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ARTINIAN (NOETHERIAN) PART OF A GOLDIE RING

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An Artinian ring with unity (a Noetherian ring) is always a Goldie ring. But the converse in not true. Here we discuss what can be thought as an Artinian (Noetherian) part of a Goldie ring and get some interesting results, mainly on right (Left) Artinian radical (i.e. the sum of all the right (left) ideals which are Artinian as right (left) R-modules) of a Goldie ring R.

1. INTRODUCTION

We know that an Artinian ring (with unity) is a Noetherian ring and a Noetherian ring is always a Goldie ring. But the converse is not true. For example the commutative integral domain e.g. $Z[X_i, i = 1, 2, ...]$, where $X_iX_j = X_jX_i$, is a Goldie ring but it is not Noetherian. However a Goldie ring may have a part which behaves as an Artinian (Noetherian) ring or an Artinian (Noetherian) module. Here, unless otherwise specified, a ring R will be a ring with unity.

A ring R satisfies 'right essential descending chain condition (r.e.d.c.c.)' if any descending chain of right essential ideals stops after a finite number of steps. Similarly the 'left essential descending chain condition (l.e.d.c.c)' may be defined. All finite rings and right (left) Artinian rings are rings with r.e.d c.c. (l.e.d.c.c.).

The sum $A(R)$ of all the Artinian right ideals of R is the 'right Artinian radical' of R (An Artinian right ideal means a right ideal of R which is Artinian as a right R-module). We have $A(R) = R$ if and only if R is a right Artinian ring and $A(R) = 0$ if and only if R has no Artinian right ideal. Similarly the 'left Artinian radical' $B(R)$ of R is defined. $A(R)$ and $B(R)$ are ideals of R. When $A(R) = B(R)$ $(= A)$ and A is Artinian ideal then A is called the 'Artinian radical of R. If R is left and right Noetherian then $A(R) = B(R)$ (page 59, Chatters and Hazarnavis[2]).

A right ((left), two sided) ideal of R is a 'countable right' (left, two sided) ideal if it is countable as a set. All the rings which are countable as sets have countable ideals. For example, let

$$F = Z/2Z \text{ and } R = \begin{bmatrix} Z & F \\ O & Z \end{bmatrix}.$$

Here R is left and right Noetherian and N is the set of strictly upper triangular elements of R. Then $A(R) = B(R) = N$ which is countable.

He we prove that if a minimal ideal I of a semiprime right Goldie ring is Artinian as a left R-module then it is a right module over a simple Artinian extension of an epimorphic image of R. Another result we prove here is : if R is a semiprime right Goldie ring satisfying the r.e.d.c.c. and a minimal ideal I of R is Noetherian as a right R-module then I is Artinian as a right R-module.

The ring $R = \begin{bmatrix} Z & Q \\ O & Q \end{bmatrix}$ is right Noetherian (Hence it is right Goldie). The right ideals $e_{12}\, Q$, $e_{22}\, Q$ of R are Artinian (in fact minimal). Now it follows that $A(R) = e_{12}\, Q + e_{22}\, Q$ and $A(R)$ is a minimal ideal of R (Moreover it is countable). Here we prove that in a semiprime right and left Goldie ring satisfying the r.e.d.c.c. and l.e.d.c.c. if $A(R)$ and $B(R)$ are minimal ideals and $A(R)$ is Noetherian as an R-module then $A(R) = B(R)$.

Other two interesting results we prove here are :

(1) In a semiprime fully left Goldie ring R if its minimal prime ideals are Noetherian as R-modules, then in some special cases, $A(R) + r\,(A(R))$ contains a regular element.

(2) If R is a semiprime fully left and fully right Goldie ring such that its minimal prime ideals are Noetherian as R-modules and its Artinian radical A is minimal and countable then A is a direct summand of R.

2. Preliminaries

A right (left) Goldie ring R is 'fully right (left) Goldie' if any homomorphic image of R is right (left) Goldie.

2.1. We now prove :

Lemma 2.1.1 — If I is a right ideal of a semiprime ring R where $R/r\,(I)$ is Artinian ring, then I is Artinian as a right R-module.

Proof : Since $R/r\,(I)$ is a right Artinian ring, it is Artinian as a right $R/r\,(I)$—module. So the submodule $I/r\,(I)$ is Artinian as a right $R/r\,(I)$—module.

If $P_1 \supseteq P_2 \supset \ldots$, is a descending chain of R-submodules of I, then $Q_1 \supseteq Q_2 \supseteq \ldots$ is a descending chain of $R/r\,(I)$ submodules of $I/r\,(I)$, where

$$Q_i = \{p_i + r\,(I) \mid p_i \in P_i\}. \text{ So } Q_t = Q_{t+1} = , \ldots, \text{ for some } t \in Z^+$$

R being semiprime, for any right ideal J ($\subseteq I$) we get $J \cap r\,(I) = 0$. And hence $P_t = P_{t+1} = \ldots$ Thus I is Artinian as a right R-module.

Lemma 2.1.2 — A semiprime right (left) Goldie ring R satisfying the r.e.d.c.c. (l.e.d.c.c.) is a quotient ring.

PROOF : Since R is semiprime right Goldie, it has a regular element, (say) $d \in R$ [Theorem 1.10 of Chatters and Hazarnavis[2]]. Then each of $dR, d^2R, d^3R, \ldots$ is right essential ideal of R (Lemma 1.11 of Chatters and Hazarnavis[2]) and R being with r.e.d.c.c. the descending chain $dR \supseteq d^2R \supseteq d^3R \supseteq \ldots$ gives $d^tR = d^{t+1}R$, for some $t \in Z^+$. Hence $R = dR$ which gives $d^{-1} \in R$. Thus R is a quotient ring. (Similarly for left Goldie ring).

If P is an ideal of R, then from the natural epimorphism $\nu : R \to R/P$ we get that for any right (left) essential ideal X of R/P, $\nu^{-1}(X) = I$ is a right (left) essential ideal of R (Goodearl[3], Proposition 1.1). [It is to be noted that a finite integral domain is a field follows from this Lemma easily]. Now we get

Lemma 2.1.3 — If R is a ring satisfying the r.e.d.c.c. (l.e.d.c.c.) and P an ideal of R, then the ring R/P also satisfies the r.e.d.c c. (l.e.d.c.c.).

From Lemma 2.1.2 and from the Lemma 12, Appendix B of Jacobson[4] we get :

Lemma 2.1.4 — If R is a semiprime right Goldie ring satisfying the r.e.d.c.c. and P is any annihilator prime ideal of R, then R/P is a quotient ring.

Now we prove the following :

Lemma 2.1.5 — For any minimal ideal I of a semiprime right Goldie ring R if $P = r(I)$ then

$$\begin{aligned} Z(I) &= \{x \in I \mid xc = 0 \text{ for some regular } c + P \in R/P\} \\ &= 0. \end{aligned}$$

PROOF : For $x, y \in Z(I)$ we get $c + P, d + P$ regular in R/P such that $xc=0$, $yd = 0$. And I being minimal, R/P is prime Goldie. So by Proposition 1.11 of Chatters and Hazarnavis[2], $(c + P) R/P$ and $(d + P) R/P$ are right essential ideals of R/P and so is their intersection $(c + P) R/P \cap (d + P) R/P$. Therefore it contains a regular element, say $e + P$. Thus $e + P = (c + P)(u + P) = (d + P)(v + P)$ for some $u, v \in R$, which gives $e - cu, e-dv \in P\ (= r(I))$. So $I(e-cu) = I(e-dv)=0$, or, $x(e - cu) = y(e - dv) = 0$, for $x, y \in I$. Therefore $xe = ye = 0$ (since $xc = yd = 0$) which gives $(x - y) e = 0$. Since $e+P$ is regular in R/P, $x-y \in Z(I)$. Again for $r \in R$,

$$\begin{aligned} (r + P)^{-1}((c + P) R/P) &= \{u + P \mid (r + P)(u + P) \in (c + P) R/P\} \\ &= \{u + P \mid ru + P \in (c + P) R/P\} \end{aligned}$$

is a right essential ideal of R/P (Goodearl[3], Proposition 1.19d). Therefore it contains a regular element, say $f + P$. Now $rf + P \in (c + P) R/p$ gives $rf - cv \in P$ for some $v \in R$. So $I(rf - cv) = 0$ which gives $xrf = 0$ (since $xc = 0$) or $xr \in Z(I)$. Thus

$Z(I)$ is an ideal of R. Moreover if $x \in Z(I)$, $(x + P)(c + P) = P$ for some regular $c + P$, which implies $x \in P$ or $Ix = 0$. Hence $Z(I)x \ (\subseteq Ix) = 0$. It therefore follows that $(Z(I))^2 = 0$. R being semiprime, we finally get $Z(I) = 0$.

Lemma 2.1.6 — If a minimal ideal of a semiprime right Goldie ring R is Artinian as a left R-module, then for any regular $d + P \in R/P$ (where $P = r(I)$) we get $I = Id$.

PROOF : Let $d + P$ be regular in R/P. Since I is Artinian as a left R-module, the chain.

$Id \supseteq Id^2 \supseteq \ldots$ stops after a finite number of steps. Let $Id^t = Id^{t+1}$, for some $t \in Z^+$. So for each $x \in I$ there is some $y \in I$ such that $(x - yd)\, d^t = 0$. Since $d^t + P$ is regular in R/P, it therefore follows that $x - yd \in Z(I)$. But $Z(I) = 0$ (Lemma 2.1.5). Hence $x = yd$ which gives $I \subseteq Id$. Hence $I = Id$.

2.2. *Lemma* 2.2.1 — (a) If I is a countable right ideal of a left Goldie ring, then there is a finite subset S of I such that $r(S) = r(I)$.

(b) Let R be a left Goldie ring and I a countable ideal of R. If I is Artinian as a right R-module then $R/r(I)$ is a right Artinian ring.

PROOF : (a) Consider $y_1 \in I$. Then $r\{y_1\} \supseteq r(I)$. Choose $y_2 \in I$, $y_2 \neq y_1$. Then $r\{y_1\} \supseteq r\{y_1, y_2\}$. Thus we get a descending chain of right annihilators,

$$r\{y_1\} \supseteq r\{y_1, y_2\} \supseteq r\{y_1, y_2, y_3\} \supseteq \ldots$$

The ring being left Goldie, the above chain stops after a finite number of steps[2] (page 2). Suppose $r\{y_1, \ldots, y_n\} = r\{y_1, \ldots, y_{n+1}\} = \ldots = r(I)$. Thus $S = \{y_1, \ldots, y_n\}$ such that $r(I) = r(S)$.

(b) From above $r(I) = r(y_1) \cap \ldots \cap r(y_n)$.

Hence by 4.5 (a) of Chatters and Hazarvanis[2], the module $R/r(I)$ is embedded in the module $y_1 R \oplus \ldots \oplus y_n R$ as a right R-module. Since I is Artinian as a right R-module and for all i, $y_i R \subseteq I$ we have that $y_i R$ is Artinian as a right R-module. Hence $R/r(I)$ is Artinian as a right R-module which implies that $R/r(I)$ is Artinian as a right $R/r(I)$ — module. Thus $R/r(I)$ is a right Artinian ring.

Lemma 2.2.2 — If the ideal in Lemma 2.2.1 (b) is a minimal ideal of R then the ring $R/r(I)$ is an Artinian ring.

PROOF : Since I is minimal, $r(I)$ is a prime ideal. Therefore $R/r(I)$ is a prime ring. Hence it is simple (ex. 14 (10 (4)) of Anderson and Fuller[1]). So it is simple left Artinian (Proposition 13.5 of Anderson and Fuller[1]). Thus $R/r(I)$ is (simple), Artinian.

2.3. *Lemma* 2.3.1 — Let R be a semiprime right Goldie ring. Then for an annihilator ideal I containing an ideal U, there is an ideal X such that $IX \subseteq U$.

PROOF : Consider the collection $\{J\}$ of all the annihilator ideals of R contained in I and containing U. Since R is right Goldie, the collection has a maximal element, say I_2 such that $I = I_1 \supseteq I_2 \supseteq U$.

Similarly we get annihilator ideals I_3, I_4, ... etc. such that $I = I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and each of these contains U. Since R is semiprime right Goldie, this descending chain is finite (Corollary 1.15 of Goodreal[3]) and so

$$I = I_1 \supseteq I_2 \supseteq \ldots \supset I_{n-1} \supseteq U = I_n.$$

Now if

$$r\,(I_j/I_{j+1}) = P_j/I_{j+1}, \text{ then } I_jP_j \subseteq I_{j+1}. \text{ Set } X = P_1P_2 \ldots P_{n-1}.$$

Then,

$$IX = (I_1P_1)\,P_2 \ldots P_{n-1} \subseteq (I_2P_2) \ldots P_{n-1}.$$

Thus

$$IX \subseteq U.$$

We now use the corresponding results of the above Lemma in the following

Lemma 2.3.2 — Let R be a semiprime fully left Goldie ring and B is an annihilator ideal. If C is an ideal of R then there is an ideal Y of R such that $YB \subseteq BC$.

PROOF : Since $BC \subseteq B$, by Lemma 2.3.1 we get an ideal Y of R such that $YB \subseteq BC$.

From Corollary 4.4 of Chatters and Hazarnavis[2] we see Goldie rings such that if R/C in above is Artinian then R/Y is Artinian.

We now here consider semiprime fully left Goldie ring R where, in above Lemma if R/C is Artinian then R/Y is Artinian.

Moreover as in the same Corollary the Goldie ring R has minimal prime ideals which are Noetherian as R-modules.

Thus we prove :

Lemma 2.3.3 — Let R be a semiprime fully left Goldie ring as above and is such that P is a minimal prime ideal which is Neotherian as an R-module and R/P is Artinian. Then the right Artinian radical $A\,(R)$ of R is not contained in P.

PROOF : Let T be the set of all ideals I of R such that R/I is Artinian. And R being semiprime Goldie it has a finite number of minimal prime ideals each of which is an annihilator ideal, say these are $P_1, \ldots, P_k$. So $P = P_i$ for some i. If $P_i \in T$ for all i, then from the monomorphism

$$R \rightarrow R/P_1 \oplus \ldots \oplus R/P_k \text{ (since } P_1 \ldots P_k = 0).$$

We get R as Artinian and hence right Artinian. So $A(R) = R$. Hence $A(R) \nsubseteq P$ (since P is a prime ideal).

Suppose for some j, $P_j \notin T$. If an annihilator ideal B and an ideal C of R be such that $B \notin T$ and $C \in T$, then by Lemma 2.3.2 there is a $D \in T$ such that $DB \subseteq BC$. Thus if $P_2 \in T$, $P_1 \notin T$ then there exists $Q_1 \in T$ such that $Q_1P_1 \subseteq P_1P_2$. Thus we have $Q_1P_1P_3 \ldots P_k \subseteq P_1P_2P_3 \ldots P_k = 0$. In this way P_j ($\notin T$) can be moved to the right in the product $P_1P_2 \ldots P_k$ until we obtain $XY=0$, where X is a product of elements of T and Y is a product of minimal prime ideals ($\notin T$).

Let $X = Q_1Q_2 \ldots Q_t$, $Q_i \in T$. Then as in Corollary 4.4 of Chatters and Hazanavis[2] R/X is Artinian. Thus $X \in T$.

Now let $Y = A_1 \ldots A_s$, where each A_i is a minimal prime ideal not contained in T. Since P is prime, $Y \notin P$.

Let $Y \supseteq Y_1 \supseteq Y_2 \supseteq \ldots$ be a descending chain of right ideals. Then this gives a chain $S \supseteq S_1 \supseteq S_2 \supseteq \ldots$ of right ideals of R/X, where $S_i = \{y_i + X \mid y_i \in Y_i\}$. Since $XY = 0$ we get $X \cap Y = 0$. (Since R is semiprime). Thus $X \cap Y_i = 0$, for all i. Since R/X is Artinian we get $S_t = S_{t+1}$ for some $t \in Z^+$. So $Y_t = Y_{t+1}$. Thus Y is Artinian as a right R-module. Therefore $Y \subseteq A(R)$. Hence $A(R) \subseteq P$.

Theorem 3.1 — If a minimal ideal I of a semiprime right Goldie ring R is Artinian as a left R-module, then it is Artinian as a right module over a simple Artinian extension of an epimorphic image of R.

PROOF : Since R/P is prime Goldie, the quotient ring Q of R/P is simple Artinian (Chatters and Hazarnavis[2], Proposition 1.28). By Lemma 2.1.6, since $I = Id$, we get $I + P = (I + P)(d + P)$ or, $I + P = (I + P)(d + P)^{-1}$, for each regular $d + P \in R/P$. It can easily be seen that if $I/P = \{i + P \mid i \in I\}$, then the map $I/P \times Q \rightarrow I/P$

$$(x + P, (r + P)(d + P)^{-1}) \rightarrow (xr + P)(d + P)^{-1}$$

makes I/P a right Q-module.

Now consider the map $I \times Q \rightarrow I$

$$(i, q) \rightarrow X,$$

where the image x is given by the condition

$$(i + P)\, q = x + P.$$

We first show that x is unique.

If for $x, y \in I$, $x + P = y + P$, then $x - y \in P$ $(= r(I))$.

Therefore $x - y \in I \cap r(I)$.

Since R is semiprime, $I \cap r(I) = 0$ which gives $x = y$.

And if $i, j \in I$ and $q \in Q$ let $(i + P)q = x + P$ and $(j + P)q = y + P$.

Then

$$((i + j) + P)q = ((i + P) + (j + P))q$$
$$= (i + P)q + (j + P)q \text{ (since } I/P \text{ is a right } Q\text{-module)}$$
$$= (x + P) + (y + P)$$
$$= (x + y) + P.$$

Thus

$$(i + j)q \, (= x + y) = iq + jq.$$

Similarly for $q, 1 \in Q, i \in I$ we get

$$i\,(q1) = (iq)1.$$

So I is a right module over the simple Artinian ring Q which is an extension as a ring of an epimorphic image R/P of R.

Theorem 3.2 — Let R be a semiprime Goldie ring satisfying the r.e.d.c.c. If minimal ideal I of R is Noetherian as a right R-module then I is Artinian as a right R-module.

Proof : If $P = r\,(I)$ then the ring R/P is prime Goldie and by Proposition 1.28 of Chatters and Hazarnavis[2] the quotient ring Q of R/P is simple Artinian, and the well defined map

$$I \; X \; R/P \rightarrow I$$
$$(i, r + P) \rightarrow ir$$

makes I a right R/P-module (Since $P = r\,(I)$). By Lemma 2.1.2, $R/P = Q$. Therefore I is a right module over the simple Artinian ring ($Q =$) R/P. Since I is Noetherian as a right R-module it is Noetherian as a right R/P module. (For any R-submodule of I is an R/P-submodule and vice-versa). Therefore I is finitely generated as a right module over the simple Artinian ring R/P. It follow that I is Artinian as a right R/P-module (Proposition 10.18 of Anderson and Fuller[1]). And hence I is Artinian as a right R-module.

We have already seen Goldie ring whose right Artinian radical is minimal and countable. Thus we consider semiprime fully left Goldie ring R where $A(R)$ is minimal, countable and Artinian as a right R-module. We now prove

Theorem 3.3 — Let R be a semiprime fully left Goldie ring as above and its minimal prime ideals are Noetherian as R-modules. Then $A(R) + r(A(R))$ contains a regular element.

Proof : Suppose for any minimal prime ideal P, $P \supseteq r\,(A(R))$. Then $P \supseteq A(R) + r(A(R))$. Since $A(R)$ is minimal, countable and Artinian as a right R-module, and $P \supseteq r(A(R))$ gives $P = r(A(R))$, by Lemma 2.2.2, R/P is Artinian which gives by Lemma 2.3.3, $A(R) \subseteq P$. So $P \supseteq A(R) + r(A(R))$. Thus in any case $P \supseteq A(R) + r(A(R))$.

If S is a semiprime left Goldie ring and I is not contained in any minimal prime ideal of it then for any left ideal K of S with $I \cap K = 0$ we get $I(KS) = 0$ which gives $KS \subseteq P$ (for all P). But the intersection of all such P being zero, we get $K = 0$. Hence I is an essential left ideal of S. Therefore it contains a regular element. Thus $A(R) + r(A(R))$ contains a regular element.

Theorem 3.4 — Let R be a semiprime right and left Goldie ring satisfying the r.e.d.c.c. and l.e.d.c.c.

If both the right Artinian radical $A(R)$ and the left Artinian radical $B(R)$ are minimal ideals and $A(R)$ is Noetherian as a right R-module as well as a left R-module (for a right Noetherian and a left Noetherian ring there are so) then $A(R) = B(R)$ $(= A)$.

PROOF : Here $A(R)$ is a minimal ideal of R and is Noetherian as a right R-module. So by Theorem 3.2 $A(R)$ is Artinian as a right R-module. Since $A(R)$ is Noetherian as a left R-module and R is semiprime left Goldie with l.e.d.c.d., we have $A(R)$ is Artinian as a left R-module. So $A(R) \subseteq B(R)$. But $B(R)$ minimal gives us $A(R) = B(R)$.

Theorem 3.5 — Let R be a semiprime fully left Goldie ring with Noetherian minimal primes where the Artinian radical A is minimal and countable. Then A is a direct summand of R.

PROOF : By above $A + r(A)$ contains a regular element say, c. Then $c = a+x$, $a \in A$, $x \in r(A)$ and wet get $Ac \supseteq Ac^2 \supseteq \ldots$. Since A is an Artinian left ideal, Ac is also Artinian left ideal. Hence for some $t \in Z^+$, $Ac^t = Ac^{t+1}$. Then for $\alpha \in A$, we get $\beta \in A$ such that $\alpha = \beta c$ (since c is regular). Thus $A \subseteq Ac$. And A being largest Artinian left ideal of R we therefore get $A = Ac$. So $a = ec$, for some $e \in A$. Then $c = a + x = ec + x$ gives $e = e^2$. And for $b \in A$ we get $b =$ be which gives $A \subseteq \text{Re}$. So $A = \text{Re}$. Consider $J = \{x - ex \mid x \in A\}$. Then J is a right ideal and J^2 $(\subseteq AJ = \text{Re}\, J) = 0$. Since R is semiprime, it therefore implies that $J=0$. Thus $x=ex$ for $x \in A$. Hence $A = eA$ which gives $A \subseteq eR \subseteq A$, i.e. $A = \text{Re}$. Thus $A = Re = eR$. Therefore e is central and so $(1 - e)$ is in centre of R and the ideal $(1 - e)R$ is such that $R = A + (1 - e)R$ and since $er \cap (1 - e)R = 0$, A is a direct summand.

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