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# LACUNARY DISTRIBUTION OF SEQUENCES

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In this paper we considered a sequence of known spaces and its inclusion which enable us to define a new type of distribution of sequence called lacunary distribution.

§1. Let $\theta = (K_r)$ be the sequence of positive integers such that

(i) $K_\theta = 0$, and $0 < K_r < K_{r+1}$

(ii) $h_r = (K_r - K_{r-1}) \to \infty$, as $r \to \infty$.

Then $\theta$ is called a lacunary sequence. The intervals determined by $\theta$ are denoted by $I = (K_{r-1}\ K_r]$. The ratio $K_r/K_{r-1}$ will be denoted by $q_r$.

We use the following known spaces of sequences of real numbers.

$C_1 = \{x = (x_k)$ : there exists '$L$' such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (x_k - L) = 0\}.$$

$| C_1 | = \{x = (x_k)$ : there exists '$L$' such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} | x_k - L | = 0\}.$$

$C_\theta = \{x = (x_k)$ : there exists '$L$' such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{K\in I_r} (x_k - L) = 0\}.$$

$| C_\theta | = \{x = (x_k)$ : There exists '$L$' such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{K\in I_r} | x_k - L | = 0\}.$$

$AC = \{x = (x_k)$ : There exists $L$ such that, uniformly in $i \geqslant 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{K=0}^{n-1} (x_{k+1} - L) = 0\}.$$

$| AC | = \{x = (x_k)$ : There exists '$L$' such that, uniformly in $i \geqslant 0$.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{K=0}^{n-1} | x_{k+1} - L = 0.\}$$

$AC_\theta = \{x = (x_k)$ : There exists '$L$' such that, uniformly in $i \geqslant 0$.

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{K\in I_r} (x_{k+i} - L) = 0\}$$

$| AC_\theta | = \{x = (x_k)$ : There exists '$L$' such that, uniformly in $i \geqslant 0$

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{K\in I_r} | x_{k+i} - L | = 0\}.$$

It is evident that

$$| C_1 | \subset C_1, \ | C_\theta | \subset C_\theta, \ | AC | \subset AC \text{ and } | AC_\theta | \subset AC_\theta.$$

The space $C_1$ is the well known (c, 1) summable space, $| C_1 |$ is the strong summble (c, 1) space. The spaces $C_\theta$ and $| C_\theta |$ spaces of lacunary and lacunary strongly convergence have been recently introduced by Freedman *et al.*[2]. The well known space $AC$ the space of all almost convergent sequences and defined by Lorentz[3] and $| AC |$ the space of strongly convergence has been recently introduced by Maddox[4-6] and also independently by Freedman *et al.*[2]. The space $AC_\theta$ and $| AC_\theta |$ the space of lacunary almost convergence and space of lacunary strongly almostly convergence has been studied by Das and Misra[1].

We record some known results :

*Theorem A*[2]—(a) $| C_1 | \subset C_\theta$, if and only if

$$\varliminf_{r} q_r > 1; \qquad \text{...(1.1)}$$

(b) $| C_\theta | \subset C_1$, if and only if

$$\varlimsup_{r} q_r < \infty \qquad \text{...(1.2)}$$

(c) $| AC | \subset C_\theta |$, for every $\theta$;

(d) $C_1 \subset C_\theta$, if and only if

$$\underline{\lim_{r}}\ q_r > 1; \qquad \ldots(1.3)$$

(e) $C_\theta \neq C_1$, for every $\theta$;

(f) $C_\theta \cap l_\infty \subset C_1$, if and only if

$$\lim_{r} q_r = 1; \qquad \ldots(1.4)$$

where $l_\infty$ is the set of all bounded sequences.

(g) $AC \subset C_\theta \cap l_\infty$, for every $\theta$.

§2. In the sequel we have the following results :

*Theorem* 1— $| AC_\theta | \Leftrightarrow | AC |$ for every '$\theta$'.

We need the following Lemma for the proof of the theorem.

*Lemma* 1—Suppose, for given $\epsilon > 0$, $\exists\ n_0$ and $p_0$ such that

$$\frac{1}{n} \sum_{j=p}^{p+n-1} | x_j - L | < \epsilon \qquad \ldots(2.1)$$

for all $n \geqslant n_0$, $p \geqslant p_0$. Then $\{x_j\} \in | AC |$.

Proof : Let $\epsilon > 0$ be given. Choose $n_0^1$, $p_0$ such that

$$\frac{1}{n} \sum_{j=p}^{p+n-1} | x_j - L | < \epsilon/2 \qquad \ldots(2.2)$$

for all $n \geqslant n_0^1$ and $p \geqslant p_0$. It is enough to prove that $\exists\ n'^{1}_{0}$ such that for $n > n'^{1}_{0}$, $0 \leqslant p \leqslant p_0$

$$\frac{1}{n} \sum_{j=p}^{p+n-1} | x_j - L | < \epsilon. \qquad \ldots(2.3)$$

Since, taking $n_0 = \max (n_0^1, n'^{1}_{0})$, (2.3) will hold for $n \geqslant n_0$ and for all $p$, which gives the result.

Once $p_0$ has been chosen, $p_0$ is fixed, so

$$\sum_{j=0}^{p_0-1} | x_j - L | = M \text{ (say)}. \qquad \ldots(2.4)$$

Now, taking $0 \leqslant p \leqslant p_0$ and $n > p_0$, we have

$$\frac{1}{n}\sum_{j=p}^{p+n-1} | x_j - L | = \frac{1}{n}\left(\sum_{j=p}^{p_0-1} + \sum_{j=p_0}^{p+n-1}\right) | x_j - L | \qquad \ldots(2.5)$$

$$\leqslant \frac{M}{n} + \frac{1}{n}\sum_{j=p_0}^{p_0+n-1} | x_j - L | \text{ (from 2.4)}$$

$$\leqslant \frac{M}{n} + \frac{\epsilon}{2} \text{ (by 2.2)}$$

Taking, $n$ sufficiently large, we can made

$$\frac{M}{n} + \frac{\epsilon}{2} < \epsilon \qquad \ldots(2.6)$$

which gives (2.3) and hence the result.

Proof of Theorem 1: Let $\{x_j\} \in | AC_\theta |$

$\Rightarrow$ Given $\epsilon > 0$, $\exists$ $r_0$, and $L$ such that

$$\frac{1}{h_r}\sum_{j=q}^{q+h_r-1} | x_j - L | < \epsilon \qquad \ldots(2.7)$$

for $r > r_0$, and $q = K_{r-1} + 1 + i$, $i \geqslant 0$.

Let $n \geqslant h_r$, write

$$n = mh_r + \theta \qquad \ldots(2.8)$$

where, $0 \leqslant \theta \leqslant h_r$, $m$ is an integer. Since $h \geqslant h_r$, $m \geqslant 1$. Now

$$\frac{1}{n}\sum_{j=q}^{q+n-1} | x_j - L | \leqslant \frac{1}{n}\sum_{j=q}^{q+(m+1)h_r-1} | x_j - L |$$

$$= \frac{1}{n}\sum_{u=0}^{m}\sum_{j=q+uh_r}^{q+(u+1)h_r-1} | x_j - L |$$

$$\leqslant \frac{m+1}{n} h_r \epsilon \qquad \ldots(2.9)$$

$$\leqslant \frac{2mh_r\epsilon}{n} \quad (m \geqslant 1).$$

For $\frac{h_r}{n} \leqslant 1$, since, $\frac{mh_r}{n} \leqslant 1$

$$\frac{1}{n}\sum_{j=q}^{q+n-1} | x_j - L | \leqslant 2\epsilon.$$

Then by Lemma 1, $| AC_\theta | \Rightarrow | AC |$. It is trivial that $| AC | \Rightarrow |AC_\theta|$ for every $\theta$. Hence, we have the result.

*Theorem* 2— (a) For some, $\theta$, $AC_\theta =\!\!/\!\!\Rightarrow l\,\infty$;

(b) For every $\theta$, $AC_\theta \cap l\,\infty \Leftrightarrow AC$.

In order to prove this theorem, we require the following lemma.

*Lemma* 2—Suppose for given $\epsilon > 0$, $\exists\ n_0, p_0$ such that

$$\frac{1}{n}\Big|\sum_{j=p}^{p+n-1} x_j - L\Big| < \epsilon$$

for all $n \geqslant n_0$, $p \geqslant p_0$. Then $\{x_j\} \in AC$.

PROOF : Let $\epsilon > 0$ be given. Choose $n_0'$ , $p_0$ such that

$$\frac{1}{n}\Big|\sum_{j=p}^{p+n-1} x_j - L\Big| < \epsilon/2 \qquad \text{...(2.10)}$$

for $n \geqslant n_0'$ $p \geqslant p_0$.

As in Lemma 1, it is enough to show, $\exists\ n_0''$ such that for $n \geqslant n_0''$ , $0 \leqslant p \leqslant p_0$, we have

$$\frac{1}{n}\Big|\sum_{j=p}^{p+n-1} x_j - L\Big| < \epsilon. \qquad \text{...(2.11)}$$

Since $p_0$ is fixed. Let

$$\sum_{j=0}^{p_0-1} | x_j - L | = M \text{ (say)}. \qquad \text{...(2.12)}$$

Now, let $0 \leqslant p \leqslant p_0$, and $n > p_0$, then

$$\frac{1}{n}\Big|\sum_{j=p}^{p+n-1} x_j - L\Big| \leqslant \frac{1}{n}\sum_{j=p}^{p_0-1} | x_j - L | + \frac{1}{n}\Big|\sum_{j=p_0}^{p+n-1} x_j - L\Big|$$

*(equation continued on p. 69)*

$$\leqslant \frac{M}{n} + \frac{1}{n} \Big| \sum_{j=p_0}^{p_0+n+p-p_0-1} x_j - L \Big| , \qquad \ldots(2.13)$$

Let $n - p_0 > n_0'$ Then for $0 \leqslant p < p_0$, we have $n + p - p_0 \geqslant n_0'$ . From (2.10)

$$\frac{1}{n+p+p_0} \Big| \sum_{p=p_0}^{p_0+n+p-p_0} x_j - L \Big| < \epsilon/2 \qquad \ldots(2.14)$$

From (2.13) and (2.14)

$$\frac{1}{n} \Big| \sum_{j=p}^{p+n-1} x_j - L \Big| \leqslant \frac{M}{n} + \frac{n+p-p_0}{n}\epsilon/2$$

$$\leqslant \frac{M}{n} + \epsilon/2$$

$< \epsilon$, for sufficiently large $n$.

Hence the result,

Proof of Theorem 2 : (a) It is enough to show $AC_\theta \;{=}\!\!/\!\!\Rightarrow l_\infty$ when, $K_r$ is even for all $r$. Let

$$x_k = (-1)^k K^\lambda. \qquad \ldots(2.15)$$

Where $\lambda$ is a constant, with $0 < \lambda < 1$. Then

$$\sum_{j=q}^{q+h_r-1} x_j,\ q \geqslant 0 \qquad \ldots(2.16)$$

will contains an even number of terms. So we can bracket the terms of the sum in pairs of consecutive terms without there being a term left over. Since, the sum of two consecutive terms is $O(K^{\lambda-1})$, it is straight forward matter to verify that $\{x_j\} \in AC_\theta$ with $L = 0$. But $\{x_j\}$ is not bounded.

*Remark* : $AC_\theta \Rightarrow l\infty$, whenever the set of values for $h_r$ includes all positive integers in any order.

(b) Let $\{x_j\} \in AC_\theta \cap l\infty$. For $\epsilon > 0$, $\exists\ r_0$ and $q_0$ such that

$$\frac{1}{h_r} \Big| \sum_{j=q}^{q+h_r-1} x_j - L \Big| < \epsilon/2 \qquad \ldots(2.17)$$

for $r \geqslant r_0$, $q \geqslant q_0$. $q = K_{r-1} + 1 + i$, $i \geqslant 0$, Now, let $n \geqslant h_r$, $m$ is an integer greater than equal to 1. Then

$$\frac{1}{n} \left| \sum_{j=q}^{q+n-1} x_j - L \right| \leqslant \frac{1}{n} \sum_{\mu=0}^{m-1} \left| \sum_{j=q+\mu h_r}^{q+(\mu+1)h_r-1} x_j - L \right|$$

$$+ \frac{1}{n} = \sum_{j=q+mh_r}^{q+n-1} | x_j - L | \qquad \text{...(2.18)}$$

Since $\{x_j\} \in l\,\infty$, let all $j$, $| x_j - L | < M$ (say).

So, from (2.17) and (2.18)

$$\frac{1}{n} \left| \sum_{j=q}^{q+n-1} x_j - L \right| \leqslant \frac{1}{n} m.\, h_r\, \epsilon/2 + \frac{Mh_r}{n}. \qquad \text{...(2.19)}$$

For, $\frac{h_r}{n} \leqslant 1$, since $\frac{mh_r}{n} \leqslant 1$, and $\frac{Mh_r}{n}$ can be made less than $\epsilon/2$ by taking $n$ sufficiently large.

So,

$$\frac{1}{n} \left| \sum_{j=q}^{q+n-1} x_j - L \right| < \epsilon \text{ for } r \geqslant r_0,\, q \geqslant q_0.$$

Hence, by Lemma 2, $AC_\theta \cap l\,\infty \Rightarrow AC$.

It is trivial that, $AC \Rightarrow AC_\theta \cap l\,\infty$.

§3. In view of the ideas of uniformly and well distributed modulo 1 of Weyl[9] and Petersen[8] and result in §2, it is natural to define uniformly and well distributed modulo 1, of the sequence of real numbers, over the lacunary sequence $\theta$ as follows :

Let $x = (x_K)$ be a sequence of real numbers such that

$$0 \leqslant x_k < 1, \forall\, k \geqslant 0. \qquad \text{...(3.0)}$$

$0 \leqslant a < b \leqslant 1$ and $I_{(a,b)}$ be the characteristic function of interval $[a,b)$. We record the definitions of Weyl[9] and Petersen[8]:

*Definition A*[9]—A sequence '$x$' is called uniformly distributed (abbreviated U.d.) if for every pair of $a$, $b$.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} I_{[a,b)}(x_K) = b - a. \qquad \text{...(3.1)}$$

*Definition* B[8]—A sequence '$x$' is called well distributed (W.d.) if for each pairs of $a$, $b$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=i}^{n+i-1} I_{[a,b)}(x_K) = b - a \qquad \ldots(3.2)$$

uniformly in $i \geqslant 0$.

Now, we define the well and uniform distributed over the lacunary sequence $\theta$ as follows :

*Definition* 1—A sequence '$x$' is called uniformly distributed over $\theta$ ($u.\ d_\theta.$) if for every pairs of $a$, $b$

$$\lim \frac{1}{h_r} \sum_{K\in I_r} I_{[a,b)}(x_K) = b - a. \qquad \ldots(3.3)$$

*Definition* 2—A sequence $x$ is called well distributed over $\theta$ (W.d$_\theta$.) if, for every pair of $a$,$b$

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{K\in I_r} I_{[a,b)}(x_{k+i}) = b - a \qquad \ldots(3.4)$$

uniformly in $i \geqslant 0$.

Let $A = (a_{nk}(j))$, $n, k, j \geqslant 0$ be the generalised three parametric real matrix with the following conditions.

$$\|A\| = \sup_{n,j} \sum_{k=0}^{\infty} |\, a_{nk}(j)\,| < \infty \qquad \ldots(3.5)$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}(j) = 1, \text{ uniformly in } j. \qquad \ldots(3.6)$$

A is almost positive

i. e.

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}^{-(j)} = 0 \text{ uniformly in } j \qquad \ldots(3.7)$$

where

$$a_{nk}^{-}(j) = \max(-a_{n,k}(j), 0).$$

$$a^{+}_{n,k} = \max (a_{n,k} (j)\ 0).$$

*Definition* $C$[7]—A sequence is said to have $A$-uniform asympotic distribution function $g\,[A -$ u.a.d.f., $g)$, if there exists, $g : [0, 1] \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}(j)\, I_{[0,b)}(x_K) = g(b). \qquad \ldots(3.8)$$

Uniformly in $j \geqslant 0$.

Note that, taking $g(t) = t,\ 0 \leqslant t \leqslant 1$ and

for all $j \geqslant 0$

$$a_{n,k}(j) = \begin{cases} \frac{1}{n}, 0 \leqslant k \leqslant n-1, \\ 0 \text{ elsewhere} \end{cases} \qquad \ldots(3.9)$$

$$a_{n,k}(j) = \begin{cases} \frac{1}{n}, \ j \leqslant K \leqslant j+n-1 \\ 0 \text{ elsewhere} \end{cases} \qquad \ldots(3.10)$$

$$a_{n,k}(j) = \begin{cases} \frac{1}{h_r}, K \in I_r, r = n \\ 0 \text{ elsewhere} \end{cases} \qquad \ldots(3.11)$$

$$a_{n,k}(j, = \begin{cases} \frac{1}{h_r}, \ K_{r-1} + j + 1 \leqslant K \leqslant K_r + j, r = n \\ 0 \text{ elsewhere.} \end{cases} \qquad \ldots(3.12)$$

Then definitions $A$, $B$,1,2 are the particular case of Definition $C$. Now we record the following from Das and Patel[7].

*Theorem B*—Let $g$ be an non-negative, non decreasing bounded continuous function from [0,1) into R. Then a sequence $(x_k)$ is $A - u.a.d.f.g$, if and only if, for every real valued continuous function $f$.

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}(j) f(x_k) = \int_0^1 f(t)\, dg(t) \qquad \ldots(3.13)$$

uniformly in $j \geqslant 0$.

*Theorem C*—Let $g$ be a non-negative, non decreasing bounded continuous function from [0, 1] into R. Then a sequence $(x_k)$ is $A - U$.a.d,f,g, if and only if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}(j)\, e^{2\pi i h x_k} = \int_0^1 e^{2\pi i h x}\, dg(x). \qquad \ldots(3.14)$$

Uniformly in $j \geqslant 0$, for all non zero integer $h$, and $i = \sqrt{-1}$.

Similar to Weyl[9] and Petersen[8] and from $B$ and $C$, We have the followings :

*Theorem* 3— (a) '$x$' is $u.\ d_\theta$. if and only if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{K \in I_r} f(x_k) = \int_0^1 f(x).\ dx \quad ..(3.15)$$

for every Riemann integrable function $f$ on [0, 1].

(b) '$x$' is $W.\ d_\theta$. if and only if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{K \in I_r} f(x_{k+i}) = \int_0^1 (x).\ dx \quad ...(3.16)$$

uniformly in $i \geqslant 0$, for every Riemann integrable function $f$ on [0, 1],

(c) '$X$' is $u.\ d_\theta$, if and only if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{K \in I_f} e^{2\pi ihx}{}_k = 0 \quad ...(3.17)$$

for all integers $h \neq \sqrt{-1}$

(d) '$x$' is $W.\ d_\theta$. if and only if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{|K \in I_r} e^{2\pi ihx}{}_{k+j} = 0 \quad ...(3.18)$$

uniformly in $j \geqslant 0$, for all integers $h \neq 0$, $i = \sqrt{-1}$.

*Theorem* 4— (a) $X$ is $u.\ d.$ $\Rightarrow$ $X$ is $u.\ d\theta$, if

$$\lim_r \inf q_r > 1 \quad ...(3.19)$$

(b) $X$ is $U, d_\theta$ $\Rightarrow$ $X$ is $U.\ d.$, if

$$\lim_r q_r = 1 \quad ...(3.20)$$

(c) For every $\theta$, (i) $W.d. \Rightarrow U.\ d_\theta$; (ii) $W.\ d_\theta \Leftrightarrow W.\ d.$

PROOF (a) : Follows from Theorem $A$ ($d$). Since $I_{[a,b]}(x)$ takes the value 0 or 1 only, we have

$$\min[(b-a), 1-(b-a)] \leqslant |I_{[a,b]}(x) - (b-a)|$$
$$\leqslant \max[(b-c), 1-(b-a)] \quad ...(3.21)$$

So $\{I_{[a\,b)}(x) - (b-a)\} \in l\,\infty$. Now $(b)$ follows from Theorem $A$ $(f)$ ($c$ (i)) follows from Theorem $A$ $(g)$ and (c (ii)) follows from Theorem 2 (b).


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