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NOTE ON THE SMALL VIBRATION OF BEAMS WITH VARYING YOUNG'S MODULUS CARRYING A CONCENTRATED MASS DISTRIBUTION

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The problem of vibration of beams or rods with varying Young's modulus has been solved assuming that the beam or rod carries a concentrated mass distribution. The effect of concentrated mass has been described by introducing Dirac δ-function in the differential equation. The methods of separation of variables and the Laplace transform have been used. Forced vibration problem has been solved after deriving the orthogonality relation for a variable density beam.

1. INTRODUCTION

The problem of vibration of beams or rods is of great importance in the theory of elasticity. The most convenient technique, to solve the homogeneous partial differential equation describing the free vibration of beams or rods, is to use the method of separation of variables. This yields a pair of ordinary differential equations. The solution of the differential equation containing spatial variables, together with the boundary conditions, gives a set of eigen functions. On the other hand, the eigen function expansion method usually gives a solution of the general initial value problem either of free vibration or of forced vibration.

Timoshenko[1] solved the problem of longitudinal vibration of a rod in which a mass was attached to the end. Hoppmann[2] studied the transverse vibration of a beam carrying a mass at the middle. His discussion did not include the general initial value problem. Moreover, neither Timoshenko[1] nor Hoppmann[2] used eigen function method, perhaps due to the difficulty in deriving the orthogonality relation. Morgan[3] showed an easy method of obtaining orthogonality relation for the cases where a Dirac δ-function described the effect of concentrated mass.

Introducing the method discussed by Morgan[3], Chen[4] obtained the solution of the vibration of beams or rods carrying a concentrated mass. Pan[5] investigated the transverse vibration of a beam carrying a system of heavy bodies. Tseitlin[6] considered the longitudinal oscillations of a semi infinite rod with a mass fixed at the end. Howson and Williams[9] derived a solution in the matrix form for the natural frequencies of vibration of a structural frame. Chang and Juan[12] derived a set of equations for the free vibration of an inclined bar with an end constraint. Chen[13] presented the general dynamic stiffness matrix of a Timoshenko beam for transverse vibrations. Park[15] studied the dynamic stability of a uniform free-free Timoshenko beam under the action of controlled force.

In all the problems mentioned above the beam or the rod was considered to be homogeneous in the sense that Young's modulus of the beam or the rod was constant throughout. But as there are plenty of materials in nature which are not elastically homogeneous, the problems of vibrations of beams or rods with varying Young's modulus or with varying density have also received the attention of numerous investigators. Marangoni *et al.*[7] obtained the transverse vibrational frequency of uniform beams in which a steady state temperature distribution introduced a coordinate dependent elastic modulus. Nayfeh[8] derived the eigen values and corresponding eigen functions of free longitudinal vibration of a finite rod whose Young's modulus and mass density were assumed to vary continuously along the rod. Among the many other problems of vibrations of elastically nonhomogeneous bodies mention may be made of the interesting works of Shahinpoor[10], Farshad and Ahmadi[11] and Ercoli and Laura[14].

The purpose of the present investigation is to apply the techniques of Chen[4] and Pan[5] to a class of beams or rods which are elastically nonhomogeneous in the sense that the Young's modulus of the beams or rods considered here, is a function of position.

In our investigation of the problem we have first considered the simple case of vibration where only a single mass is placed at the middle of the rod and then the more general case where a system of heavy bodies have been placed at different points on the rod and driving forces and driving moments have also been taken into account. We refer these two cases as Case I and Case II respectively.

Thus the present problem may be considered as an extension of the works of Chen[4] and also of Pan[5].

2. Formulation and the Solution of the Problem in Case I

Let us consider the free vibration of a simply supported beam of nonhomogeneous material having length $2L$. Let the beam carry a concentrated mass M at its middle. The equation of motion, for Euler Bernoulli's description of the beam, may be written as,

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2} \right) + \left[\rho + M\,\delta\,(x - L)\right]\frac{\partial^2 y}{\partial t^2} = 0 \qquad \text{...(1)}$$

where E is the Young's modulus and ρ is the mass per unit length of the beam, and the δ-function satisfies the relation,

$$\int_0^{2L} \delta(x - L)\, dx = 1. \qquad \text{...(2)}$$

The boundary conditions of the problem are,

$$y(0, t) = y(2L, t) = y''(0, t) = y''(2L, t) = 0 \qquad \text{...(3)}$$

where the primes indicate differentiation of y with respect to x.

Thus our problem is to find y from (1) subject to the boundary conditions (3).

If we assume a series solution of (1) in the form,

$$y = \sum_{n=1}^{\infty} c_n^* \, \psi_n(x) \sin p_n t \qquad \text{...(4)}$$

we get the equation for $\psi_n(x)$ as

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 \psi_n}{dx^4} \right) - p_n^2 \left[\rho + M\,\delta(x - L) \right] \psi_n = 0 \qquad \text{...(5)}$$

subject to the boundary conditions

$$\psi_n(0) = \psi_n(2L) = \psi_n''(0) = \psi_n''(2L) = 0. \qquad \text{...(6)}$$

Let the Young's modulus E and the density ρ of the beam be functions of position given by

$$E = E_0 \, e^{\alpha x} \qquad \text{...(7)}$$

$$\rho = \rho_0 \, e^{\alpha x}$$

α being any real quantity having the dimension of reciprocal of length.

Equations (5) and (6) define an eigen value problem. Let us proceed to solve it by the method of Laplace transform. Denoting the transformed quantity by a bar over the corresponding function, we may write,

$$\bar{\psi}_n = \frac{(s + \alpha)^2}{s^4 + 2s^3\alpha + s^2\alpha^2 - k_n^4} \psi_n'(0) + \frac{1}{s^4 + 2s^3\alpha + s^2\alpha^2 - k_n^4} \psi_n'''(0)$$

$$+ \frac{M p_n^2 \, e^{-L\alpha} \, \psi_n(L)}{E_0 I} \; \frac{e^{-Ls}}{s^4 + 2s^3\alpha + s^2\alpha^2 - k_n^4} \qquad \text{...(8)}$$

where $k_n^4 = \rho_0 \, p_n^2 / E_0 \, I$ and s is the Laplace transform parameter.

While deriving equation (8) two of the boundary conditions (6) namely $\psi_n(0)=0$ and $\psi_n''(0) = 0$ have been used.

Taking inversion of (8) we get,

$$\psi_n(x) = \psi_n'(0) \, f(x) + \psi_n'''(0) \, g(x) + \frac{Mp_n^2 \, e^{-L\alpha} \, \psi_n(L)}{E_0 \, I} h(x) \qquad \ldots(9)$$

where

$$f(x) = \exp(-\alpha x/2) \, [A_1 \cos \beta_n x + C_1 \cosh \gamma_n x + \xi_1 \sin \beta_n x + \eta_1 \sinh \gamma_n x]$$

$$g(x) = \exp(-\alpha x/2) \, [A_2 \cos \beta_n x + C_2 \cosh \gamma_n x + \xi_2 \sin \beta_n x + \eta_2 \sinh \gamma_n x]$$

$$h(x) = g(x - L) \, H(X - L)$$

$$\xi_i = \left(B_i - \frac{A_i \, \alpha}{2}\right) \frac{1}{\beta_n}, \; \eta_i = \left(D_i - \frac{C_i \, \alpha}{2}\right) \frac{1}{\gamma_n} \quad (i = 1, 2)$$

$$A_1 = -\frac{\alpha}{2k_n^2}, \; C_1 = \frac{\alpha}{2k_n^2}$$

$$B_1 = \tfrac{1}{2}\left(1 - \frac{\alpha^2}{k_n^2}\right), \; D_1 = \tfrac{1}{2}\left(1 + \frac{\alpha^2}{k_n^2}\right)$$

$$A_2 = C_2 = 0$$

$$B_2 = -\frac{1}{2k_n^2}, \; D_2 = \frac{1}{2k_n^2}$$

$$\beta_n^2 = k_n^2 - \frac{\alpha^2}{4}, \; \gamma_n^2 = k_n^2 + \frac{\alpha^2}{4}.$$

$H(x - L)$ being a unit step function at $x = L$.

The constants $\psi_n'(0)$ and $\psi_n'''(0)$ in (9) may be determined from the remaining two boundary conditions of (6) i.e., $\psi_n(2L) = 0$ and $\psi_n''(2L) = 0$. The results obtained are

$$\psi_n'(0) = L_1 \frac{Mp_n^2 e^{-L\alpha} \psi_n(L)}{E_0 I} \qquad \ldots(10)$$

$$\psi_n'''(0) = M_1 \frac{Mp_n^2 e^{-L\alpha} \psi_n(L)}{E_0 I}$$

where

$$L_1 = \frac{g(2L)\, h''(2L) - g''(2L)\, h(2L)}{f(2L)\, g''(2L) - f''(2L)\, g(2L)}$$

$$M_1 = \frac{f''(2L)\, h(2L) - f(2L)\, h''(2L)}{f(2L)\, g''(2L) - f''(2L)\, g(2L)}.$$

Thus in view of (10) eqn. (9) becomes,

$$\psi_n(x) = \frac{Mp_n^2 e^{-L\alpha} \psi_n(L)}{E_0 I}\Big[L_1 f(x) + M_1 g(x) + h(x)\Big]. \qquad \ldots(11)$$

If we let $x = L$ in (11) the following equation results,

$$\psi_n(L)\Big[Mp_n^2 e^{-L\alpha}\Big\{L_1 f(L) + M_1 g(L) + h(L)\Big\} - E_0 I\Big] = 0. \ldots(12)$$

Since $\psi_n(L) = 0$ corresponds to the trivial solution of y, it follows that for nontrivial solutions

$$Mp_n^2 e^{-L\alpha}\Big\{L_1 f(L) + M_1 g(L) + h(L)\Big\} - E_0 I = 0. \qquad \ldots(13)$$

Equation (13) gives an infinite set of eigenvalues p_n, each of which corresponds to the eigen function $\psi_n(x)$. Substituting their values in (4) we get the solution of the problem in this case. The arbitrary constants c_n^* are to be determined from the initial conditions of the problem.

At this stage it may be important to note the reduction of the function $\psi_n(x)$ in (9) and the equation (13) giving the eigen values p_n, as $\alpha \to 0$. Thus making $\alpha \to 0$, (9) and (13) reduce to

$$\psi_n(x) = \frac{\psi_n'(0)}{2k_n}\Big[\sin k_n x + \sinh k_n x\Big] + \frac{\psi_n'''(0)}{2k_n^3}\Big[\sinh k_n x - \sin k_n x\Big]$$

$$+ \frac{Mp_n^2 \psi_n(L)}{2E_0 I k_n^3} H(x - L)\Big[\sinh k_n(x - L) - \sin k_n(x - L)\Big]$$

$$\ldots(9a)$$

and

$$Mp_n^2 (\tan k_n L - \tanh k_n L) - 4 E_0 I k_n^3 = 0 \qquad ...(13a)$$

which are in complete agreement with the equations (8) and (12) respectively, obtained by Chen[4] for an elastically homogeneous beam.

2.1. *The Orthogonality Relation*

It has been suggested by Morgan[8] that if the function $\rho + M\delta(x - L)$ be treated as a weighting function then the eigen functions $\psi_n(x)$ are orthogonal with respect to this weighting function.

Let us assume

$$\epsilon(x) = \rho + M\delta(x - L) \qquad ..(14)$$

then

$$\int_0^{2L} \epsilon(x)\, \psi_n(x)\, \psi_m(x)\, dx = \frac{E_0 I}{p_n^2 - p_m^2} \int_0^{2L} e^{\alpha x} \Big[\psi_m(x)\, \psi_n''''(x) - \psi_n(x)\, \psi_m''''(x)$$

$$+ 2\alpha\, \psi_m(x)\, \psi_n'''(x) - 2\alpha\, \psi_n(x)\, \psi_m'''(x)$$

$$+ \alpha^2\, \psi_m(x)\, \psi_n''(x) - \alpha^2\, \psi_n(x)\, \psi_m''(x) \Big] dx = 0$$

$$\text{for } m \neq n. \qquad ...(15)$$

Verification of eqn. (15) is a very simple task.

Now if we substitute (14) in (15) we get,

$$\int_0^{2L} e^{\alpha x}\, \psi_n(x)\, \psi_m(x)\, dx = -\frac{M}{\rho_0}\, \psi_n(L)\, \psi_m(L). \qquad ...(16)$$

Let

$$\int_0^{2L} \epsilon(x)\, \psi_n^2(x)\, dx = \frac{E_0 I}{p_n^2} \int_0^{2L} e^{\alpha x} \Big[\psi''(x) \Big]^2 dx = N_n. \qquad ...(17)$$

N_n is the normalization constant for the system of eigen functions.

2.2. *The Forced Vibration*

Let us determine the response of the beam to an impulsive load.

For a unit impulsive load the differential equation of the forced vibration may be written as,

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2} \right) + \Big[\rho + M\,\delta(x - L) \Big] \frac{\partial^2 y}{\partial t^2} = \delta(t)\, \delta(x - L). \qquad ...(18)$$

Let

$$y = \sum_{n=1}^{\infty} q_n(t)\, \psi_n(x). \qquad \text{...(19)}$$

$\psi_n(x)$ is the eigen function which we have already determined and the function $q_n(t)$ is to be determined. Substituting (19) in (18), multiplying the resulting equation by ψ_m and then integrating from 0 to $2L$ we obtain,

$$\int_0^{2L} \psi_m \frac{\partial^2}{\partial x^2} \left(EI \sum_{n=1}^{\infty} \psi_n'' \, q_n\right) dx + \int_0^{2L} \psi_m\, \epsilon(x) \left(\sum_{n=1}^{\infty} \psi_n \ddot{q}_n\right) dx$$

$$= \delta(t) \int_0^{2L} \psi_m\, \delta(x - L)\, dx. \qquad \text{...(20)}$$

Now ψ_n satisfies eqn. (5) as well as the orthogonality relation. Hence eqn. (20) reduces to

$$\ddot{q}_n + p_n^2 q_n = \frac{\psi_n(L)}{N_n} \delta(t) \qquad n = 1, 2, \ldots \qquad \text{...(21)}$$

The particular solution of equation (21) is given by

$$q_n = \frac{\psi_n(L)}{N_n\, p_n} \sin p_n t. \qquad \text{...(22)}$$

For a general forcing function $F(t)$ we get, by convolution, the particular solution as,

$$y(t) = \sum_{n=1}^{\infty} \frac{\psi_n(L)\, \psi_n(x)}{N_n\, p_n} \int_0^t \sin p_n (t - \xi)\, F(\xi)\, d\xi. \qquad \text{...(23)}$$

Here it should be noted that by virtue of the orthogonality relation the solution of the initial-value problem can easily be obtained by expanding the initial conditions into series of eigen functions.

3. Formulation and the Solution of the Problem in Case II

When a uniform beam, carrying a system of heavy bodies placed at the points $x = a_i$, is subjected to a system of driving forces $F_j(t)$ and driving moments $G_k(t)$ acting at points b_j and c_k respectively, its equation of motion may be written as,

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2} \right) - \frac{\partial}{\partial x}\left[\sum_{i=1}^{R} J_i\, \delta(x - a_i) \frac{\partial^3 y}{\partial x\, \partial t^2}\right]$$

$$+ \left[\rho + \sum_{i=1}^{R} M_i\, \delta(x - a_i)\right] \frac{\partial^2 y}{\partial t^2} = \sum_{j=1}^{s} F_j(t)\, \delta(x - b_j)$$

(*equation continued on p.* 82)

$$+ \sum_{k=1}^{T} G_k(t)\, \delta'(x - c_k). \qquad \ldots(24)$$

In eqn. (24) M_i and J_i respectively denote the mass and the rotary inertia of the ith body; R, S, T are the total number of heavy bodies, the total number of driving forces and the total number of driving moments respectively.

For free vibration of the beam the right-hand side of (24) is zero. In this case if we assume a series solution in the form

$$y(x, t) = \sum_{n=1}^{\infty} \psi_n(x) \Big[P_n \sin p_n t + Q_n \cos p_n t \Big] \qquad \ldots(25)$$

then the reduced homogeneous equation, when (7) is taken into account, yields the following spatial equation,

$$\frac{\partial^2}{\partial x^2}\Big[E_0 I e^{\alpha x} \psi_n'' \Big] + p_n^2 \left\{ \Big[\sum_{i=1}^{R} J_i\, \delta(x - a_i)\, \psi_n' \Big]' - \Big[\rho_0 e^{\alpha x} + \sum_{i=1}^{R} M_i\, \delta(x - a_i) \Big] \psi_n \right\} = 0. \qquad \ldots(26)$$

Thus in this case our problem is to find ψ_n from (26) subject to the boundary conditions (6).

As in Case I, noting the boundary conditions (6) and taking Laplace transform of (26) we obtain,

$$\bar{\psi}_n = \frac{(s + \alpha)^2}{s^4 + 2s^3\alpha + s^2\alpha^2 - k_n^4}\, \psi_n'(0) + \frac{1}{s^4 + 2s^3\alpha + s^2\alpha^2 - k_n^4}\, \psi_n'''(0)$$

$$- \frac{s}{s^4 + 2s^3\alpha + s^2\alpha^2 - k_n^4}\, \frac{p_n^2}{E_0 I} \sum_{i=1}^{R} J_i\, \psi_n'(a_i)\, e^{-a_i\alpha} . e^{-a_i s}$$

$$+ \frac{1}{s^4 + 2s^3\alpha + s^2\alpha^2 - k_n^4}\, \frac{p_n^2}{E_0 I} \sum_{i=1}^{R} M_i\, \psi_n(a_i)\, e^{-a_i\alpha} . \, e^{-a_i s}. \qquad \ldots(27)$$

Taking inversion of (27) we get,

$$\psi_n(x) = \psi_n'(0) f(x) + \psi_n'''(0) g(x) - \frac{p_n^2}{E_0 I} \sum_{i=1}^{R} J_i\, \psi_n'(a_i)\, e^{-a_i\alpha} . f_1(x - a_i)$$

(equation contiuned on p. 83)

$$\times \frac{p_n^2}{E_0 I} \sum_{i=1}^{R} M_i \psi_n(a_i) e^{-a_i \alpha} g_1(x - a_i) \qquad \ldots(28)$$

where

$$f_1(x) = e^{-\frac{\alpha}{2}x} H(x) [A_3 \cos \beta_n x + C_3 \cosh \gamma_n x$$
$$+ \xi_3 \sin \beta_n x + \eta_3 \sinh \gamma_n x]$$

$$g_1(x) = g(x) H(x)$$

$$\xi_3 = \left(B_3 - \frac{A_3 \alpha}{2}\right) \frac{1}{\beta_n} \qquad \eta_3 = \left(D_3 - \frac{C_3 \alpha}{2}\right) \frac{1}{\gamma_n}$$

$$A_3 = -\frac{1}{2k_n^2}, \; C_3 = \frac{1}{2k_n^2}$$

$$B_3 = D_3 = 0.$$

If we impose remaining two of the boundary conditions (6) namely $\psi_n \; (2L) = 0$ and $\psi_n'' \; (2L) = 0$ on (28) we obtain the following equations :

$$\psi_n' \; (0) \; f(2L) + \psi_n''' (0) g(2L) - \frac{p_n^2}{E_0 I} \sum_{i=1}^{R} J_i \psi_n' \; (a_i) e^{-a_i \alpha} f_1(2L - a_i)$$

$$+ \frac{p_n^2}{E_0 I} \sum_{i=1}^{R} M_i \psi_n(a_i) e^{-a_i \alpha} g_1(2L - a_i) = 0 \qquad \ldots(29)$$

and

$$\psi_n' \; (0) f''(2L) + \psi_n''' (0) g''(2L) - \frac{p_n^2}{E_0 I} \sum_{i=1}^{R} J_i \psi_n' \; (a_i) e^{-a_i \alpha}$$

$$\times f_1'' \; (2L - a_i) + \frac{p_n^2}{E_0 I} \sum_{i=1}^{R} M_i \psi_n(a_i) e^{-a_i \alpha}$$

$$\times g_1'' \; (2L - a_i) = 0. \qquad \ldots(30)$$

Substituting $x = a_q - \epsilon$ in (28) and its derivative, and letting $\epsilon \to 0$, two systems of equations which represent the consistency conditions are obtained.

$$-\psi_n(a_q) + \psi_n' \; (0) f(a_q) + \psi_n''' (0) g(a_q) - \frac{p_n^2}{E_0 I} \sum_{i=1}^{q-1} J_i \psi_n' \; (a_i)$$

(equation continued on p. 84)

$$\times\, e^{-a_i\alpha} f_1(a_q - a_i) + \frac{p_n^2}{E_0 I} \sum_{i=1}^{q-1} M_i \psi_n(a_i)\, e^{-a_i\alpha} g_1(a_q - a_i) = 0,$$

$$q = 1, 2, 3, \ldots R \qquad \ldots(31)$$

$$-\psi_n'(a_q) + \psi_n'(0) f'(a_q) + \psi_n'''(0) g'(a_q) - \frac{p_n^2}{E_0 I} \sum_{i=1}^{q-1} J_i \psi_n'(a_i)$$

$$\times\, e^{-a_i\alpha} f_1'(a_q - a_i) + \frac{p_n^2}{E_0 I} \sum_{i=1}^{q-1} M_i \psi_n(a_i)\, e^{-a_i\alpha}$$

$$\times\, g_1'(a_q - a_i) = 0,\ q = 1, 2, 3, \ \ , R. \qquad \ldots(32)$$

Equations (29) to (32) form a system of homogeneous equations in $(2R+2)$ unknowns $\psi_n'(0)$, $\psi_n'''(0)$, $\psi_n(a_i)$, $\psi_n'(a_i)$. For a nontrivial solution of the problem the determinant of the coefficient matrix X of the foregoing system of equations must vanish, leading to the frequency equation as

$$\det(X) = 0. \qquad \ldots(33)$$

Equation (33) is an equation determining k_n.

Substituting each k_n obtained from (33) back into equations (29) to (32) the values of $\psi_n'(0)$, $\psi_n'''(0)$, $\psi_n(a_i)$ and $\psi_n'(a_i)$ are determined. Corresponding to the set of eigen values k_n, the eigen functions are obtained from equation (28).

After obtaining $\psi_n(x)$, and p_n from k_n and substituting their values in (25) the solution of the free vibration is obtained. The constants P_n and Q_n are determined from the initial conditions $\psi(x, 0)$ and $\dot{\psi}(x, 0)$. This may be done by expanding these initial conditions in the obtained eigen functions and comparing the coefficients.

3.1. *Orthogonality condition and eigenfunction expansion*

We may write eqn. (26) in the following form :

$$-\frac{\partial^2}{\partial x^2}\left[e^{\alpha x}\,\psi_n''\right] = \lambda_n\left\{\left[\sum_{i=1}^{R} J_i\,\delta(x - a_i)\,\psi_n'\right]' - \left[\rho_0 e^{\alpha x} + \sum_{i=1}^{R} M_i\,\delta(x - a_i)\right]\psi_n\right\} \qquad \ldots(34)$$

where

$$\lambda_n = \frac{p_n^2}{E_0 I} = \frac{k_n^4}{\rho_0}.$$

For $m \neq n$ we may deduce from (34) the relation,

$$\lambda_n \int_0^{2L} \Big\{ \Big[-\sum_{i=1}^{R} J_i \, \delta(x - a_i) \, \psi_n' \Big]' \psi_m$$

$$+ \Big[\rho_0 \, e^{\alpha x} + \sum_{i=1}^{R} M_i \, \delta(x - a_i) \Big] \psi_n \, \psi_m \Big\} \, dx$$

$$- \lambda_m \int_0^{2L} \Big\{ \Big[-\sum_{i=1}^{R} J_i \, \delta(x - a_i) \, \psi_m' \Big]' \psi_n$$

$$+ \Big[\rho_0 \, e^{\alpha x} + \sum_{i=1}^{R} M_i \, \delta(x - a_i) \Big] \psi_n \, \psi_m \Big\} \, dx$$

$$= \int_0^{2L} e^{\alpha x} \Big[\psi_n'''' (x) \, \psi_m (x) - \psi_m'''' (x) \, \psi_n (x) + 2\alpha \, \psi_n''' (x) \, \psi_m (x)$$

$$- 2\alpha \, \psi_m''' (x) \, \psi_n (x) + \alpha^2 \, \psi_n'' (x) \, \psi_m (x)$$

$$- \alpha^2 \, \psi_m'' (x) \, \psi_n (x) \Big] \, dx = 0. \qquad \text{...(35)}$$

Since for $m \neq n$ the right-hand side of (35) is equal to zero and both the integrals on the left hand side yield same result, we may write eqn. (35) as

$$\int_0^{2L} \rho_0 \, e^{\alpha x} \, \psi_n \, \psi_m \, dx + \sum_{i=1}^{R} M_i \, \psi_n (a_i) \, \psi_m (a_i)$$

$$+ \sum_{i=1}^{R} J_i \, \psi_n' (a_i) \, \psi_m' (a_i) = 0. \qquad \text{...(36)}$$

In deriving (36) we have used the properties that,

$$\delta (x - a) = 0 \text{ for } x \neq a \qquad \text{...(37)}$$

and

$$\int_{-\infty}^{+\infty} \varphi (x) \, \delta (x - a) \, dx = \varphi (a).$$

For the special case, $J_i = 0$, $M_1 = M$ and $M_i = 0$ for $i \neq 1$

(36) reduces to,

$$\int_0^{2L} \rho_0 e^{\alpha x} \psi_n \psi_m \, dx + M \psi_n (a) \psi_m (a) = 0 \qquad \text{...(38)}$$

which agrees with the condition (16) in Case I.

With the orthogonality condition (36) it can be shown that any arbitrary continuously differentiable function $h^* (x)$, defined in the interval $[0, 2L]$, has a Fourier-type expansion in the obtained eigen functions :

$$h^* (x) = \sum_{n=1}^{\infty} \zeta_n \psi_n (x)$$

with its Fourier coefficients ζ_n given by the formula,

$$\zeta_n = \frac{\int_0^{2L} \rho_0 e^{\alpha x} h^* \psi_n \, dx + \sum_{i=1}^{R} M_i h^* (a_i) \psi_n (a_i) + \sum_{i=1}^{R} J_i h^{*\prime} (a_i) \psi_n' (a_i)}{\int_0^{2L} \rho_0 e^{\alpha x} \psi_n^2 \, dx + \sum_{i=1}^{R} M_i \left[\psi_n (a_i) \right]^2 + \sum_{i=1}^{R} J_i \left[\psi_n' (a_i) \right]^2} \qquad \text{...(39)}$$

3.2 *Forced vibration*

Let us expand $y (x, t)$ in the eigen functions,

$$y (x, t) = \sum_{n=1}^{\infty} q_n (t) \psi_n (x). \qquad \text{...(40)}$$

After substituting (40) in (24) and then multiplying (24) by $\psi_m (x)$ and integrating we get,

$$E_0 I \sum_{n=1}^{\infty} q_n (t) \int_0^{2L} \left\{ \frac{\partial^2}{\partial x^2} \left[e^{\alpha x} \psi_n'' \right] \right\} \psi_m \, dx$$

$$+ \sum_{n=1}^{\infty} \ddot{q}_n (t) \int_0^{2L} \left\{ - \left[\sum_{i=1}^{R} J_i \delta (x - a_i) \psi_n' \right]' \right.$$

$$\left. + \left[\rho_0 e^{\alpha x} + \sum_{i=1}^{R} M_i \delta (x - a_i) \right] \psi_n \right\} \psi_m \, dx$$

(*equation continued on p.* 87)

$$= \sum_{j=1}^{S} F_j(t) \int_0^{2L} \delta(x - b_j)\, \psi_m\, dx$$

$$+ \sum_{k=1}^{T} G_k(t) \int_0^{2L} \delta'(x - c_k)\, \psi_m\, dx \qquad ...(41)$$

using (34) and the orthogonality condition (36) the general solution for the forced vibration is obtained as

$$y(x, t) = \sum_{n=1}^{\infty} (P_n \sin p_n t + Q_n \cos p_n t)\, \psi_n(x)$$

$$+ \sum_{n=1}^{\infty} \frac{\psi_n(x)}{p_n N_n} \int_0^{t} \Big[\sum_{j=1}^{S} \psi_n(b_j)\, F_j(\tau)$$

$$- \sum_{k=1}^{T} \psi_n'(c_k)\, G_k(\tau) \Big] \sin p_n (t - \tau)\, d\tau \qquad ...(42)$$

where

$$N_n = \int_0^{2L} \rho_0\, e^{\alpha x}\, \psi_n^2\, dx + \sum_{i=1}^{R} M_i \Big[\psi_n(a_i) \Big]^2$$

$$+ \sum_{i=1}^{R} J_i \Big[\psi_n'(a_i) \Big]^2. \qquad ...(43)$$

If the forcing functions are sinusoidal $F_j(t) = F_j \sin \omega_j t$ and $G_k(t) = G_k \sin \omega_k t$, (42) may be integrated into,

$$y(x, t) = \sum_{n=1}^{\infty} (P_n \sin p_n t + Q_n \cos p_n t)\, \psi_n(x)$$

$$+ \sum_{n=1}^{\infty} \frac{\psi_n(x)}{p_n N_n} \Big\{ \sum_{j=1}^{S} \psi_n(b_j)\, F_j \Big(\frac{\omega_j}{\omega_j^2 - p_n^2} \sin p_n t$$

$$- \frac{p_n}{\omega_j^2 - p_n^2} \sin \omega_j t \Big) - \sum_{k=1}^{T} \psi_n'(c_k) G_k \Big(\frac{\omega_k}{\omega_k^2 - p_n^2} \sin p_n t$$

(equation continued on p. 88)

$$- \frac{p_n}{\omega_k^2 - p_n^2} \sin \omega_k t \Big) \Big\}. \qquad \ldots(44)$$

The constants P_n and Q_n, as mentioned earlier, are to be determined from the initial conditions.

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