Indian J. pure appl. Math., 20 (5) : 439-445, May 1989

RADICAL GOLDIE NEAR-RINGS

KHANINDRA CHANDRA CHOWDHURY

Department of Mathematics, Gauhati University, Guwahati 781014

(*Received* 10 *May* 1988)

Here we prove the existence of the classical near-ring of right quotients of the sub near-ring D of the distributive elements of a prime Goldie Abelian near-ring K which is such that some power of each element of K belongs to D and further its non nilpotent elements are distributive.

§1. We have already introduced the notion of a Goldie near-ring and have established some results on prime and semiprime Goldie near-rings, proved Goldie Theorem analogue for Abelian Goldie near-rings in which non nilpotent elements are distributive[1].

In this paper we discuss a radical Goldie near-ring. If A is a subnear-ring of a near-ring K then K is radical over A (or K is A-radical) if a power of each element in K lies in A. In an Abelian near-ring in which non nilpotent elements are distributive[1] K is D-radical, where D is the subnear-ring of distributive elements of K. Here we establish that if K is a prime Goldie Abelian near-ring which is radical over the sub-near-ring D of distributive elements of K and further its non nilpotent elements are distributive, then D has a classical near-ring of right quotients. Unless otherwise specified a near-ring K will contain unity and $a.0 = 0$ for all $a \in K$.

§2. A countable ordered family $\{A_1, A_2, \ldots,\}$ of subsets of a near-ring K is an independent family if for all $n \in N$, $A_i \cap (\sum_{k \neq i} A_k) = 0$, were $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n$.

A right (left) K-subset A of (right) near-ring K is a subset of K such that $ak\ (ka) \in A$ for $a \in A$, $k \in K$. If A is a right and a left K-subset of K then A is a K-subset of K. A is a right (left) essential K-subset of K if for any non zero right (left) K-subset of K has a nonzero intersection with A. A near-ring K is semiprime if K has no nonzero nilpotent K-subset, K is prime if O is a prime K-subset (i. e. for any two K-subsets A, B of K, $AB = 0 \Rightarrow A = 0$ or $B = 0$).

A Goldie near-ring is a (right) near-ring K with a.c.c. on right annihilator K-subsets and which has no infinite independent family of right K-subsets of it.

In an Abelian near-ring K, the subset D of distributive elements of K is a sub-near-ring of K (it is actually a ring). The following two Lemmas are easy.

Lemma 2.1.1— Let K be an Abelian near ring and S be a set of distributive elements of K. Then the right annihilator K-subset $r_K\ (S)$ is a right ideal of K.

Lemma 2.1.2— If A is a subnear-ring of a near-ring K and $S \subseteq A$, then $r_A\ (S) = r_K\ (S) \cap A$.

Lemma 2.2.1— Let K be a near-ring. Suppose $k \in K$ is such an element that for each $x \in K$ there exists $n \in Z^+$ (depending on x) satisfying the condition $kx^n\ k = 0$. (In Lemma 2.2.7 we see that such a condition is justified). Then there exists a nonzero element a of K with $a^2 = 0$ and satisfying the same hypothesis as k.

Proof : Assume $x = k$. Then $k^\alpha = 0$ for some $\alpha > 2$. For $k^2 = 0$ we get the required result on taking $k = a$. And if $k^2 \neq 0$ let α be minimal such that $k^\alpha = 0$. Then $a = k^{\alpha-1}\ (\neq 0)$ satisfies the required hypothesis.

Lemma 2.2.2— Let K be a semiprime Goldie near-ring and $k \in K$, $k \neq 0$ be as in Lemma 2.2.1.

Then there exists $a \in K$ with $a^2 = 0$ such that a satisfies same hypothesis as k and for any $m \in K$ with $m^2 = 0$ we have $ama = 0$.

Proof : By above Lemma there exists $a \in K\ a \neq 0$, $a^2 = 0$ satisfying the same hypothesis as k. Let $m \in K$ be such that $m^2 = 0$. Then by given hypothesis there exists $\alpha \geqslant 1$ such that for any $x \in K$ we have

$$a\ (m + maxa)^\alpha\ a = 0, \text{ or } a\ (m + maxa)^{\alpha-1}\ (m + maxa)\ a = 0$$

or, $a\ (m + maxa)^{\alpha-1}\ (ma) = 0$ (since right distributivity holds in K and $a^2 = 0$). Using the fact that $m^2 = 0$ and repeating in like way we get finally $(xama)^\alpha = 0$.

Thus, $Kama$ is a nil left K-subset of K. And K being semi-prime we therefore get that $ama = O\ (1 \in K)$.

Lemma 2.2.3— Let K be as in Lemma 2.2.2. Then there exists $a \in K$, $a \neq 0$, $a^2 = 0$ such that a satisfies the same hypothesis as k in the above Lemma and for any $p, q \in K$, $pq = 0$ implies $paq = 0$.

Proof : From above Lemma we get $a \in K$ such that $a^2 = 0$, $a \neq 0$ and a satisfies same hypothesis as k and for any $m \in K$ with $m^2 = 0$ we have $ama = 0$.

Now let $p, q = K$ and $pq = 0$. Then for any $x \in K$, $(qxp)^2 = 0$ which gives $a\ (qxp)\ a = 0$ for any $x \in K$. Thus $(Kpaq)^2 = 0$ from which $Kpaq = 0$ follows for K is semiprime Goldie. Therefore $paq = 0$ (since $1 \in K$).

Lemma 2.2.4— Let K be a semiprime Goldie near-ring and $k \in K$ be such that for each $x \in K$, there exists an $n \in Z^+$ (depending on x) such that $kx^n\ k = 0$. Then $k = 0$.

Proof : If $k \neq 0$, then by Lemma 2.2.3 we get $a \in K$, $a \neq 0$, $a^2 = 0$ such that a satisfies the hypothesis as in the Lemma. Now for any $x \in K$ we get $n \in Z^+$ such that $ax^n a = 0$. If $n = 1$ then $axa = 0$. Therefore $(xa)^2 = 0$. And if $n > 1$ then $ax^n a = 0$ gives $(ax^{n-1}) a (xa) = 0$. By the hypothesis satisfied by a we get $(ax^{n-1}) a (xa) = 0$. And this gives $ax^{n-2} (xa)^2 = 0$. In like manner we get $ax^{n-3} (xa)^3 = 0$. Finally we get $(xa)^{n+1} = 0$. Thus in any case we get $(xa)^{n+1} = 0$. So Ka is a nilpotent left K-subset of K. And K being semiprime it therefore follows that $Ka = 0$ which gives $a = 0$, a contradiction. Hence $k = 0$.

Lemma 2.2.5— Let K be a prime Goldie near-ring. Suppose $a, b \in K$, $a \neq 0$ and for each $x \in K$ there exists an $n \in Z^+$ (depending on x) such that $ax^n b = 0$. Then $b = 0$.

Proof : Let $\rho = \{y \in K \mid y\ x^{n(x)}\ b = 0,\ x \in K\}$. Since for any $k \in K$, $kyx^{n(x)} b = 0$ is a left K-subset of K. And from Lemma 2.2.4 $(by)\, x^{n(x)}\, (by) = 0$ gives $by = 0$. Thus $b\, \rho = 0$ which gives $(KbK)\ (\rho K) \subseteq Kb\, \rho\, K\, (= 0)$. And since $\rho K \neq 0$ and K is prime it therefore follows that $KbK = 0$. And hence $b = 0$.

Lemma 2.2.6— Let K be a prime Goldie near-ring and A be any subnear-ring K. If K is A-radical, then

(i) for any A-subset I, J of A with $IJ = 0$, $I \neq 0$; we get $J = 0$;

(ii) if $a \in A$ and $r_A(a) = 0$ then $r_K(a) = 0$.

Proof : (i) Let $a \in I$, $a \neq 0$ and $b \in J$; K being A-radical for $x \in K$ there exists $n \in Z^+$ such that $x^n \in A$. Now $ax^n b \in IAJ \subseteq IJ\ (= 0)$. Thus $ax^n b = 0$, $a \neq 0$. Therefore by Lemma 2.2.5 we get $b = 0$, So $J = 0$.

(ii) If $r_K(a) \neq 0$, let $x \in r_K(a)$, $x \neq 0$. Then for some $n \in Z^+$ we have $x^n \in A$ and $ax^n = 0$. So $x^n \in r_A(a)$ which means that x is nilpotent. Thus $r_K(a)$ is a nil right K-subset of K. And K being prime it therefore follows that $r_K(a) = 0$.

Lemma 2.2.7— Let K be a semiprime Goldie near-ring and A a subnear-ring of K such that K is A-radical. Then

(i) A cannot have any nonzero nil right (left) A-subset;

(ii) if for $a_1, a_2 \in A$ we have $a_1 A \cap a_2 A = 0$, then $a_1 K \cap a_2 K = 0$.

Proof : First we show that A cannot have nilpotent right A-subset. Let ρ be a nilpotent right A-subset of A. Without loss of generality we may assume $\rho^2 = 0$ and let $a \in \rho$, $x \in K$. We get $n \in Z^+$ such that $x^n \in A$. And then $ax^n a \in \rho A \rho \subseteq \rho\rho\ (= 0)$. Thus for any $x \in K$ there exists $n \in Z^+$ such that $ax^n a = 0$. And by Lemma 2.2.4 we get $a = 0$. Therefore $\rho = 0$. Now we show that A can not have nonzero nil right A subset. For this we show that a satisfies the a.c.c. for right annihilators.

Consider an ascending chain

$r_A(S_1) \subseteq r_A(S_2) \subseteq \ldots$ of right annihilator; where $S_1, S_2, \ldots, \subseteq A$. Then

$r_K(l_A(r_A(S_1))) \subseteq r_K(l_A(r_A(S_2))) \subseteq \ldots$. Since K is Goldie, we therefore get $t \in Z^+$ such that

$$r_K(l_A(r_A(S_t))) = r_K(l_A(r_A(S_{t+1}))) = \ldots$$

or,

$$A \cap (r_K(L_A(r_A(S_t))) = A \cap (r_K(l_A(r_A(S_{t+1}))) = \ldots.$$

And because of Lemma 2.1.2 we get

$$r_A(l_A(r_A(S_t))) = r_A(l_A(r_A(S_{t+1}))) = \ldots.$$

Hence

$$r_A(S_t) = r_A(S_{t+1}) = \ldots .$$

(ii) Consider the right A-subset $\rho = a_1 K \cap a_2 A$. Let $x \in \rho$. Then $x = a_1 k_1 = a_2 k_2$ where $k_1 \in K$, $k_2 \in A$. Since K is A-radical there exists $n \geqslant 1$ such that $(k_1 a_1)^n \in A$. Since $a_1, k_2 \in A$ we get $a_1 (k_1 a_1)^n = (a_1 k_1)^n a_1 = (a_2 k_2)^n a_1 \in a_1 A \cap a_2 A$ $(= 0)$.

Therefore

$$a_1 (k_1 a_1)^n = 0.$$

Hence

$$x^{n+1} = (a_1 k_1)(a_1 k_1)^n = a_1 (k_1 a_1)^n k_1 = 0.$$

Thus ρ is a nil right A-subset of A. Therefore $\rho = 0$. So $a_1 K \cap a_2 A = 0$.

Now let $x \in a_1 k \cap a_2 K$ and suppose $x = a_1 t_1 = a_2 t_2$, where $t_1, t_2 \in K$. Since K is A-radical we get $m \in Z^+$ such that $(t_2 a_2)^m \in A$. Then $(a_1 t_1)^m a_2 = (a_2 t_2)^m a_2 = a_2 (t_2 a_2)^m \in a_1 K \cap a_2 A$ $(= 0)$. So $x^{m+1} = (a_1 t_1)^{m+1} = 0$. Thus $a_1 K \cap a_2 K$ is a nil right K-subset of K. K being semiprime, we therefore get $a_1 K \cap a_2 K = 0$.

The following Lemmas have been proved in Chowdhury[1].

Lemma 2.2.8 : For any right essential K-subset A of a near-ring K, if $a \in A$ then

$$a^{-1} A = \{x \in K \mid ax \in A\}$$

is a right essential K-subset of K (Corollary 2.3.3 of Chowdhury[1]).

Lemma 2.2.9— Let K be a Goldie near-ring whose non-nilpotent elements are distributive. If $x \in K$ be such that $r(x) = 0$, then xK is a right essential K-subset of K (Lemma 2.4.1 of Chowdhury[1]).

Lemma 2.2.10— A semiprime right Goldie near-ring is right non-singular (Lemma 2.4.3 of Chowdhury[1]).

Lemma 2.2.11— If K is a semi prime Goldie near-ring where non nilpotent elements are distributive then every right essential K-subset of K contains a regular element. (Lemma 2.4.4 of Chowdhury[1]).

Lemma 2.2.12— Let K be a semiprime Abelian Goldie near-ing such that its non-nilpotent elements are distributive. Then, K-satisfies the d.c.c. for right annihilator ideals (Lemma 2.4.5 of Chowdhury[1]).

Lemma 2.3.1—Let K be a prime Goldie Abelian near-ring which is such that its non-nilpotent elements are distributive. If D is the sub near ring of distributive elements of K, then D satisfies the D.C.D. on right annihilators.

PROOF : If $S \subseteq D$, then $r_D(S) = r_K(S) \cap D$.

Now let $r_D(S_1) \supseteq r_D(S_2) \supseteq \ldots$, where $S_1, S_2, \ldots \subseteq D$ be a decending chain of right annihilators in D. Then

$$r_K(l_D(r_D(S_1))) \supseteq r_K(l_D(r_D(S_2))) \supseteq \ldots$$

Here $r_K(l_D(r_D(S_1)))$, ... etc. are right ideals. Therefore by Lemma 2.2.12, these exists $t \in Z^+$ such that

$$r_K(l_D(r_D(S_t))) = r_K(l_D(r_D(S_{t+1}))) = \ldots$$

Therefore $D \cap r_K(l_D(r_D(S_t))) = D \cap r_K(l_D(r_D(S_{t+1}))) = \ldots$

And by Lemme 2.1.2 we get $r_D(l_D(r_D(S_t))) = r_D(l_D(r_D(S_{t+1})))$ which gives $r_D(S_t) = r_D(S_{t+1}) = \ldots$

Thus D satisfies the D.C.C. on right annihilators.

Lemma 2.3.2— Let K be as above. If for some $a \in D$, we have $r_D(a) \neq 0$ then aD is not a right essential D-subset of D.

PROOF : If $aK \leqslant_e K$, then by Lemma 2.2.11 $r_K(a) = 0$ and therefore $r_D(a) = r_K(a) \cap D = 0$, a contradiction. So $aK \nleqslant_e K$. If possible let $aD \leqslant_e D$ and consider any nonzero right K-subset C of K. Then C is not nil and let c be a non-nilpotent element of C. By hypothesis $c \in D$. Since $1 \in D$, $cD \neq 0$. So $aD \cap cD \neq 0$.

Now

$$aK \cap C \supseteq aK \cap cK \supseteq aD \cap cD \neq 0.$$

Thus $aK \leqslant_e K$, a contradiction. Hence $aD \nleqslant_e D$.

Lemma 2.3.3— Let K and D be as above and K is D-radical. Then a right D-subgroup λ of D which is also a right essential D-subset of D contain a regular element.

PROOF : Choose $a \in \lambda$ so that $r_D(a)$ is minimal in D (Lemma 2.3.1). We claim that $r_D(a) = 0$.

If $r_D(a) \neq 0$ then by Lemma 2.3.2 $aD \nleq_e D$. So there is a nonzero right D-subset J of D such that $aD \cap J = 0$. Since $\lambda \leqslant_e D$,

$$\lambda \cap J \neq 0. \text{ Write } J' = \lambda \cap J.$$

Then

$$aD \cap J = aD \subset (\lambda \cap J) = (aD \cap J) \cap \lambda = 0.$$

If $x \in J'$ and $d \in r_D(x + a)$, then $xd + ad = 0$

Or,

$$ad = -xd = y(-d) \text{ (since } x \in D)$$

$$= 0 \text{ (for } aD \cap J = 0).$$

Thus $d \in r_D(a) \cap r_D(x)$. Therefore $r_D(x + a) \subseteq r_D(a) \cap r_D(x) \subseteq r_D(a)$.

And $x + a \in \lambda$ (since λ is a subgroup of D).

Hence we get, by minimality of $r_D(a)$, $r_D(x + a) = r_D(a)$.

So $r_D(x + a) = r_D(a) \cap r_D(x) = r_D(a)$ which gives $r_D(a) \subseteq r_D(x)$. Hence $x\, r_D(a) = 0$. Therefor we get $J'\, r_D(a) = 0$. Thus $(DJ')\,(Dr_D(a)) \subseteq DJ'\, r_D(a)\ (=0)$. So $Dr_D(a) = 0$ for $DJ' \neq 0$ (since $1 \in D$) (Lemma 2.2.6 (i)). It follows that $r_D(a)=0$. Therefore $r_K(a) = 0$ (Lemma 2.2.6 (ii)). New let $c \in l_K(a)$. Then $ca = 0$ which gives $caK = 0$. By Lemma 2.2.9 $aK \leqslant_e K$ and therefore $c \in Z(K)$. And K being right non singular (Lemma 2.2.10), we get $c = 0$. Thus $l_K(a) = 0$. It follows that a is regular in D.

§3. We now prove the main result.

Theorem 3.1 : Let K be a prime Goldie Abelian near-ring which is D-radical and whose non nilpotent elements are distributive. Then D has a classical near-ring of right quotients.

PROOF: Let $a, b \in D$ and a be regular in D. Then $r_D(a) = 0$. By Lemma 2.2.6 (ii) we get $r_K(a) = 0$. So $aK \leqslant_e K$ (Lemma 2.2.9). Now if $aD \leqslant_e D$, then there exists a non zero right D-subset ρ of D such that $aD \cap \rho = 0$. Let $x \in \rho$, $x \neq 0$, we then get $aD \cap xD \subseteq aD \cap \rho\ (= 0)$ or $aD \cap xD = 0$ which gives $aK \cap xK = 0$ (Lemma 2.2.7 (ii)) put $aK \leqslant_e K$ then gives $xK = 0$ which is a contradiction (since $1 \in K$). So $aD \leqslant_e D$.

Now let $\lambda = \{x \in D \mid bx \in aD\}$. Then by Lemma 2.2.8, $\lambda \leqslant_e D$. And λ is a subgroup of D (Since b is distributive and aD is a right ideal of D). Thus λ is a right D-subgroup of D. So by Lemma 2.3.3, λ contains a regular element, say a'. Hence $ba' = ab'$, for some $b' \in D$. Thus the right Ore condition is satisfied with respect to the set of regular elements in D. Therefore by Theorem 3.3[1] D has a classical near ring of right quotients.

Acknowledgement

The author wishes his gratitude to Professor B. K. Tamuli, Head of the Mathematics department (retd.), Gauhati University, for his helfp in the work.

Reference

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