Indian J. pure appl. Math., **20** (4) : 395-402, **April 1989**

ON TEMPERATURE-RATE DEPENDENT THERMOELASTIC LONGITUDINAL VIBRATIONS OF AN INFINITE CIRCULAR CYLINDER

D. YADAIAH AND RAM KUMAR SHUKLA

Department of Mathematics, Kakatiya University, Warangal 506 009

(*Received* 26 *May* 1988)

Green and Lindsay[1] have given the linearized temperature-rate dependent thermoelasticity theory. Using this theory the problem of thermoelastic longitudinal vibrations of an infinite circular cylinder has been solved and the results have been obtained in terms of potential functions. It is interesting to note that due to the temperature rate dependent theory the amplitude of both the elastic and thermal waves are higher than that of conventional theory. We further observe that if we put $\gamma = \gamma^* = 0$ in the results of this paper we arive at the results of the conventional theory of thermoelasticity.

1. INTRODUCTION

The thermoelasticity theory which includes the temperature-rate among the constitutive variables has aroused considerable interest in recent years. This theory which was developed by Green and Lindsay[1] and by Suhubi[2] is a generalisation of the conventional coupled thermoelasticity theory[3] and predicts a finite speed for the propagation of thermal signals. Several problems revealing interesting phenomena characterizing this theory have been considered earlier[4-6]. Because of the experimental evidence available in favour of the finiteness of heat propagation speed[7], these studies are of practical relevance too.

The problem of thermoelastic vibrations of a circular cylinder was discussed by Chadwick[8] and the solutions have been obtained in terms of potential functions. In this paper an attempt has been made to study the problem of thermoelastic longitudinal vibration of an infinite circular cylinder in the context of linearised temperature—rate dependent thermoelasticity theory and the solutions have been obtained in terms of the potential functions. It is interesting to note that due to the temperature rate dependent theory the amplitude of both the elastic and thermal waves are higher than that of conventional theory. We further observe that if we put $\gamma = \gamma^* = 0$ in the results of this paper we arrive at the results of the conventional theory of thermoelasticity discussed by Chadwick[8].

2. BASIC EQUATIONS

In the context of the linearised temperature—rate dependent thermoelastic theory of Green and Lindsay[1], the equations governing the displacement vector $\bar{U}$, the stress tensor σ_{ij} and the temperature deviation θ, in a homogeneous and isotropic solid, in tensor notation are given as follows.

$$\sigma_{ij} = \lambda \, \delta_{ij} \, u_{k,k} + \mu \, (u_{i,j} + u_{j,i}) - \frac{\alpha}{\chi_T} \, (\theta + \gamma \, \dot{\theta}) \, \delta_{ij} \qquad \text{...(2.1)}$$

$$\rho \, \ddot{\bar{U}} = \mu \, \nabla^2 \, \bar{U} + (\lambda + \mu) \text{ grad div } \bar{U} - \frac{\alpha}{\chi_T} \text{grad } (\theta + \gamma \dot{\theta}) \qquad \text{...(2.2)}$$

$$k \, \nabla^2 \, \theta = \rho c_\epsilon \, (\dot{\theta} + \gamma^* \, \ddot{\theta}) + \frac{\alpha \theta_0}{\chi_T} \dot{u}_{k,k}. \qquad \text{...(2.3)}$$

Here ρ is the mass density and λ and μ are the isothermal Lame's constants given by

$$\lambda = \frac{E_T \, \nu_T}{(1 + \nu_T) \, (1 - 2\nu_T)}, \; \mu = \frac{E_T}{2 \, (1 + \nu_T)} \qquad \text{...(2.4)}$$

in which E_T and ν_T are respectively isothermal Young's modulus and isothermal Poisson's ratio and

$$\alpha = \left(\frac{\partial \, u_{k,k}}{\partial \theta} \right)_\sigma \qquad \text{...(2.5)}$$

is the coefficient of volume expansion.

The isothermal compressibility is given by

$$\chi_T = 3 \left(\frac{\partial \, u_{k,k}}{\partial \, \sigma_{kk}} \right)_\theta = 3 \left(\frac{1 - 2\nu_T}{E_T} \right) \qquad \text{...(2.6)}$$

from which we see that E_T, ν_T as their suffixes indicate, are the isothermal elastic constants. Further C_ϵ is the specific heat at constant strain, k is the thermal conductivity, θ_0 is the initial uniform temperature and γ, γ^* are thermal constants characteristic of the theory. A superposed dot denotes partial differentiation with respect to time t. It is assumed that the body forces and heat sources are absent. Expressing the displacement vector $\bar{U}$ as the sum of irrotational and solenoidal components

$$\bar{U} = \bar{\nabla} \, \phi + \text{curl } \bar{A}. \qquad \text{.. (2.7)}$$

Equations (2.2) (2.3) are found to be equivalent to the set

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{E_T \, (1 - \nu_T)}{(1 + \nu_T) \, (1 - 2\nu_T)} \, \nabla^2 \, \phi - \frac{E_T \, \alpha}{3\rho \, (1 - 2\nu_T)} \, (\theta + \gamma \, \dot{\theta})$$

$$k \, \nabla^2 \, \theta = \rho C_\epsilon \, (\dot{\theta} + \gamma^* \, \ddot{\theta}) + \frac{E_T \, \theta_0 \, \alpha}{3 \, (1 - 2\nu_T)} \frac{\partial}{\partial t} \nabla^2 \, \phi \qquad \text{...(2.9)}$$

$$\frac{\partial^2 \, \bar{A}}{\partial t^2} = \frac{E_T}{2\rho \, (1 + \nu_T)} \, \nabla^2 \, \bar{A}. \qquad \text{...(2.10)}$$

From equation (2.8) we observe that the velocity of longitudinal elastic waves in a medium with zero coefficient of expansion is given by

$$V_T = \left[\frac{E_T (1 - \nu_T)}{(1 + \nu_T)(1 - 2\nu_T)} \right]^{1/2}. \quad \ldots(2.11)$$

We shall refer to V_T as the isothermal velocity. Further the velocity of transverse elastic waves[8] is given by

$$V_s = \left[\frac{E_T}{2\,\rho(1 + \nu_T)} \right]^{1/2}. \quad \ldots(2.12)$$

3. Thermoelastic Vibrations of a Circular Cylinder

We denote by (r, θ, z) cylindrical polar coordinates referred to the axis of the cylinder and make use of potential functions. In longitudinal disturbances the tangential component of displacement vanishes identically and since, in addition, field quantities do not depend upon the angular coordinates no confusion will arise from the use of θ to denote the temperature perturbation. The vector potential $\bar{A}$ is of the form $(0, \psi, 0)$ and the displacement components are given in terms of the scalar functions $\phi\,(r, z, t)$, $\psi\,(r, z, t)$ by

$$u_r = \frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z},\; u_\theta = 0,\; u_z = \frac{\partial\phi}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\,\psi). \quad \ldots(3.1)$$

The thermoelastic equations (2.8) – (2.10) now take the form

$$\frac{\partial^2 \phi}{\partial t^2} = V_T^2 \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\alpha}{\rho\chi_T}(\theta + \gamma\dot{\theta}) \right. \quad \ldots(3.2)$$

$$\rho C_\epsilon (\dot{\theta} + \gamma^* \ddot{\theta}) + \frac{\alpha\theta_0}{\chi_T}\frac{\partial}{\partial t}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right)$$

$$= k \left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} \right) \quad \ldots(3.3)$$

$$\frac{\partial^2 \psi}{\partial t^2} = V_s^2\, \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\left(\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2} \right). \quad \ldots(3.4)$$

Inserting into equations (3.2) – (3.4) the solution

$$[\theta, \phi, \psi] = [\hat{\theta}(r), \hat{\phi}(r), \hat{\psi}(r)]\; e^{i(\eta z - \omega t)}. \quad \ldots(3.5)$$

In these expressions η and ω are, in general, complex quantities so that the wave length is $2\pi/\mathrm{Re}\,\eta$ and the period $2\pi/\mathrm{Re}\,\omega$.

$$\frac{d^2 \hat{\phi}}{dr^2} + \frac{1}{r}\frac{d\hat{\phi}}{dr} + \left(\frac{\omega^2}{V_T^2} - \eta^2 \right)\hat{\phi} = \frac{\alpha\,(1 - i\gamma\omega)\hat{\theta}}{\rho\chi_T\, V_T^2} \quad \ldots(3.6)$$

$$\frac{d^2 \hat{\theta}}{dr^2} + \frac{1}{r}\frac{d\hat{\theta}}{dr} + \left[\frac{i\omega\rho C_\epsilon}{k}(1 - i\,\gamma^*\omega) - \eta^2 \right]\hat{\theta}$$

(equation continued on p. 398)

$$= - \frac{i\omega\alpha\theta_0}{k\chi_T}\left(\frac{d^2\hat{\theta}}{dr^2} + \frac{1}{r}\frac{d\hat{\phi}}{dr} - \eta^2\hat{\phi}\right) \quad \text{...(3.7)}$$

$$\frac{d^2\hat{\psi}}{dr^2} + \frac{1}{r}\frac{d\hat{\psi}}{dr} + \left[\left(\frac{\omega^2}{V_s^2} - \eta^2\right) - \frac{1}{r^2}\right]\hat{\psi} = 0. \quad \text{...(3.8)}$$

Solving these equations, we get

$$\phi = A\,J_0\left(r\sqrt{\xi_1^2 - \eta^2}\right) + BJ_0\left(r\sqrt{\xi_2^2 - \eta^2}\right)\exp \alpha(i(\eta z - \omega t)) \quad \text{...(3.9)}$$

$$\theta = \frac{\rho\chi_T V_T^2}{\alpha(1 - i\gamma\omega)}\left[A\left(\frac{\omega^2}{V_T^2} - \xi_1^2\right)J_0\left(r\sqrt{\xi_1^2 - \eta^2}\right) + B\left(\frac{\omega^2}{V_T^2} - \xi_2^2\right)J_0\left(r\sqrt{\xi_2^2 - \eta^2}\right)\right]\exp(i(\eta z - \omega t)) \quad \text{...(3.10)}$$

$$\psi = C\,J_1\left(r\sqrt{\xi_3^2 - \eta^2}\right)\exp(i(\eta z - \omega t)) \quad \text{...(3.11)}$$

where ξ_1^2, ξ_2^2 are the roots of

$$\xi^4 - \left[\frac{\omega^2}{V_T^2} + \frac{i\omega\rho C_\varepsilon}{k}\left\{(1 + \epsilon) - i\omega(\epsilon\gamma + \gamma^*)\right\}\right]\xi^2 + \frac{i\omega^3\rho C_\varepsilon}{k V_T^2}(1 - i\omega\gamma^*) = 0. \quad \text{...(3.12)}$$

The equation corresponding to (3.12) from Chadwick[8] is

$$\xi^4 - \left[\frac{\omega^2}{V_T^2} + \frac{i\omega\rho C_\varepsilon}{k}(1 + \epsilon)\right]\xi^2 + \frac{i\omega^3\rho C_\varepsilon}{k V_T^2} = 0 \quad \text{...(3.12A)}$$

$$\epsilon = \frac{\alpha^2\theta_0}{\chi_T^2 C_\varepsilon \rho^2 V_T^2} \quad \text{...(3.13)}$$

and

$$\xi_3^2 = \frac{\omega^2}{V_s^2}. \quad \text{...(3.14)}$$

The Bessel function Y_0, Y_1 of the second kind are excluded from the solution by the requirement that u_r, u_z, θ shall be finite at the axis.

We find from (3.12) and (3.12A) that due to the introduction of the temperature rate theory the values of $\xi_1^2 + \xi_2^2$ and $\xi_1^2 \ \xi_2^2$ are reduced by amounts of $i\omega\,(\epsilon\gamma + \gamma^*)$ and $i\omega\gamma^*$ respectively. This implies a reduction in the values of ξ_1^2 and ξ_2^2 which by equations (3.9) and (3.10) cause increase in the amplitude of both elastic and thermal waves.

The constants A, B, C are found by substituting (3.9) — (3.11) into boundary conditions which, since the surface of the cylinder is free from mechanical and thermal constraints, take the form

$$\sigma_{rr} = 0,\ \sigma_{rz} = 0 \text{ at } r = a. \qquad \ldots(3.15)$$

Equations (2.1) and (3.15) give

$$\sigma_{rr} = \rho V_T^2 \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} - 2\rho V_s^2 \left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r \partial z} \right) - \frac{\alpha}{\chi_T}(\theta + \gamma\dot{\theta}) = 0 \right.$$

$$\sigma_{rz} = \rho V_s^2 \left(2\frac{\partial^2 \phi}{\partial r\, \partial z} + \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\psi}{r^2} - \frac{\partial^2 \psi}{\partial z^2} \right) = 0.$$

Thermal boundary condition is

$$\frac{\partial \theta}{\partial r} + h\,\theta = 0$$

at $r = a$ ...(3.16)

where h is the heat transfer coefficient of the boundary $r = a$.

The three homogeneous linear equations connecting A, B, C are found to be

$$A\left[\left(2 - \frac{\omega^2}{V_s^2\ \eta^2}\right) J_0(a\eta H_1) + \frac{2H_1}{a\eta}\ J_1(a\eta\ H_1)\right]$$

$$+ B\left[\left(2 - \frac{\omega^2}{V_s^2\ \eta^2}\right) J_0(a\ \eta\ H_2) + \frac{2H_2}{a\eta}\ J_1(a\ \eta\ H_2)\right]$$

$$- 2iCH_3\left[J_0(a\ \eta\ H_3) - \frac{1}{a\eta H_3}\ J_1(a\eta H_3)\right] = 0 \qquad \ldots(3.17)$$

$$2i\ [AH_1\ J_1(a\eta H_1) + BH_2\ J_1(a\eta H_2) - C\left(2 - \frac{\omega^2}{V_s^2\ \eta^2}\right) J_1(a\eta H_3) = 0$$

...(3.18)

$$A\left[\frac{h}{\eta} J_0(a\eta H_1) - H_1 J_1(a\eta H_1)\right]\left[\left(\frac{\omega^2}{V_T^2\ \eta^2} - 1 - H_1^2\right)\right]$$

$$+ B\left[\frac{h}{\eta} J_0(a\eta H_2) - H_2 J_1(a\eta H_2)\right]\left(\frac{\omega^2}{V_T^2\ \eta^2} - 1 - H_2^2\right) = 0 \qquad \ldots(3.19)$$

where

$$H_i^2 = \frac{\xi_i^2}{\eta^2} - 1, \qquad i = 1, 2, 3. \qquad \ldots(3.20)$$

Eliminating A, B, C between equations (3.17) — (3.19) we obtain

$$\left(2 - \frac{\omega^2}{V_s^2\ \eta^2}\right)^2 J_1(a\eta H_3)\left[H_1\left(H_1^2 + 1 - \frac{\omega^2}{V_T^2\ \eta^2}\right) J_0(a\eta H_2) J_1(a\eta H_1)\right.$$

$$\left. - H_2\left(H_2^2 + 1 - \frac{\omega^2}{V_T^2\ \eta^2}\right) J_9(a\eta H_1) J_1(a\eta H_2)\right]$$

$$+ \frac{2}{a\eta} H_1 H_2\left(H_1^2 - H_2^2\right) J_1(a\eta H_1) J_1(a\eta H_2)$$

$$\times \left\{2aH_3 J_0(a\eta H_3) - \frac{\omega^2}{V_s^2\ \eta^2} J_1(a\eta H_3)\right\}$$

$$= \frac{h}{\eta}\left[\left(2 - \frac{\omega^2}{V_s^2\ \eta^2}\right)^2\left(H_1^2 - H_2^2\right) J_0(a\eta H_1)\right.$$

$$\times J_0(a\eta H_2) J_1(a\eta H_3) - \frac{2}{a\eta}\left\{2\, a\eta H_3 J_0(a\eta H_3)\right.$$

$$\left. - \frac{\omega^2}{V_s^2\ \eta^2} J_1(a\eta H_3)\right\} \times H_1\left(H_2^2 + 1 - \frac{\omega^2}{V_T^2\ \eta^2}\right)$$

$$\times J_0(a\eta H_2) J_1(a\eta H_1) - H_2\left(H_1^2 + 1 - \frac{\omega^2}{V_T^2\ \eta^2}\right)$$

$$\left.\times J_0(a\eta H_1) J_1(a\eta H_2)\right\}\Bigg]. \qquad \ldots(3.21)$$

Equations (3.12), (3.14), (3.20) and (3.21) together determine the wave number the wave number η as a function of the frequency ω.

We observe that the form of equation (3.21) is the same as the frequency equation obtained by Chadwick[8]. The difference lies only in the values of H_1 and H_2 which in the present case depend on γ and γ^* also and if we put $\gamma = \gamma^* = 0$ the two equations become identical and the results tally.

If we replace V_T by V_P and put

$$H_1 = \sqrt{\frac{V^2}{V_p^2} - 1}, \qquad H_2 = \infty \qquad \text{...(3.22)}$$

where

$$V = \frac{\omega}{\eta}. \qquad \text{...(3.23)}$$

Equation (3.21) reduces to

$$\left(2 - \frac{V^2}{V_s^2}\right)^2 J_0\left(a\eta\sqrt{\frac{V^2}{V_p^2} - 1}\right) J_1\left(a\eta\sqrt{\frac{V^2}{V_s^2} - 1}\right)$$

$$+ \left(\frac{2}{a\eta}\sqrt{\frac{V^2}{V_p^2} - 1}\right) J_1\left(a\eta\sqrt{\frac{V^2}{V_p^2} - 1}\right)$$

$$\times \left[\left(2a\eta\sqrt{\frac{V^2}{V_s^2} - 1}\right) J_0\left(a\eta\sqrt{\frac{V^2}{V_s^2} - 1}\right)\right.$$

$$\left. - \frac{V^2}{V_s^2} J_1\left(a\eta\sqrt{\frac{V^2}{V_s^2} - 1}\right)\right] = 0. \qquad \text{...(3.24)}$$

Equation (3.24) is the final result of the Pochammer Chree analysis of the longitudinal modes of Vibrations of a circular cylinder.

To find the order of magnitude of the discrepancy between equation (3.24) and the exact frequency relation (3.21) we expand the roots of equation (3.12) in powers of the reduced frequency χ[1]. From equation (3.20) we obtain

$$H_1 = \sqrt{\frac{V^2}{V_p^2} - 1}\ [1 + O(\chi)] \qquad \text{...(3.25)}$$

$$H_2 = \frac{1 + i}{\sqrt{2\chi}} \frac{V}{V_p} (1 + \epsilon) [1 + O(\chi)] \qquad \text{...(3.26)}$$

using the relation $V_p = V_T \sqrt{1 + \epsilon}$ ...(3.27)

(1) $\chi = \frac{\omega}{\omega^*}$ where $\omega = \frac{\rho C_\epsilon V_T^2}{k}$ is characteristic frequency of the solid[8].

It follows that

$$J_0\,(a\eta H_1) = J_0\left(a\eta\sqrt{\frac{V^2}{V_p^2} - 1}\right) + O\,(\chi) \text{ as } \chi \to 0$$

$$J_0\,(a\eta H_2) = (\tfrac{1}{2}\chi)^{1/2}\left\{\frac{V_p}{\pi a\eta V\,(1+\epsilon)}\right\}^{\frac{1}{2}}$$

$$\times \exp\left\{\frac{a\eta}{\sqrt{2\chi}}\frac{V}{V_p}\,(1+\epsilon)\,(1-i) + \frac{1}{8}\pi\, i\right\}, \qquad \ldots(3.28)$$

with similar results for the first order Bessel functions $J_1\,(a\eta H_1)$ $J_1\,(a\eta H_2)$. Entering these expressions into equation (3.21) it is found that each sider educes to multiple of $L + O(\chi)$ where L is the left hand side of equation (3.24). Thus in the limit $\chi \to 0$. V becomes independent of χ and h and we recover the classical frequency relation (3.24). Since this equation has been shown to differ from the thermoelastic frequency relation (3.21) by terms of order χ, the error sincurred by using the Pochammer—Chree analysis in the interpretation of experimental results are generally quite insignificant.

References

1. A.E. Green and K.A. Lindsay, *J. Elasticity* **2** (1972), 1.
2. E. S. Suhubi, *Thermoelastic solids*. In *Continuum Physics* (ed. A.C. Eringen) Vol. II, Part II, Chap. II. Academic Press, New York, 1975.
3. M.A. Biot, *J. Appl. Phys.* **27** (1956), 240.
4. A. E. Green *Mathematica* **19** (1972), 68.
5. V.K. Agarwal, *Acta. Mech.* **31**(1979), 185.
6. D. S. Chandrashekaraiah, *Proc. Indian Acad. Soc. (Math. Sci.)*, **89** (1980), 43.
7. C. C. Ackerman and R. A. Guyer, *Ann. Phys.* **50** (1968), 128.
8. P. Chadwick, *Thermoelasticity—The Dynamical theory*. In *Progress in Solid Mechanics*, Vol. I. Amestardam, 1960.