S.W. Wong, *Bull. Inst. Math. Acad. Sinica* **3** (1975), 283–309.