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LINEAR STABILITY AND THE RESONANCE FOR THE TRIANGULAR LIBRATION POINTS FOR THE DOUBLY PHOTOGRAVITATIONAL ELLIPTIC RESTRICTED PROBLEM OF THREE BODIES

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In the present paper we have found the range of values of μ and e for the linear stability of the triangular points for the doubly photo-gravitational elliptic restricted problem of three bodies and it has been shown that some resonances of the third and the fourth order exist which will need special investigation for the determination of complete stability of the lihration points under our reference.

1. INTRODUCTION

Simmons *et al*[14], studied the existence and linear stability of the libration points for the photo-gravitational circular restricted problem of three bodies. They found the ranges of the linear stability as well for all the various libration points. Manju and Choudhry[9] studied the non-linear stability for resonance as well as for non-resonance cases for the isosceles triangular libration points when only one of the gravitating bodies was taken to be radiating and the reduction factor very small equal to that for the sum. The authors[5] studied the non-linear stability of the triangular libration points for non-resonance case for the doubly photo-gravitational restricted problem of three bodies where both of the gravitating bodies were assumed to be radiating and similar to the assumption of Simmons and others in the mentioned paper we took the reduction factors to vary from $-\infty$ to 1. In another paper[6] we have studied the non-linear stability of the same libration points in the presence of the third and the fourth order resonances.

In the present paper we have studied the linear stability and its range for the various admissible values of the eccentricity of the elliptic orbit. We have also shown that resonances of the third and the fourth order will exist under our range of linear stability. So for the study of non-linear stability we shall have to take into consideration the resonance as well as non-resonance cases. Our present paper may be claimed to be a generalization over our mentioned papers in the sense that we consider here elliptic restricted problem and over Markeev's work[10] in the sense that we have here assumed both of the gravitating bodies to be radiating as well as where Markeev has taken them only gravitating. We have not investigated the existence of the various libration points in detail. However, the existence of three collinear and two triangular libration points can be shown. The present paper studies the linear stability of the triangular libration points only. A similar study for the collinear libration points will be interesting.

The present problem which was first taken up by Radziavskii[13] was studied in detail by Simmons *et al.*[14]. They remained restricted to the circular case and contented themselves with the linear stability only. Realising its far-reaching consequences we took up the investigation of the problem in full detail. The results are of immediate importance in stellar dynamics, but it is not less important for the solar system. Regarding the two reduction factors to be arbitrary parameters, the problem reduces to one where the gravitational forces may be taken to vary from the classical case ($\alpha = \beta = 1$) to the trivial case ($\alpha = \beta = 0$). Negative values for α and β can be available only for stellar system. Thus we find that the results are equally applicable to the solar system as well as to the stellar system. The circular restricted problem is only a first approximation of the problem. The natural problem is the elliptic restricted problem. To us it appears that the solution will be of far-reaching consequences applicable to solar as well as to stellar system.

2. Coordinates of the Triangular Libration Points

Let us refer our coordinates to Nechvil's coordinate system $(C, \xi\, \eta\, \zeta)$ (Duboshin[3]). We shall choose the sum of the two finite masses for the unit mass, the unit of time so that the constant of gravitation $k^2 = 1$ and for the unit length, the parameter p of the elliptic orbit. Then the equations of motion may be written as

$$\left.\begin{aligned} \frac{d\xi}{dv} &= \frac{\partial H}{\partial p_\xi}, \quad \frac{d\eta}{dv} = \frac{\partial H}{\partial p_\eta}, \quad \frac{d\zeta}{dv} = \frac{\partial H}{\partial p_\zeta} \\ \frac{dp_\xi}{dv} &= -\frac{\partial H}{\partial \xi}, \quad \frac{dp_\eta}{dv} = -\frac{dH}{\partial \eta}, \quad \frac{dp_\zeta}{\partial \eta} = -\frac{dH}{\partial \zeta} \end{aligned}\right\} \qquad \ldots(1)$$

where

$$H = \frac{1}{2}\left(p_\xi^2 + p_\eta^2 + p_\zeta^2\right) + p_\xi\,\eta - p_\eta\,\xi + \frac{e\cos v}{2(1 + e\cos v)}(\xi^2 + \eta^2 + \zeta^2) - \frac{W}{1 + e\cos v}$$

$$W = \alpha\,(1 - \mu)/r_1 + \beta\,\mu/r_2$$

$$r_1 = \sqrt{(\xi + \mu)^2 + \eta^2 + \zeta^2}\,, \quad r_2 = \sqrt{(\xi + \mu - 1)^2 + \eta^2 + \zeta^2}$$

μ, $1-\mu$ = the masses of the finite bodies ($0 < \mu < \frac{1}{2}$),

α, β = the reduction factors due to the radiation pressure exerted by the two finite bodies,

$(-\mu, 0, 0)$ = the coordinates of the larger finite body,

$(1-\mu, 0, 0)$ = the coordinates of the smaller finite body.

For the coordinates of the libration points, we consider the equations

$$\frac{\partial H}{\partial p_\xi} = \frac{\partial H}{\partial p_\eta} = \frac{\partial H}{\partial p_\zeta} = \frac{\partial H}{\partial \xi} = \frac{\partial H}{\partial \eta} = \frac{\partial H}{\partial \zeta} = 0 \qquad \text{...(2)}$$

whence we find that $r_1 = \alpha^{1/3}$, $r_2 = \beta^{1/3}$, is a solution of the equations. If L_4 and L_5 be the corresponding libration points, then clearly they are seen to form triangels with the finite masses for the other vertices. For convenience, if we put $\alpha = \delta_1^3$ and and $\beta = \delta_2^3$ and $(\xi_{Li}, \eta_{Li}, \zeta_{Li}, p_{\xi_{Li}}, p_{\eta_{Li}}, p_{\zeta_{Li}})$ $(i = 4, 5)$ be taken for the coordinates of Li $(i = 4, 5)$, then

$$\xi_{L_4} = \xi_{L_5} = \frac{\delta_1^2 + 1 - \delta_2^2}{2} - \mu, \quad p_{\xi_{L_4}} = -p_{\xi_{L_5}} = -\delta_1 \delta_2 \sqrt{b}$$

$$\eta_{L_4} = \eta_{L_5} = \delta_1 \delta_2 \sqrt{b}, \quad p_{\eta_{L_4}} = p_{\eta_{L_5}} = \frac{\delta_1^2 + 1 - \delta_2^2}{2} - \mu$$

$$\zeta_{L_4} = \zeta_{L_5} = 0, \quad p_{\zeta_{L_4}} = p_{\zeta_{L_5}} = 0$$

$$b = 1 - \left(\delta_1^2 + \delta_2^2 - 1\right)^2 \Big/ 4\,\delta_1^2 + \delta_2^2 .$$

Different from the classical case[3] here the triangles will be ordinary triangles whose sides will be of lengths δ_1, δ_2 and 1.

3. Characteristic Roots and the First Order Stability

We shall restrict our study to the planar case alone. Since L_5 is symmetrical to L_4, the nature of motion near L_5 will be the same as near L_4, so we shall consider the motion near L_4 alone. Taking (q_i, p_i) $(i = 1, 2)$ for the variations in the coordinates of L_4, the variational equations may be written as

$$\frac{dq_i}{dv} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dv} = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2) \qquad \text{...(3)}$$

where

$$H = \sum_{k=0}^{\infty} H_k, \quad H_0 = \text{constant}, \quad H_1 = 0$$

$$H_2 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + p_1 q_2 - q_1 p_2 + \frac{e\cos v}{2(1+e\cos v)}\left(q_1^2 + q_2^2\right)$$
$$+ \frac{q_1^2}{2(1+e\cos v)}\left[1 - \frac{3}{4}\left\{(1-\mu)\,\xi^2\,\delta_1^{-2} + \mu\,\eta^2\,\delta_2^{-2}\right\}\right]$$
$$- \frac{3\sqrt{b}\,q_1 q_2}{2(1+e\cos v)}\left[(1-\mu)\,\xi\,\delta_2\,\delta_1^{-1} + \mu\,\eta\,\delta_1\,\delta_2^{-1}\right]$$
$$- \frac{q_2^2}{2(1+e\cos v)}\left[3b\left\{(1-\mu)\,\delta_2^2 + \mu\,\delta_1^2\right\} - 1\right] \qquad \ldots(4)$$

$$H_3 = \frac{1}{1+e\cos v}\left[\frac{q_1^3}{1b}\left\{(1-\mu)\,\xi\,(5\xi^2 - 12\delta_1^2)\,\delta_1^{-4}\right.\right.$$
$$\left. + \mu\,\eta\,(5\eta^2 - 12\delta_2^2)\,\delta_2^{-4}\right\} + \frac{3}{8}\,q_1^2\,q_2\sqrt{b}\left\{(1-\mu)\,\delta_2\,\delta_1^{-3}\,(5\xi^2 - 4\delta_1^2)\right.$$
$$\left. + \mu\,\delta_1\,\delta_1^{-3}\,(5\eta^2 - 4\delta_2^2)\right\} + \frac{3}{4}\,q_1\,q_2^2\left\{\xi\,\delta_1^{-2}\,(1-\mu)\,(5b\,\delta_2^2 - 1)\right.$$
$$\left. + \eta\,\delta_2^{-2}\,\mu\,(5b\,\delta_1^2 - 1)\right\} + \frac{1}{2}\,q_2^3\sqrt{b}\left\{(1-\mu)\,\delta_2\,\delta_1^{-1}\,(5\,\delta_2^2\,b - 3)\right.$$
$$\left.\left. + \mu\,\delta_1\,\delta_2^{\,1}\,(5\,\delta_1^2\,b - 3)\right\}\right]. \qquad \ldots(5)$$

$$H_4 = \frac{1}{1+e\cos v}\left[-\frac{1}{8}\,q_1^4\left\{(1-\mu)\,\delta_1^{-6}\left(3\,\delta_1^4 - \frac{15}{2}\xi^2\,\delta_1^2 + \frac{35}{16}\,\xi^4\right)\right.\right.$$
$$\left. + \mu\,\delta_2^{-6}\left(3\,\delta_2^4 - \frac{15}{2}\,\eta^2\,\delta_2^2 + \frac{35}{16}\,\eta^4\right)\right\}$$
$$+ \frac{5}{4}\,q_1^3\,q_2\sqrt{b}\left\{(1-\mu)\,\delta_2\,\delta_1^{-3}\,\xi\left(3 - \frac{7}{4}\,\xi^2\,\delta_1^{-2}\right)\right.$$
$$\left. + \mu\,\delta_1\,\delta_2^{-3}\,\eta\left(3 - \frac{7}{4}\,\eta^2\,\delta_2^{-2}\right)\right.$$
$$+ \frac{5}{4}\,q_1\,q_2^3\sqrt{b}\left\{(1-\mu)\;\xi\,\delta_2\,\delta_1^{-3}\left(3 - 7\,\delta_2^2\,b\right)\right.$$
$$\left. + \mu\,\eta\,\delta_1\,\delta_2^{-3}\left(3 - 7\,\delta_1^2\,b\right)\right\}$$
$$+ \frac{3}{4}\,q_1^2\,q_2^2\left\{(1-\mu)\,\delta_1^{-2}\left(-1 + 5\,\delta_2^2\,b + \frac{5}{4}\,\xi^2\,\delta_1^{-2}\right.\right.$$
$$\left. - \frac{35}{4}\;\xi^2\,\delta_1^{-2}\,\delta_2^2\,b\right) + \mu\,\delta_2^{-2}\left(-1 + 5\,\delta_1^2 + \frac{5}{4}\,\eta^2\,\delta_2^{-2}\right.$$
$$\left.\left. - \frac{35}{4}\,\eta^2\,\delta_1^2\,\delta_2^{-2}\,b\right)\right\} - \frac{q_1^4}{8}\left\{(1-\mu)\,\delta_1^{-2}\left(3 - 30b\,\delta_2^2 + 35\,\delta_2^4\,b^2\right)\right.$$

(*equation continued on p. 407*)

$$+ \mu \, \delta_2^{-2} \left(3 - 30b \, \delta_1^2 + 35 \, \delta_1^4 b^2 \right) \Big\} \Big] . \qquad \ldots(6)$$

H_k = the sum of the terms of the kth degree homogeneous in the variables q_1, q_2, p_1 and p_2

$$\xi = \delta_1^2 + 1 - \delta_2^2 \quad \text{and} \quad \eta = \delta_1^2 - 1 - \delta_2^2 .$$

To find the characteristic roots we shall follow Bennet's method[1] where he expands λ, H_2 as well as the general solution of the first order variational equation in powers of e and also uses

$$(1 + e \cos v)^{-1} = 1 - e \cos v + e^2 \cos^2 v - e^3 \cos^3 v + \ldots\ldots + (-1)^r e^r \cos^r v + \ldots\ldots \qquad \ldots(1)$$

If the general solution be put as

$$y = y^{(0)} + y^{(1)} e + y^{(2)} e^2 + \ldots\ldots\ldots \qquad \ldots(8)$$

then equating the coefficients of different powers of e on the two sides of the variational equation, we get differential equations for $y^{(0)}$, $y^{(1)}$, $y^{(2)}$ and so on and they can be solved successively. Now putting these solutions in the differential equations for $y^{(0)}$, $y^{(1)}$, $y^{(2)}$ and so on, the coefficients $\lambda^{(0)}$, $\lambda^{(1)}$, $\lambda^{(2)}$ are calculated. which may be given as

$$\lambda^{(1)} = \pm \left[\tfrac{1}{2} \pm \tfrac{1}{2} \{ 1 - 36 \, \mu \, (1 - \mu) \, b \}^{1/2} \right] \qquad \ldots(9)$$

$$\lambda^{(1)} = 0 \quad \text{and} \quad \lambda^{(2)} = \frac{\alpha \, \{\lambda^{(0)}\}^2 + \beta}{\gamma \, \{\lambda^{(0)}\}^2 + \delta} \, \lambda^{(0)}$$

where α, β, γ and δ are constants depending on μ.

In order that the roots (9) may be purely imaginary, it will be necessary that

$$36 \, \mu \, (1 - \mu) \, b \leqslant 1 \Leftrightarrow \mu \leqslant 0.0285954 \qquad \ldots(10)$$

$\mu = 0.0285954\ldots\ldots$ corresponds to the resonance case with equal frequencies for $b = 1$ and $e = 0$. This limit coincides with the limit of stability found by Lanzano[7]. It may be noted for distinct δ_1 and δ_2 we shall have different ranges of stability given by the different values of μ for $e = 0$ shown by us in our earlier paper[6].

Lukyanov[8] uses Bennet's method as well as he adopts another method by expanding $1/(1 + e \cos v)$ in Fourier series given as

$$1/(1 + e \cos v) = \frac{1}{\sqrt{1 - e^2}} (1 + 2 \, \epsilon \cos v + 2 \, \epsilon^2 \cos 2v + \ldots\ldots)$$

where

$$\epsilon^r = \frac{1}{\pi}\int_0^\pi \cos rv\, d\, v/(1 + e\cos v)$$

and then proceeding similar to Bennet he calculates the coefficients $\bar{\lambda}^{(0)}$, $\bar{\lambda}^{(1)}$, $\bar{\lambda}^{(2)}$ and so on, of the charasteristic exponent $\bar{\lambda}$. By calculating the characteristic exponents by the two methods, he finally compares the ranges of the linear stability, obtained by the two methods and he shows that the ranges coincide with those found by Danby[4]. If $\bar{\lambda}^{(0)}$ be the characteristic root corresponding to the case $\epsilon = 0$, then

$$\bar{\lambda}^{(0)} = \pm\left[-\frac{1}{2}\left(4 - \frac{3}{\sqrt{1-e^2}}\right) \pm \frac{1}{2}\left\{\left(4 - \frac{3}{\sqrt{1-e^2}}\right)^2 - \frac{36\,\mu\,(1-\mu)\,b}{3-e^2}\right\}^{1/2}\right]^{1/2} \quad \ldots(11)$$

whence we find that for purely imaginary value of $\bar{\lambda}^{(0)}$

$$e < 0.66144\ldots\ldots \text{ and } (4 - 3/\sqrt{1-e^2})^2 \geqslant 36\,\mu\,(1-\mu)\,b/(1-e^2). \quad \ldots(12)$$

It gives

$$\mu \leqslant \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - (25 - 16\,e^2 - 24\sqrt{1-e^2})/36\,b}. \quad \ldots(13)$$

The equality corresponds to resonance cases with equal frequencies for different values of b. We shall not study these cases in further details. The values of μ denoted by $\mu\,(e)$ basing on the range of vales of b varying from 0 to 1 are given in Table I.

4. Normalization of the Hamiltonian Function H_2

Taking into view our subsequent studies we shall need the normalization of H_2 given by (4). For convenience we shall write H_2 in the form

$$H_2 = H_2^{(0)} + H_2^{(1)} \quad \ldots(14)$$

where

$$H_2^{(0)} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + p_1\,q_2 - q_1\,p_2 + \frac{1}{2}\,q_1^2\,(1 - A) - \frac{1}{2}\,B\,q_1\,q_2 + \frac{1}{2}\,q_2^2\,(1 - C) \quad \ldots(14a)$$

$$H_2^{(1)} = (e\cos v/2\,(1 + e\cos v))\left(A\,q_1^2 + B\,q_1\,q_2 + C\,q_2^2\right) \quad \ldots(14b)$$

TABLE I

The values of μ (e)

b	e	$\mu(e)$	b	e	$\mu(e)$	b	e	$\mu(e)$	b	e	$\mu(e)$
0.00	00—										
	0.7	imaginary	.3	.3	0.0659684	.6	00	0.0486645	.8	.5	0.00753561
0.05	00—07	,,	.3	.4	0.0429195		.1	0.046633		.6	0.00139082
.1	00	,,	.3	.6	0.203579		.2	0.0407785		.7	0.00071481
.1	.1	,,	.3	.6	0.0037175		.3	0.0318208	.9	00	0.0318805
.1	.2	0.376283	.3	.7	0.0019084		.4	0.0209788		.1	0.0305737
.1	.3	0.244755	.4	00	0.0750817		.5	0.0100732		.2	0.026795
1	.4	0.143956	.4	.1	0.0007185		.6	0.00185529		.3	0.00209989
.1	.5	0.0639155	.4	.2	0.0625911		.7	0.00095330		.4	0.0138852
.1	.6	0.011237	.4	.3	0.0485716	.7	00	0.0413961		.5	0.00669264
.1	.7	0.005747		.4	0.0318206		.1	0.03968		.6	0.00123609
.2	00	0 1667		.5	0.0151883		.2	0.03473		.7	0.000635
.2	.1	0 158496		.6	0.0278553		.3	0.02714	.1	00	0.0285954
.2	.2	0.135784		.7	0.00143063		.4	0.017918		.1	0.0274273
.2	.3	0.1030426	.5	00	0.0590414		.5	0.00862		.2	0.0240476
.2	.4	0.0659679		.1	0.0565477		.6	0.00159		.3	0.0188399
.2	.5	0.0308681		.2	0 0493768		.7	0.000817		.4	0.0124789
.2	.6	0.0058676		.3	0.0384482	.8	00	0.0360196		.5	0.0060192
.2	.7	0.0028654		.4	0.0252858		.1	0.0345365		.6	0.0011123
.3	00	0.1032539		.5	0.0121128		.2	0.0302519		.7	0.000571
.3	.1	0.0986483		.6	0.0022272		.3	0.0236663			
.3	.2	0.0853556		.7	0.00114428		.4	0.0156489			

$$\left.\begin{aligned} A &= \frac{3}{4}\left\{(1-\mu)\,\xi^2\,\delta_1^{-2} + \mu\,\eta^2\,\delta_2^{-2}\right\} \\ B &= \frac{3\sqrt{b}}{\delta_1\,\delta_2}\left\{\delta_2^2\,\xi\,(1-\mu) + \delta_1^2\,\eta\,\mu\right\} \\ C &= 3b\left\{(1-\mu)\,\xi_2^2 + \mu\,\delta_1^2\right\}. \end{aligned}\right\} \qquad \ldots(15)$$

For normalisation we shall introduce the transformation

$$(q_1, q_2, p_1, p_2) = \left(q_1', q_2', p_1', p_2'\right) N \qquad \ldots(16)$$

where N is the same as used in Manju and Choudhry[9].

Here

$$\omega_1^2 = -\left\{\lambda_{1,2}^{(0)}\right\}^2 \text{ and } \omega_2^2 = -\left\{\lambda_{3,4}^{(0)}\right\}^2$$

we shall call ω_1 and ω_2 as the frequencies and they are given by the equation

$$\omega^4 - \omega^2 + 9\mu(1-\mu)\,b = 0. \quad \text{...(17)}$$

The transformation (16) reduces the Hamiltonian (14) to the form

$$H_2^1 = \frac{1}{2}\left(p_1^{12} + \omega_1^2 q_1^{12}\right) - \frac{1}{2}\left(p_2^{12} + \omega_2^2 q_2^{12}\right) + \frac{e\cos v}{1+e\cos v}\sum_{v+\mu=2} a_{v\mu}^1\, q_1^{1v_1}\, q_2^{1v_2}\, p_1^{1\mu_1}\, p_2^{1\mu_2} \quad \text{...(18)}$$

where $v = (v_1, v_2)$, $\mu = (\mu_1, \mu_2)$ and v_1, v_2, μ_1 and μ_2 so vary that $v + \mu = v_1 + v_2 + \mu_1 + \mu_1 = 2$ and for simplicity we shall not write the ranges again and the coefficients $a'_{v\mu}$ are given as follows :

$$a'_{2000} = \frac{1}{2}a_1^2\left(A + B c_1 + C c_1^2\right),\quad a'_{0200} = \frac{1}{2}a_2^2\left(A + B c_2 + C c_2^2\right)$$

$$a'_{0200} = \frac{1}{2}C a_1^2 b_1^2,\quad a'_{0002} = \frac{1}{2}C a_2^2 b_2^2$$

$$a'_{1100} = a_1 a_2\left\{A + \frac{1}{2}B(c_1 + c_2) + C c_1 c_2\right\}$$

$$a'_{1010} = a_1^2 b_1\left(\tfrac{1}{2}B + C c_1\right),\quad a'_{1001} = -a_1 a_2 b_2\left(\tfrac{1}{2}B + C c_1\right)$$

$$a'_{0110} = a_1 a_2 b_1\left(\tfrac{1}{2}B + C c_2\right),\quad a'_{0011} = -a_1 a_3 b_1 b_2 C$$

$$a'_{0101} = -a_2^2 b_2\left(\tfrac{1}{2}B + C c_2\right).$$

Next we shall introduce the following transformations :

$$q'_i = \frac{1}{\sqrt{\omega_i}}\tilde{q}_i,\quad p'_i = \sqrt{\omega_i}\,\tilde{p}_i\ (i = 1, 2) \quad \text{...(19)}$$

which transform $H_2^{\prime(0)}$ to $\tilde{H}_2^{(0)}$ given as

$$\tilde{H}_2^{(0)} = \tfrac{1}{2}\omega_1\left(\tilde{p}_1^2 + \tilde{q}_1^2\right) - \tfrac{1}{2}\omega_2\left(\tilde{p}_2^2 + \tilde{q}_2^2\right) \quad \text{...(20)}$$

and $H_2^{\prime(1)}$ to $\tilde{H}_2^{(1)}$ given as

$$H_2^{(1)} = \frac{e\cos v}{1+e\cos v}\sum_{v+\mu=2} a_{v\mu}\,\tilde{q}_1^{v_1}\,\tilde{q}_2^{v_2}\,\tilde{p}_1^{\mu_1}\,\tilde{p}_2^{\mu_2} \quad \text{...(21)}$$

where $\tilde{a}_{\nu\mu}$ can be easily calculated. For the convenience of subsequent calculations we shall introduce the complex conjugate canonic variables given as

$$q_j'' = \tilde{p}_j + i\,\tilde{q}_j,\ p_j'' = \tilde{p}_j - i\,\tilde{q}_j\ (j = 1, 2). \qquad \text{...(22)}$$

Consequently the Hamiltonian will be reduced to the form $H_2'' = 2i\,\tilde{H}_2$

where

$$H_2'' = i\,\omega_1\,q_1''\,p_2'' - i\,\omega_2\,q_2''\,p_2''$$
$$+ 2i\,\frac{e\cos v}{1 + e\cos v}\sum_{\nu+\mu=2} a_{\nu\mu}''\,g_1^{''\nu_1}\,q_2^{''\nu_2}\,p_1^{''\mu_1}\,p_2^{''\mu_2} \qquad \text{...(23)}$$

The coefficients in H_2'' are such that $a_{\nu\mu}'' = \bar{a}_{\mu\nu}''$

where the bar sign denotes the complex conjugate quantity. Other coefficients are given as follows :

$$a_{2000}'' = \tfrac{1}{4}\,(-\,\tilde{a}_{2000} + \tilde{a}_{0020} - i\,\tilde{a}_{1010})$$

$$a_{0200}'' = \tfrac{1}{4}\,(-\,\tilde{a}_{0200} + \tilde{a}_{0002} - i\,\tilde{a}_{0101})$$

$$a_{1100}'' = \tfrac{1}{4}\,(-\,\tilde{a}_{1100} + \tilde{a}_{0011} - i\,\tilde{a}_{1001} - i\,\tilde{a}_{0110})$$

$$a_{1001}'' = \tfrac{1}{4}\,(-\,\tilde{a}_{1100} + \tilde{a}_{0011} - i\,\tilde{a}_{1001} - i\,\tilde{a}_{0110})$$

$$a_{1010}'' = \tfrac{1}{2}\,(\tilde{a}_{2000} + \tilde{a}_{0020}),\quad \tilde{a}_{0101}'' = \tfrac{1}{2}\,(\tilde{a}_{0200} + \tilde{a}_{0002}). \qquad \text{.. (24)}$$

Next we shall find the transformation

$$\left(q_j'', p_j''\right) \rightarrow \left(q_j^{**}, p_j^{**}\right) \qquad \text{...(25)}$$

reducing the Hamiltonian (23) to the normal form in complex conjugate variables given as

$$H_2^{*}\left(q_j^{**}, p_j^{**}\right) = i\,\lambda_1\,q_1^{**}\,p_1^{**} + i\,\lambda_2\,q_2^{**}\,p_2^{**}. \qquad \text{...(26)}$$

Let this transformation be given by the generating function

$$q_1''\,p_1^{**} + p_2''\,q_2^{**} + S\left(q_1'', q_2'', p_1^{**}, p_2^{**}, v\right)$$

where

$$S = \sum_{\nu+\mu=2} S_{\nu\mu}\, q_1''^{\nu_1}\, q_2''^{\nu_2}\, p_1^{**\mu_1}\, p_2^{**\mu_2}$$

and $s_{\nu\mu}$ are to be chosen 2π-periodic functions of ν. The relation between the variables q_j'', p_j'' and q_j^{**}, p_j^{**} are given as

$$q_j^{**} = q_j'' + \frac{\partial s}{\partial p_j^{**}}\,, \quad p_j'' = p_j^{**} + \frac{\partial s}{\partial p_j''}$$

whence we the identity

$$H_2^{**}\left(q_j'' + \frac{\partial s}{\partial p_j^{**}}\,,\, p_j^{**}\,,\, \nu \right) - H_2''\left(q_j''\,,\, p_j^{**} + \frac{\partial s}{\partial p_j''}\,,\, \nu \right) = \frac{\partial s}{\partial \nu}\,.$$

On expanding and equating the coefficients of equal powers on the two sides, we shall get

$$H_2^{**}\left(q_j'', p_j'', \nu\right) + \sum_{j=1}^{2} \frac{\partial s}{\partial p_j^{**}} \frac{\partial H_2^{**}}{\partial q_j''} + \frac{1}{2}\left[\left(\frac{\partial s}{\partial p_1^{**}}\right)^2 \frac{\partial^2 H_2}{\partial q_1''^2}\right.$$

$$\left. + 2\frac{\partial s}{\partial p_1^{**}} \frac{\partial s}{\partial p_2^{**}} \frac{\partial^2 H_2^{**}}{\partial q_1''\, \partial q_2''} + \left(\frac{\partial s}{\partial p_2^{**}}\right)^2 \frac{\partial^2 H_2^{**}}{\partial q_2''^2}\right]$$

$$- H_2''\left(q_j'', p_j^{**}, \nu \right) - \sum_{j=j}^{2} \frac{\partial s}{\partial q_j''} \frac{\partial H_2''}{\partial p_j^{**}}$$

$$- \frac{1}{2}\left[\left(\frac{\partial s}{\partial q_1''}\right)^2 \frac{\partial^2 H_2''}{\partial p_1^{**2}} + 2\left(\frac{\partial s}{\partial q_1''}\right)\left(\frac{\partial s}{\partial q_2''}\right)\right.$$

$$\left. \times \frac{\partial^2 H_2''}{\partial p_1^{**}\, \partial p_2^{**}} + \left(\frac{\partial s}{\partial q_2''}\right)^2 \frac{\partial^2 H_2^{**}}{\partial p_2^{**2}}\right]$$

$$= \sum_{\nu+\mu=2} \frac{ds_{\nu\mu}}{d\nu}\, q_1''^{\nu_1}\, q_2''^{\nu_2}\, p_1^{**\mu_1}\, p_2^{**\mu_2} \qquad \ldots(27)$$

whence we get

$$i\,\lambda_1\,q_1''\,p_1^{**} + i\,\lambda_2\,q_2''\,p_2^{**} + i \sum_{\nu+\mu=2} (\mu_1\,\lambda_1 + \mu_2\,\lambda_2)\,s_{\nu\mu}\,q_1^{''\nu_1}\,q_2^{''\nu_2}$$

$$\times\, p_1^{**\mu_1}\,p_2^{**\mu_2} - i\,\omega_1\,q_1''\,p_1^{**} + i\,\omega_2\,q_2''\,p_2^{**}$$

$$-\,2i\,\frac{e\cos\nu}{1+e\cos\nu}\sum_{\nu+\mu=2} a''_{\nu\mu}\,q_1^{''\nu_1}\,q_2^{''\nu_2}\,p_1^{**\mu_1}\,p_2^{**\mu_2}$$

$$-\sum_{\nu+\mu=2}(\nu_1\,\omega_1 - \nu_2\,\omega_2)\,s_{\nu\mu}\,q_1^{''\nu_1}\,q_2^{''\nu_2}\,p_1^{**\mu_1}\,p_2^{**\mu_2}$$

$$= \sum_{\nu+\mu=2}\frac{ds_{\nu\mu}}{dv}\,q_1^{''\nu_1}\,q_2^{''\nu_2}\,p_1^{**\mu_1}\,p_2^{**\mu_2}\,. \qquad \ldots(28)$$

Restricting only upto the second order terms in e, we may write (28) as

$$i\,\lambda_1\,q_1''\,p_1^{**} + i\,\lambda_2\,q_2''\,p_2^{**} + i\sum(\mu_1\,\lambda_1 + \mu_2\,\lambda_2)\,[e\,s^{(1)} + e^2\,s^{(2)})$$

$$-\,i\,\omega_1\,q_1''\,p_1^{**} + i\,\omega_2\,q_2''\,p_2^{**} - 2i\left[e\cos\nu - \frac{e^2}{2}(1+\cos 2\nu)\right]$$

$$\times\sum a''_{\nu\mu}\,q_1^{''\nu_1}\,q_2^{''\nu_2}\,p_1^{**\mu_1}\,p_2^{**\mu_2}$$

$$-\,i\,\Sigma\,(\nu_1\,\omega_1 - \nu_2\,\omega_2)\,(e\,s^{(1)} + e^2\,s^{(2)})$$

$$= e\,\frac{ds^{(1)}}{dv} + e^2\,\frac{ds^{(2)}}{dv} \qquad \ldots(29)$$

where

$$s_{\nu\mu} = e\sum s^{(1)}_{\nu\mu} + e\sum s^{(2)}_{\nu\mu}\,.$$

Let us write

$$\left.\begin{aligned}\lambda_1 &= \lambda_1^{(0)} + e\,\lambda_1^{(1)} + e^2\,\lambda_1^{(2)} + \ldots\\ \text{and}\quad \lambda_2 &= \lambda_2^{(0)} + e\,\lambda_2^{(1)} + e^2\,\lambda_2^{(2)} + \ldots\end{aligned}\right\} \qquad \ldots(30)$$

and equate the coefficients of the equal powers in e, then we shall have

$$\lambda_1^{(0)} = \omega_1,\ \lambda_2^{(0)} = \omega_2,\ i\,\lambda_1^{(1)} + i\,\lambda_2^{(1)} + i\sum\left(\mu_1\,\lambda_1^{(0)} + \mu_2\,\lambda_2^{(0)}\right)s^{(1)}$$

$$-\,2i\cos\nu\,a''_{\nu\mu} - i\sum(\nu_1\,\omega_1 - \nu_2\,\omega_2) = \frac{ds^{(1)}}{dv} \qquad \ldots(31)$$

whence we shall have the following relations.

$$\left.\begin{aligned}
\frac{ds^{(1)}_{1010}}{dv} &= i\,\lambda^{(1)}_1 + i\,\lambda^{(0)}_1\, s^{(1)}_{1010} - 2i\cos v\; a''_{1010} - i\,\omega_1\, s^{(1)}_{1010} \\
\frac{ds^{(1)}_{0101}}{dv} &= i\,\lambda^{(1)}_2 + i\,\lambda^{(0)}_2\, s^{(1)}_{0101} - 2i\cos v\; a''_{0101} - i\,\omega_2\, s^{(1)}_{0101} \\
\frac{ds^{(1)}_{\nu\mu}}{ds} &+ i\,[(\nu_1 - \mu_1)\,\omega_1 - (\nu_2 - \mu_2)\,\omega_2]\; s^{(1)}_{\nu\mu} = -\,2i\cos v\, a''_{\nu\mu}
\end{aligned}\right\} \quad \ldots(32)$$

On integration, we shall have

$$\begin{aligned}
s^{(1)}_{1010} &= i\,\lambda^{(1)}_1\; v - 2i\sin v\; a''_{1010} \\
s^{(1)}_{0101} &= -\,2i\sin v\; a''_{0101} + i\,\lambda^{(1)}_2\; v \\
s^{(1)}_{\nu\mu} &= \frac{2i\, a''_{\nu\mu}\,[\sin v + i\,\{(\nu_1 - \mu_1)\,\omega_1 - (\nu_2 - \mu_2)\,\omega_2\}\cos v]}{[(\nu_1 - \mu_1)\,\omega_1 - (\nu_2 - \mu_2)\,\omega_2]^2 - 1}.
\end{aligned} \quad \ldots(33)$$

By virtue of periodicity of $s^{(1)}_{1010}$ and $s^{(1)}_{0101}$ it follows that $\lambda^{(1)}_1 = \lambda^{(1)}_2 = 0$ and the relations (33) completely determine S as a complex-valued function restricted to the first order terms in e alone.

Now let us find a real-valued transformation $(\tilde{q}_j, \tilde{p}_j) \rightarrow (q_j^*, p_j^*)$ reducing the Hamiltonian $\tilde{H}_2 = \tilde{H}^{(0)}_2 + \tilde{H}^{(1)}_2$ to the normal form given as

$$H_2 = \tfrac{1}{2}\lambda_1\,(q_1^{*2} + p_1^{*2}) + \tfrac{1}{2}\,\lambda_2\,(q_2^{*2} + p_2^{*2}) \quad \ldots(34)$$

Let this transformation be given by means of the generating function $\tilde{q}_1\; p_1^* + \tilde{q}_2\; p_2^* + K(\tilde{q}_j, p_j^*, v)$, where K is restricted to the order of e alone. From the transformation formula

$$q_j^* = \tilde{q}_j + \frac{\partial K}{\partial p_j^*},\quad \tilde{p}_j = p_j^* + \frac{\partial K}{\partial \tilde{q}_j^*}$$

we may obtain by implicit function theorem and using the fact that all the variables are small

$$\tilde{q}_j = q_j^* - \frac{\partial K}{\partial p_j^*},\quad \tilde{p}_j = p_j^* + \frac{\partial K}{\partial q_j^*} \quad \ldots(35)$$

where K^* will be obviously of the order of e.

By the formula (26) correct to the order of e we have

$$q_j'' = q_j^{**} - \frac{\partial S^{**}}{\partial p_j^{**}} \quad p_j'' = p_j^{**} + \frac{\partial S^{**}}{\partial q_j^{**}} \qquad \ldots(36)$$

where $S^{**} = S^{(1)}(q^{**}, p^{**}, v)$. Further taking into account the relation between the complex canonic variables with the real ones

$$q_j'' = \tilde{p}_j + i\tilde{q}_j, \quad p_j'' = \tilde{p}_j - i\tilde{q}_j$$

$$q_j^* = p_j^* + iq_j^*, \; p_j^* = p_j^* - iq_j^*$$

and denoting the function $S^{(1)}(p^* + iq^*, p^* - iq^*, v)$ by $W(q^*, p^*, v)$, we shall obtain by the formula (36),

$$\tilde{q}_j = q_j^* - \frac{1}{2i}\frac{\partial W}{\partial p_j^*}, \; \tilde{p}_j = p_j^* + \frac{1}{2i}\frac{\partial W}{\partial q_j^*}. \qquad \ldots(37)$$

Comparing (35) and (37), we obtain $K^* = \frac{1}{2}w\ldots$ ...(38)

and the function $K = \sum k_{\nu_\mu} q_1^{*\nu} q_2^{*\nu} p_1^{*\mu} p_2^{*\mu 2}$ will be realvalued. With the help of the formulae (4b), (19) and (33) its coefficients may be found which are given as follows.

$$\begin{aligned}
k_{2000} &= \frac{1}{2i}\left(-s_{2000}^{(1)} - s_{0020}^{(1)} + s_{1010}^{(1)}\right)\\
k_{0200} &= \frac{1}{2i}\left(-s_{0200}^{(1)} - s_{0002}^{(1)} + s_{0101}^{(1)}\right)\\
k_{0020} &= \frac{1}{2i}\left(s_{2000}^{(1)} + s_{0020}^{(1)} + s_{1010}^{(1)}\right)\\
k_{0002} &= \frac{1}{2i}\left(s_{0200}^{(1)} + s_{0002}^{(1)} + s_{0101}^{(1)}\right)\\
k_{1100} &= \frac{1}{2i}\left(s_{1100}^{(1)} + s_{1001}^{(1)} + s_{0110}^{(1)} - s_{0011}^{(1)}\right)\\
k_{1010} &= s_{2000}^{(1)} - s_{0020}^{(1)}\\
k_{1001} &= \tfrac{1}{2}\left(s_{1100}^{(1)} + s_{1001}^{(1)} - s_{0110}^{(1)} - s_{0011}^{(1)}\right)\\
k_{0110} &= \tfrac{1}{2}\left(s_{1100}^{(1)} - s_{1001}^{(1)} + s_{0110}^{(1)} - s_{0011}^{(1)}\right)\\
k_{0101} &= s_{0200}^{(1)} - s_{0002}^{(1)}\\
k_{0011} &= \frac{1}{2i}\left(s_{1100}^{(1)} + s_{1001}^{(1)} + s_{0110}^{(1)} + s_{0011}^{(1)}\right).
\end{aligned} \qquad \ldots(93)$$

Thus the transfarmation of the Hamiltonian H_2 to the normal form given by (34) correct to the first order of eccentricity has been found. It is obtained through the transformations (16), (19) and (35) and the coefficients of the generating function K are given by (39).

5. Resonance Cases

Taking into view the complete study of the stability of the triangular libration points we shall need an investigation if resonances are present. Since we aim to apply KAM-theorem for stability which needs mainly the third and the fourth order terms, we shall only examine it the resonances of the third and the fourth order exist.

If a curve giving resonances be plotted for different values of e, then we shall need the value of λ_1 and λ_2 at least correct to $O(e^2)$, since $\lambda_1^{(1)} = \lambda_2^{(1)} = 0$. The quantities $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ are found by the periodicity conditions of the functions $s_{1010}^{(2)}$ and $s_{0101}^{(2)}$. For it let us take up the expansion (27) and here equating the coefficients with e^2, we shall get

$$\frac{ds_{1010}^{(2)}}{dv} = -2i\cos v\left(4a''_{0020}\, s_{2000}^{(1)} + a''_{1010}\, s_{1010}^{(1)} + a''_{1001}\, s_{1001}^{(1)} + a''_{0110}\, s_{1100}^{(1)}\right) + 2i\cos^2 v\, a''_{1010} + i\lambda_1^{(2)}$$

$$\frac{ds_{0101}^{(2)}}{dv} = -2i\cos v\left(4a''_{0002}\, s_{0200}^{(1)} + a''_{0110}\, s_{1001}^{(1)} + a''_{0101}\, s_{0101}^{(1)} + a''_{0011}\, s_{1100}^{(1)}\right) + 2i\cos^2 v\, a''_{0101} + i\lambda_2^{(2)}.$$

Substituting into the right-hand side expressions of these equations the value of the functions $s_{\mu\nu}^{(1)}$ given by (33) and choosing $\lambda_1^{(2)}$ and $\lambda_1^{(2)}$ such that the constant terms on the right-hand side are equal to zero (the condition of periodicity of $s_{1010}^{(2)}$ and $s_{0101}^{(2)}$) we shall obtain after some manipulations using the formulae (14b), (18) and the equation

$$\omega^4 - \omega^2 + 9\mu(1-\mu)b = 0$$

the following expressions for $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ and obtained.

$$\lambda_1^{(2)} = -\frac{\omega_1 \omega_2 (6\omega_1^2 - 7)}{4(4\omega_1^2 - 1)(2\omega_1^2 - 1)},$$

$$\lambda_2^{(2)} - \frac{\omega_2 \omega_1^2 (6\omega_2^2 - 7)}{4(4\omega_2^2 - 1)(2\omega_2^2 - 1)}. \qquad \ldots(40)$$

These values coincide in forms with Markeev's[10] values.

Let the value of μ giving the resonance $k_1 \lambda_1 + k_2 \lambda_2 = N$ for small e and correct to $O(e^2)$ be given as $\mu = \mu^{(0)} + e^2 \mu^{(2)}$ where $\mu^{(0)}$ is the value of μ when $e = 0$ and $\mu^{(2)}$ is the contribution to the value of μ when $e \neq 0$ and it is considered correct to $O(e^2)$. Letting $\lambda_1 = \lambda_1(\mu^{(0)} + e^2 \mu^{(2)})$ and $\lambda_2 = \lambda_2(\mu^{(0)} + e^2 \mu^{(2)})$, we have on expansion by Taylor's theorem that

$$\lambda_1 = \lambda_1^{(0)} + e^2 \lambda_1^{(2)} + e^2 \mu^{(2)} \left(\frac{d\lambda_1}{d\mu}\right)_0$$

and similarly,

$$\lambda_2 = \lambda_2^{(0)} + e^2 \lambda_2^{(2)} + e^2 \mu^{(2)} \left(\frac{d\lambda_2}{d\mu}\right)_0$$

where $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ are given by (40) and the suffix (0) denotes that the value is to be taken when $\mu = \mu^{(0)}$. Putting these values in $k_1 \lambda_1 + k_2 \lambda_2 = N$ and equating the coefficient of e^2 to zero, we shall have

$$\mu^{(2)} = \frac{k_1 \lambda_1^{(2)} + k_2 \lambda_2^{(2)}}{k_2 \frac{d\omega_2}{d\mu} - k_1 \frac{d\omega_2}{d\mu}}.$$

The value of $\mu^{(2)}$ is calculated on putting $\mu = \mu^{(0)}$ on the right-hand side. The values of $\mu^{(0)}$ and $\mu^{(2)}$ for the different resonances of the third order are given in Tables II and III. In Table IV we have denoted by $\mu(e^2)$ the values of μ giving the limit for the linear stability for different values of b and e under the resonance case $3\lambda_2 = -1$. The values have been calculated on the formula $\mu = \mu^{(0)} + e^2 \mu^{(2)}$. e has been taken to vary from $e = 0.0$ to $e = 0.7$. We have here compared the values of $\mu(e)$ and $\mu(e^2)$ for the resonance under reference and we have examined if the resonance exists for a given value of b and e and if the study of stability will be necessary in the particular resonance case when higher order variational terms are taken into consideration. The conclusions are listed in the succeeding section.

We shall examine the following resonances of the third order if they exist under our range of linear stability given by Table 3.1 :

TABLE II

The value of $\mu^{(0)}$

b	$3\lambda_2 = -1$ $\mu^{(0)}$	$\lambda_1 + 2\lambda_2 = 0$ $\mu^{(0)}$	$2\lambda_1 + \lambda_2 = 1$ $\mu^{(0)}$	$\lambda_1 - 2\lambda_2 = 2$ $\mu^{(0)}$	$3\lambda_2 = -2$ $\mu^{(0)}$
.1	0.1255	0.23126	imaginary	imaginary	imaginary
.2	0.0583	0.09861	0.15071	0.15071	0.1641
.3	0 03803	0.06326	0.09421	0.09421	0.1018
.4	0.02823	0.04662	0.06872	0.06872	0.07407
.5	0.02245	0.03692	0.05413	0.0513	0.05826
.6	0.01864	0.03056	0.04466	0.04466	0.04303
.7	0.01593	0.02608	0.03802	0.03802	0.04086
.8	0.01391	0.02274	0.033095	0.033095	0.03556
.9	0.01235	0.02016	0.029303	0.029303	0.03147
1	0.0111	0.01811	0 026291	0.026291	0.02823
.75	0.014853	0 24294	0.0353854	0.0353854	0.038026
$\mu^{(0)}$	$\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{8}{925b}}$	$\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{4}{225b}}$	$\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{16}{625b}}$	$\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{16}{625b}}$	$\mu^{(0)} = \frac{1}{2} - \sqrt{\frac{1}{4} \sim \frac{20}{729b}}$

TABLE III

The values of $\mu^{(2)}$

b	$3\lambda_2 = -1$	$\lambda_1 + 2\lambda_2 = 0$	$2\lambda_1 + \lambda_2 = 1$	$\lambda_1 - 2\lambda_2 = 2$	$3\lambda_2 = -2$
.1	−0.83513	−3.8097591	imaginary	imaginary	imaginary
.2	−0.35404	−1.2733528	0.51445	1.5010664	0.568121
.3	−0.22567	− 0.780191	0.295214	0.8613774	0.3198794
4	−0.165736	−0.56367	0.2083243	0.6078504	0.2242903
.5	−0.130984	−0.441488	0.1612059	0.470368	0.1730103
.6	−0.10829	−0.3629225	0.1315443	0.3838212	0.140912
.7	−0.09290	−0.308136	0.111132	0.3242611	0.1188955
8.	−0.080427	−0.267732	0.0962145	0.2807356	0.1028464
9.	−0.071262	−0.236704	0.084835	0.2475324	0.09621
1	−0.063972	−0.212128	0.075866	0.2213626	0.0809987
$\mu^{(2)}$	$\mu^{(2)} = -\frac{1596}{25515b(1-2\mu)}$	$\mu^{(2)} = -\frac{138}{675b(1-2\mu)}$	$\mu^{(2)} = \frac{0.0718769}{b(1-2\mu)}$	$\mu^{(2)} = \frac{0.209723}{b(1-2\mu)}$	$\mu^{(2)} = \frac{0.0764256}{b(1-2\mu)}$

(i) $3\lambda_2 = -1$, (ii) $\lambda_1 + 2\lambda_2 = 0$, (iii) $2\lambda_2 + \lambda_2 = 1$,

(iv) $\lambda_1 - 2\lambda_2 = 2$, (v) $3\lambda_2 = -2$.

The corresponding values are given by Tables IV—VIII.

Coming to resonances of the fourth order we may proceed similarly and it may be seen that the study of the stability will be needed for the following resonance.

(i) $4\lambda_2 = -1$, (ii) $\lambda_1 + 3\lambda_2 = 0$, (iii) $\lambda_1 - 3\lambda_2 = 2$,

(iv) $2(\lambda_1 + \lambda_2) = 1$, (v) $3\lambda_1 + \lambda_2 = 2$, (vi) $3\lambda_1 - \lambda_2 = 3$,

(vii) $\lambda_1 + 3\lambda_2 = -1$, (viii) $4\lambda_1 = 3$.

TABLE IV

The values for $\mu\ (e^2)$

				$(3\lambda_2 = -1)$				
b	$e = 00$	$e = 0.1$	$e = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.7$
0.1	0.1255	0.117	0.092	0.05	negative	—ve	—ve	—ve
0.2	0.058	0.055	0.044	0.026	0.002	—ve	—ve	—ve
0.3	0.038	0.036	0.029	0.018	0.002	—ve	—ve	—ve
0.4	0.028	0.027	0.022	0.013	0.002	—ve	—ve	—ve
0.5	0.022	0.021	0.017	0.011	0.001	—ve	—ve	—ve
0.6	0.019	0.018	0.014	0.009	0.001	—ve	—ve	—ve
0.7	0.016	0.015	0.012	3.008	0.001	—ve	—ve	—ve
0.8	0.014	0.013	0.011	0.007	0.001	—ve	—ve	—ve
0.9	0.012	0.012	0.009	0.006	0.001	—ve	—ve	—ve
1.0	0.012	0.010	0.009	0.005	0.001	—ve	—ve	—ve

—ve = negative

TABLE V

The values for $\mu\ (e^2)$

				$(\lambda_1 + 2\lambda_2 = 0)$				
b	$e = 0.0$	$e = 0.1$	$c = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.8$
0.1	0.231	0.193	0.079	—ve	—ve	—ve	—ve	—ve
0.2	0.099	0.0859	0.048	—ve	—ve	—ve	—ve	—ve
0.3	0.063	0.055	0.032	—ve	—ve	—ve	—ve	—ve
0.4	0.047	0.041	0.024	—ve	—ve	—ve	—ve	—ve
0.5	0.037	0.033	0.019	—ve	—ve	—ve	—ve	—ve
0.6	0.031	0.027	0.016	—ve	—ve	—ve	—ve	—ve
0.7	0.026	0.023	0.014	—ve	—ve	—ve	—ve	—ve
0.8	0.023	0.020	0.012	—ve	—ve	—ve	—ve	—ve
0.9	0.020	0.018	0.017	—ve	—ve	—ve	—ve	—ve
1.0	0.018	0.016	0.001	—ve	—ve	—ve	—ve	—ve

As shown by Moser[2], the motion under the resonances $\lambda_1 - 3\lambda_2 = 2$ and $3\lambda_1 - \lambda_2 = 3$ cannot lead to instability and the cases $\lambda_1 + 3\lambda_2 = 0$ and $\lambda_1 + 3\lambda_2 = -1$ reduce to the case $\lambda_1 + 3\lambda_2 = m$ (an integer). Thus it remains to study the stability only for the following five cases :

$$\text{(i) } 4\lambda_2 = m, \quad \text{(ii) } \lambda_1 + 3\lambda_2 = m, \quad \text{(iii) } 2(\lambda_1 + \lambda_2) = m,$$

$$\text{(iv) } 3\lambda_1 + \lambda_2 = m, \quad \text{(v) } 4\lambda_1 = m,$$

where m is an integer.

TABLE VI

The values for μ (e^2)

$(2\lambda_1 + \lambda_2 = 1)$

b	$e = 0.0$	$e = 0.1$	$e = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.7$
0.1	Imag.	Imag.	Imog.	Imag.	Imag.	Imag.	Imag.	Imag.
0.2	0.151	0.156	0.017	0.197	0.233	0.279	0.339	0.403
0.3	0.094	0.097	0.106	0.121	0.141	0.168	0.200	0.239
0.4	0.069	0.071	0.077	0.087	0.102	0.121	0.144	0.171
0.5	0.054	0.056	0.061	0.686	0.080	0.094	0.112	0.133
0.6	0.045	0 046	0.050	0.066	0.066	0.078	0.092	0.109
0.7	0.038	0.039	0.042	0.048	0.056	0.066	0.078	0.092
0.8	0.033	0.034	0.037	0.042	0.048	0.057	0.068	0.081
0.9	0.029	0.030	0.033	0.037	0.043	0.051	0.060	0.070
1.0	0.026	0.027	0.029	0.033	0.036	0.045	0.054	0.063

Imag. = Imaginary

TABLE VII

The values for μ (e^2)

$(\lambda_1 - 2\lambda_2 = 2)$

b	$e = 0.0$	$e = 0.1$	$e = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.7$
0.1	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.
0.2	0.151	0.166	0.211	0.286	0.391	0.526	0.691	0.886
0.3	0.094	0.103	0.129	0.172	0.232	0.340	0.404	0.516
0.4	0.069	0.0748	0.093	0.123	0.166	0.221	0.288	0.367
0.5	0.054	0.0588	0.0729	0.0965	0.129	0.1717	0.2235	0.2846
0.6	0.045	0.048	0.060	0.079	0.106	0.141	0.183	0.233
0,7	0.038	0.0413	0.051	0.0672	0.09	0.1191	0.1548	0.1969
0.8	0.033	0.0368	0.0443	0.0623	0.085	0.1142	0.1498	0.192
0.9	0.029	0.032	0.0392	0.0515	0 0689	0.0912	0.1184	0.1506
1.0	0.026	0.0285	0.035	0.046	0.062	0.082	0.106	0.1348

Imag. = Imaginary

TABLE VIII

The values for u (e²)

				(3 = − 2)				
b	$e = 0.0$	$e = 0.1$	$e = 0.2$	$e = 0.3$	$e = 0.4$	$e = 0.5$	$e = 0.6$	$e = 0.7$
0.1	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.	Imag.
0.2	0.104	0.170	0.187	0.215	0.255	0.306	0.369	0.443
0.3	0.102	0.105	0.115	0.131	0.153	0.182	0.217	0.259
0.4	0.074	0.077	0.083	0.094	0.110	0.130	0.155	0.184
0.5	0.058	0.060	0.065	0.074	0.086	0.102	0.121	0.143
0.6	0.048	0.049	0.054	0.061	0.071	0.083	0.092	0.117
0.7	0.041	0.042	0.046	0.052	0.060	0.071	0.084	0.099
0.8	0.036	0.037	0.040	0.045	0.052	0.061	0.072	0.086
0.9	0.031	0.032	0.035	0.040	0.047	0.056	0.066	0.079
1.0	0.028	0.029	0.331	0.036	0.041	0.048	0.057	0.068

Imag. = Imaginary

6. CONCLUSIONS

In section 3 we found out the characteristic exponents correct to $O(e^2)$ and we have calculated the values of μ giving the range of linear stability for different values of e. These values denoted by $\mu(e)$ have been given by Table I. In section 4 we have normalized the second order terms by Birkhoff's transformations and also we have calculated the various terms as resulting after transformations. In section 5 we have found the values of μ corresponding to different types of resonances. e has been taken to vary from 0.0 to 0.7 and b from 0.1 to 1.0. From Table IV we find that the resonances of the type of $3\lambda_2 = -1$ exists for $e < 0.5$ and in all such cases the motion is stable in the linear sense since the corresponding values of $\mu(e)$ is greater than $\mu(e^2)$. So for the investigation of non-linear stability this type of resonance case has to be taken into account. The negative values of $\mu(e^2)$ indicate that the resonance case will not exist for our range of values of $\mu(e)$ under consideration.

Table V shows the resonance case $\lambda_1 + 2\lambda_2 = 0$ exists for $b = 0.1$ only for values of e such that $0.1 < e < 0.3$, and for $b \geqslant 0.2$ it exists for $e \leqslant 0.2$. Since the values of $\mu(e^2)$ corresponding to the resonance are all less than the value $\mu(e)$ corresponding to the range of linear stability, for non-linear stability the study of the resonance case will be necessary.

Table VI shows that the resonance case $2\lambda_1 + \lambda_2 = 1$ will not exist for $b = 0.1$ and for $b \geqslant 0.2$, the resonance exists for all e from 0.0 to 0.7, but the the linear stability is possible only for $e < 0.2$, and for $e \geqslant 0.2$ the motion will be unstable since even the linear stability does not hold.

Table VII shows that for $b = 0.1$ the resonance $\lambda_1 - 2\lambda_2 = 2$ does not exist but for values of b it does exist. The values of $\mu\ (e^2)$ are greater than the corresponding values $\mu\ (e)$ for the linear stability. It shows that the motion can be stable only for the values of e such that $e < 0.1$, and for $e > 0.1$ the motion will be unstable.

Table VIII shows that in the resonance case $3\lambda_2 = -2$, the motion can be stable only for $e < 0.1$, and for $e > 0.1$ the motion will be unstable since $\mu\ (e) < \mu\ (e^2)$ for all b and all e.

Similarly resonances of the fourth order will exist for some values of b and e and for non-linear stability these cases have to be taken into account. We have not calculated the corresponding values of $\mu\ (e^2)$, but it is guessed that the resonances of the fourth type will exist, similar to the classical case as examined by Markeev[10].

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