Indian J. pure appl. Math., **20** (5) : 455–465, May 1989

A TRANSFORMATION OF THE FINSLER METRIC BY AN *h*-VECTOR

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(*Received 5 May* 1988)

The propose of the present paper is to find the relation between v-curvature tensors with respect to $C\Gamma$ of the Finsler spaces (M^n, L) and (M^n, L^*) where $L^*(x, y)$ is obtained from $L(x, y)$ by the transformation.

$$L^{*2}(x, y) = L^{*}(x, p) + (X_i, Y^i)^2$$

where $X_i(x, y)$ is an h-vector in (M^n, L). The relation in n-fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L^*) have also been obtained.

1. INTRODUCTION

Let $F^n = (M^n, L)$ be an in n-dimensional Finsler space, where M^n is an n-dimensional differentiable manifold and $L(x, y)$ is the Finsler fundamental function. Matsumoto[7] introduced the transformations of Finsler metric :

$$\bar{L}(x, y) = L(x, y) + X_i y^i \qquad \ldots(1.1)$$

$$L^{*2}(x, y) = L^2(x, y) + (X_i y^i)^2 \qquad \ldots(1.2)$$

and obtained the relation between the imbedding class numbers of tangent Riemannian spaces to (M^n, L), $(M^n \bar{L})$ and (M^n, L^*). It has been assumed that the functions X_i in (1.1) and (1.2) are functions of coordinate only. Since a concurrent vector field is a function of coordinate only, assuming X_i as a concurrent vector field, Matsumoto[4] has studied the $R3$-likeness of Finsler spaces (M^*, L) and $(M^n, \bar{L})$, Singh and Prasad[14] and Prasad *et al.*[10] generalized the concept of concurrent vector field and introduced the semi parallel and concircular vector fieldswhich are functions of coordinate only. Assuming X_i as a concircular vector field, Prasad, Singh and Singh[10] has studicd the $R3$-likeness of (M^n, L) and $(M^n, \bar{L})$.

If $L(x, y)$ is a metric function of Riemannian space, than $\bar{L}(x, y)$ reduces to the metric function of Rander's space. Such a Finsler metric was first introduced by

Rander's[12] from the stand point of general theory of relativity and applied to the theory of electron microscope by ingraden[2] who first named it a Rander's space. The geometrical properties of the space have been studied by various authors. Numata[9] studied the properties of $(M^n, \bar{L})$ which is obtained from Minkowskian space (M^n, L) by the transformation (1.1). In all these works the functions X_i are assumed to be a function of coordinate only.

Izumi[1] while studying the conformal transformation of Finsler spaces, introduced the h-vector X_i which is v-covariantly coastant with respect to Cartan's connection $C\Gamma$ and satisfies $L\, C^h_{ij}\, X_h = \rho\, h_{ij}$. Thus the h-vector X_i is not only a function of coordinate but it is a function of directional argument satisfying $L\, \partial_i X_i = \rho\, h_{ij}$, Prasad *et al.*[11] has obtained the relation between imbedding class numbers of (M^n, L) and $(M^n, \bar{L})$ where $\bar{L}\,(x, y)$ is obtained from $L\,(x, y)$ by the transformation (1.1) under the assumption that X_i is an h-vector in (M^n, L).

2. The Finsler Space (M^n, L^*)

Let X_i be a vector field in the Finsler space (M^n, L). If X_i satisfy the conditions

$$X_i \mid_j = 0 \qquad \ldots(2.1)$$

$$L\, C^h_{ij}\, X_h = \rho\, h_{ij} \qquad \ldots(2.2)$$

then the vector neld X_i is called an h-vector[1]. Here $\mid_j$ denote the v-covariant differentiation with respect to Cartan connection $C\Gamma$, C^h_{ij} is the Cartan's C-tensor, h_{ij} is the angular metric tensor and ρ is function given by

$$\rho = \frac{1}{n-1}\, L\, C^i\, X_i \qquad \ldots(2.3)$$

where C^i is torsion vector $C_{jk}\, g^{jk}$. The first of the following lemmas has been proved in[1] while the other two is a direction consequence of the definition of h-vector.

Lemma 2.1—If X_i is an h-vector than the function ρ and $X^*_i = X_i - \rho\, l_i$ are independent functions of y.

Lemma 2.2—The magnitude X of an h-vector X_i is independent function of y.

Lemma 2.3—For an h-vector X_i we have $S_{hijk} X^h = 0$ where S_{hijk} is v-curvature tensor of Cartan's connection $C\Gamma$.

Let X_i is an h-vector in the Fnsler space (M^n, L) and (M^n, L^*) be another Finsler space whose fundamental metric function $L^*\,(x, y)$ is defined by

$$L^{*2}\,(x, y) = L^2\,(x, y) + \beta^2\,(x, y) \qquad \ldots(2.4)$$

where $\beta(x, y) = X_i y^i$. Since X_i is h-vector from (2.1) and (2.2) we get

$$\dot{\partial}_j X_i = L^{-1} \rho h_{ij}$$

which after using the indicatory property of h_{ij} $\dot{\partial}_j \beta = X_j$. Thus differentiation of (2.4) with respect to y^i gives

$$L^* l^*_i = L l_i + \beta X_i \qquad ...(2.5)$$

where $l^*_i = \dot{\partial}_i L^*$ is the normalized element of support in (M^n, L^*). The quantities of (M^n, L^*) will be denoted by star letters

Since $\dot{\partial}_j l_i = L^{-1} h_{ij}$, differentiation of (2.5) with y^j and application of (2.4) give

$$h^*_{ij} + l^*_i l^*_j = \sigma h_{ij} + l_i l_j + X_i X_j \qquad ...(2.6)$$

where

$$\sigma = \left(1 + \frac{\beta \rho}{L}\right). \qquad ...(2.7)$$

Hence we have

$$g^*_{ij} = g_{ij} + (1 - \sigma) l_i l_j + X_i X_j. \qquad ...(2.8)$$

From (2.8), the relation between the contravariant components of the fundamental metric tensors can be derived as follows :

$$g^{*ij} = \sigma^{-1} g^{ij} - \frac{(1 - \sigma)\beta}{L \lambda} (l^i X^j + l^j X^i) + \frac{(1 - \sigma)(X^2 + \sigma)}{\lambda} l^i l^j + \frac{1}{\lambda} X^i X^j \qquad ...(2.9)$$

where

$$\lambda = \left\{ \frac{\sigma (1 - \sigma) \beta^2}{L^2} - X^2 - \sigma \right\} \qquad ...(2.10)$$

and X is the magnitude of the vector X^i $(= g^{ij} X_j)$.

From the Lemma 2.1 and relation (2.7) we get

$$\dot{\partial}_i \sigma = \frac{\rho}{L} m_i \qquad ...(2.11)$$

where

$$m_i = X_i - \frac{\beta}{L} l_i. \qquad ...(2.12)$$

Since $\partial_j\ l_i = L^{-1} h_{ij}$, differentiating (2.8) with respect to y^k and using (2.4), (2.11) and (2.12) we get

$$C^*_{ijk} = \sigma\, C_{ijk} + \frac{\rho}{2L}(h_{ij}\, m_k + h_{ki}\, m_j + h_{jk}\, m_i). \qquad \text{...(2.13)}$$

From the definition of m_i, it is evident that

(a) $m_i\, l^i = 0$

(b) $m_i\, X^i = m_i\, m^i = X^2 - \dfrac{\beta^2}{L^2}$

(c) $h_{ij}\, m^i = h_{ij}\, X^i = m_j$

(d) $C^h_{ij}\, m_h = L^{-1}\, \rho\, h_{ij}.$ ...(2.14)

From (2.9), (2.2), (2.13) and (2.14) we get

$$C^{*h}_{ij} = C^h_{ij} + \frac{\rho}{2L\sigma}\left(h_{ij}\, m^h + h^h_j\, m_i + h^h_i\, m_j\right)$$
$$- \frac{(1-\sigma)\,\beta\,\rho}{L^2\,\lambda}\left[\left\{\sigma + \tfrac{1}{2}\left(X^2 - \frac{\beta^2}{L^2}\right)\right\} h_{ij}\, l^h + m_i\, m_j l^h\right]$$
$$+ \frac{\rho}{L\lambda}\left[\left\{\sigma + \tfrac{1}{2}\left(X^2 - \frac{\beta^2}{L^2}\right)\right\} h_{ij}\, X^h + m_i\, m_j\, X^h\right)\Big]. \qquad \text{...(2.15)}$$

We shall now find the v-corvature tensor S^*_{hijk} of (M^n, L^*) which with respect to the Cartan's connection $C\,\Gamma$, is defined as

$$S^*_{hijk} = C^*_{hkm}\, C^{*m}_{ij} - C^*_{hjm}\, C^{*m}_{ik}. \qquad \text{...(2.16)}$$

Firstly from (2.13) and (2.15) we have

$$C^*_{hkm}\, C^{*m}_{ij} = C_{hkm}\, C^m_{ij} + \alpha_1\, h_{ij}\, h_{hk} + \frac{\rho}{2L}(C_{ijk}\, m_h$$
$$+ C_{ijh}\, m_k + C_{ihk}\, m_j + C_{jhk}\, m_i)$$
$$+ h_{hk}\, m_i\, m_j\left\{\frac{\rho^2\,\sigma}{2L^2\,\lambda} + \frac{\rho^2}{2L^2\,\sigma} + \frac{\rho^2}{2L^2\,\lambda}\right.$$
$$\left.\times\left(X^2 - \frac{\beta^2}{L^2}\right)\right\} + h_{ij}\, m_h\, m_k\left[\frac{\rho^2}{2L^2\sigma} + \frac{\rho^2}{2L^2\,\lambda}\left\{\sigma\right.\right.$$
$$\left.\left.+ \tfrac{1}{2}\left(X^2 - \frac{\beta^2}{L^2}\right)\right\}\right] + \frac{\rho^2}{4\,L^2\,\lambda}(h_{jh}\, m_i\, m_k + h_{ih}\, m_j$$

(*equation continued on p.* 459)

$$\times\ m_k + h_{jr}\, m_i\, m_h + h_{ik}\, m_j\, m_h) + \frac{\rho^2}{L^2\lambda}\, m_i\, m_j\, m_h\, m_k \qquad ...(2.17)$$

where

$$\alpha_1 = \frac{\rho^2}{L^2} + \frac{\rho^2\sigma^2}{L^2\lambda} + \frac{\rho^2 \sigma}{L^2\lambda}\left(X^i - \frac{\beta^2}{L^2}\right) + \frac{\rho^2}{4L^2\sigma}\left(X^2 - \frac{\beta^2}{L^2}\right) + \frac{\rho^2}{4L^2\lambda}\left(X^2 - \frac{\beta^2}{L^2}\right)^2. \qquad ...(2.18)$$

Thus from (2.16) we get

$$S^*_{hijk} = \sigma\, S_{hijk} + h_{ij}\, d_{hk} + h_{hk} + h_{hk}\, d_{ij} - h_{ik}\, d_{hj} - h_{hj}\, d_{ik} \qquad ...(2.19)$$

where

$$d_{ij} = \tfrac{1}{2}\, \alpha_1\, h_{ij} + \alpha_2\, m_i\, m_j \qquad ...(2.20)$$

$$\alpha_2 = \frac{\rho^2\sigma}{L^2\lambda} + \frac{\rho^2}{2L^2\lambda} + \frac{\rho^2}{2L^2\lambda}\left(X^2 - \frac{\beta^2}{L^2}\right) - \frac{\rho^2}{4L^2\lambda} \qquad ...(2.21)$$

3. Hypersurface of (M^n, L)

Let (M^{n-1}, L) be a hypersurface of (M^n, L) given by the equation

$$x^i = x^i\,(u^\alpha). \qquad ...(3.1)$$

Let us suppose that the functions (3.1) are atleast of class C^3 in u^α and the projection factors $\beta^j_\alpha = \frac{\partial x^j}{\partial u^\alpha}$ are such that their matrix has maximal rank $n - 1$. The fundamental metric function L (u, v) of the hypersurface is given by

$$\mathrm{L}\,(u^\alpha, v^\alpha) = L\,(x^i\,(u^\alpha)\, B^i_\alpha\, v^\alpha)$$

where v^α is the element of support for the hypersurface for which

$$y^i = B^i_\alpha\, V^\alpha.$$

Thus if l^α denote the normalized vector using the element of support then

$$l^i = B^i_\alpha\, l^\alpha. \qquad ...(3.2)$$

If $g_{hj}\,(x, y)$ denotes the metric tensor of (M^n, L), the induced metric tensor of (M^{n-1}, L) is given by

$$g_{\alpha\beta}\,(u, v) = g_{hj}\,(x, y)\, B^h_\alpha\, B^j_\beta\,. \qquad ...(3.3)$$

The inverse of (3.3) is denoted by $g^{\alpha\beta}(u, v)$ by means of which we define the quantities

$$B^{\alpha}_{i}\ (n, v) = g^{\alpha\beta}(u, v)\, g_{ij}\ (x, y)\ B^{j}_{\beta}\,. \qquad ...(3.4)$$

The unit normal vector $N^j(x, y)$ of (M^{n-1}, L) is determined by the relations

$$g_{hj}(x, y)\, B^{h}{}_{\beta} N^{j}(x, y) = 0,\ g_{hj}(x, y)\, N^{h}(x, y)\ N^{j}(x, y) = 1. \qquad ...(3.5)$$

We have the following identity from (3.3), (3.4) and (3.5)

$$B^{\alpha}_{j}\ B^{j}_{\beta} = \delta^{\alpha}_{\beta}\ ,\ B^{j}_{\alpha}\ B^{\alpha}_{h} + N^{j} N_{h} = \delta^{j}_{h} \qquad ...(3.6)$$

where

$$N_i = g_{ij}(x, y)\, N^j.$$

If $C_{hjk}(u, y)$ denotes the (*h*) *hv*-torsion tensor of (M^n, L) the induced (*h*) *hv*-torsion tensor $C_{\alpha\beta\gamma}(u, v)$ of (M^{n-1}, L) is given by

$$C_{\alpha\beta\gamma}(u, v) - C_{hjk}(x, y)\ B^{h}_{\alpha}\ B^{j}_{\beta}\ B^{R}_{\gamma} \qquad ...(3.7)$$

from which we obtain

$$C^{\alpha}_{\beta\gamma} = B^{\alpha}_{i}\ \ C^{j}_{jk}\ \ B^{j}_{\beta}\ B^{k}_{\gamma}\,. \qquad ...(3.8)$$

The relative *v*-covarient derivative of the projection factor B^{i}_{β} with respect to the induced Cartan connection $I\ C\ \Gamma$ is defined as[6]

$$B^{i}_{\beta}\ |\ \gamma = -B^{i}_{\alpha}\ \ C^{i}_{\beta\gamma} + C^{i}_{hk}\ \ B^{h}_{\beta}\ \ B^{k}_{\gamma}\,. \qquad ...(3.9)$$

This tensor is normal to (M^{n-1}, L). Therefore we may write

$$B^{i}_{\beta|\gamma} = M_{\beta\gamma}\, N^{i}. \qquad ...(3.10)$$

From (3.9), it is clear that $M_{\beta\gamma}$ is symmetric in β and γ and it may be written as

$$M_{\beta\gamma} = C_{ijk}\, N^{i}\ B^{j}_{\beta}\ \ B^{k}_{\gamma}\,. \qquad ...(3.11)$$

The tangent vector-space M^{n-1}_{x} to M^{n-1} at every point $x^i\ (= u^\alpha)$ of the hypersurface is considered as the Riemannian space $(M^{n\ 1}_{x}, g_x)$ with the Riemannian metric $g_x = g_{\alpha\beta}(u, v)\, dv^{\alpha}\, dv^{\beta}$. The components of the (*h*) *hv*-torsion tensor $C^{\alpha}_{\beta\gamma}$ will be the Christoffel symbols associated with g_x. If M^{n}_{x} is the tengent vector space to M^n at x^i

($= u^\alpha$) the (M_x^{n-1}, g_x) will be the hypersurface of (M_x^n, g_x) given by (3.2a) where $g_x = g_{ij}(x, y)\, dy^i\, dy^j$ is the Riemannian metric of M_x^n The quantities $M_{\beta\gamma}$ given in (3.11) will be considered as the coefficients of second fundamental forms of tangent Riemannian space (M_x^{n-1}, g_x).

In general the coefficients of the rth fundamental forms of (M^{n-1}, g_x) are defined as[13]

$$C_{(1)\alpha\beta} = g_{\alpha\beta}$$

$$C_{(2)\alpha\beta} = M_{\alpha\beta}$$

$$C_{(r)\alpha\beta} = C_{(r-1)\alpha\delta}\ M_\beta^\delta\ (2 \leqslant r \leqslant n)$$

where

$$M_\beta^\delta = g^{\alpha\delta}\ M_{\alpha\beta}.$$

4. h-Vector Fields in $(M^{n-1}\ L)$

At the point of (M^{n-1}, L), the vector field X_i may be written as

$$X_i = X_\alpha\, B_i^\alpha + \mu\, N_i \qquad \ldots(4.1)$$

where

$$\text{(a)}\quad X_\alpha = X_i\, B_\alpha^i \qquad \text{(b)}\quad \mu = X_i\, N^i. \qquad \ldots(4.2)$$

Since $X_i|_\beta = X_i|_j\ B_\beta^j$, we have from (4.2 (a)) and (3.10)

$$X_\alpha|_\beta = X_i|_j\, B_\alpha^i\ B_\beta^j + \mu\, M_{\alpha\beta}. \qquad \ldots(4.3)$$

From (3.1) and (3.6) we get

$$L\, C_{\beta\gamma}^\alpha\, X_\alpha = L\, C_{ij}^h\, X_h\ B_\beta^i\ B_\gamma^j - L\, \mu\, M_{\beta\gamma}. \qquad \ldots(4.4)$$

If X_i is a concurrent vector field in F_n then in view of (2.1), (2.2), (3.2b) and (3.5) eqns. (4.3) and (4.4) reduce to

$$X_\alpha|_\beta = \mu\, M_{\alpha\beta},\ L\ C_{\beta\gamma}^\alpha\ X_\alpha = \rho h_{\beta\gamma} - L\, \mu\, M_{\beta\gamma}$$

These relations yield the

Theorem 4.1—If X_i is an h-vector field in (M^n, L) the vector field $X_\alpha = X_i B^i_\alpha$ is also an h-vector field in $(M^{n-1} L)$ if and only if

(i) X_i is tangential to the hypersurface (M^{n-1}, L)

or

(ii) $M_{\alpha\beta} = 0$.

The hyperplane of first, second and third kinds are defined[6]. In a hyper plane of third kind $M_{\alpha\beta}$ vanishes[6]. Thus :

Theorem 4.2—If X_i is an h-vector field in (M^n, L) then vector field $X_i B^i_\alpha$ is also an h-vector field in a hyperplane of third kind.

In the following we assume that X_i is tangential to $(M^{n-1}; L)$, so that

$$\text{(a)} \quad X_i = X_\alpha B^\alpha_i \qquad \text{(b)} \quad X^i = X^\alpha B^i_\alpha \qquad \text{...(4.5)}$$

where

$$X^\alpha = g^{\alpha\beta} X_\beta.$$

5. The n-Fundamental Forms of Hypersurface of (M^n, L^*)

Let (M^{n-1}, L^*) be a hypersurface of (M^n, L^*) given by the same equation (3.1). The relations (2.8), (2.9), (3.2b), (3.3) and (4.5) yield

$$g^*_{\alpha\beta} = \sigma g_{\alpha\beta} (1 - \sigma) l_\alpha l_\beta + X_\alpha X_\beta \qquad \text{...(5.1)}$$

$$g^{*\alpha\beta} = \sigma^{-1} g^{\alpha\beta} - \frac{(1-\sigma)}{L\lambda} (l^\alpha X^\beta + l^\beta X^\alpha)$$
$$+ (1-\sigma) \frac{(X^2 + \sigma^2)}{\lambda} l^\alpha l^\beta + \frac{1}{\lambda} X^\alpha X^\beta. \qquad \text{...(5.2)}$$

From (3.2b), (3.3) and (2.5) we also have

$$h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta , \; h_{ij} B^i_\alpha N^j = 0 \qquad \text{...(5.3)}$$

and

$$m_{\alpha\beta} = X_\alpha - \beta/L \, l_\alpha = m_i B^i_\alpha . \qquad \text{...(5.4)}$$

It is to be noted that if N^{*i} is a unit normal vector to (M^{n-1}, L^*) then it is not normal to (M^{n-1}, L).

We may write

$$N^{*i} = N^{\alpha} B^{i}_{\alpha} + D N^{i}. \qquad ...(5.5)$$

To obtain N^{α} and D we use (3.8), (3.5) and (5.1). Thus we get

$$g^{*}_{\alpha\beta} N^{\alpha} + D\mu X_{\beta} = 0 \qquad ...(5.6)$$

$$g^{*\alpha\beta}_{\alpha\beta} N^{\alpha} N^{\beta} + D^{2} \sigma + \mu (2D X_{\alpha} N^{\alpha} + D^{2} \mu) = 1. \qquad ...(5.7)$$

If X_i is tangential to (M^{n-1}, L) then $\mu = 0$ and eqns. (5.6) and (5.7) give $N^{\alpha} = 0$, $D = \sigma^{-1/2}$

From (5.5) it follows that

$$N^{*i} = \sigma^{-1/2} N^{i}. \qquad ...(5.8)$$

Hence we have the following :

Theorem 5.1—Let (M^{n}, L^{*}) be a Finsler space obtained from (M^{n}, L) by the transformation (2.4). If (M^{d-1}, L^{*}) and (M^{n-1}, L) are hypersurfaces of these spaces and X_i is tangential to the hypersurface (M^{n-1}, L) then the vector normal to (M^{n-1}, L) is also normal to (M^{n-1}, L^{*}).

Now we estoblish the following :

Theorem 5.2—Let (M^{n}, L^{*}) be a locally Minkowskian space obtained from locally Minkowskian space (M^{n}, L) by the transformation (2.4). Let (M^{n-1}, L^{*}) and (M^{n-1}, L) be hyper surfaces of (M^{n}, L^{*}) and (M^{d}, L) resspectively. If X_i is tangential to the hypersurfaces (M^{n-1}, L) and (M^{n}_{x}, g_x), (M^{n}, g^{*}_{x}), (M^{n-1}_{x}, g_x), (M^{n-1}_{x}, g^{*}_{x}) are tangent Riemannian spaces to (M^{n}, L); (M^{n}, L^{*}), (M^{n-1}, L), $(M^{n-1} L^{*})$ respectively, then we have the following :

(i) Second fundamental forms of (M^{n-1}_{x}, g_x) and (M^{n-1}_{x}, g^{*}_{x}) are proportional

(ii) Every asymptotic direction of $(M^{n-1}_{x} . g_x)$ is asymptotic direction of $(M^{n-1}_{x}$ g^{*}_{x}).

(iii) The rth fundamental tensors of (M^{n-1}_{x}, g_x) and (M^{n-1}_{x}, g^{*}_{x}) are related by

$$C^{*}{}_{(r)\alpha\beta} = \sigma^{3-r/2} \left[C_{(r)\alpha\beta} + \sum_{m=2}^{r-1} P_{(m)\beta}\, Q_{(r+m-1)\alpha}\right], \; 3 \leqslant r \leqslant n \qquad ...(5.9)$$

where

$$P_{(m)\alpha} = \sqrt{\frac{\sigma}{\lambda}}\, C_{(m)\alpha\varepsilon} X^{\varepsilon}, \quad 2 \leqslant m \leqslant n-1 \qquad ...(5.10)$$

$$R_{(m)} = \sqrt{\frac{\sigma}{\lambda}}\ P_{(m)\varepsilon}\, X^{\varepsilon}, \quad 2 \leqslant m \leqslant n-1 \qquad \text{...(5.11)}$$

$$Q_{(2)\alpha} = P_{(2)\alpha} \qquad \text{...(5.12a)}$$

$$Q_{(r)\alpha} = P_{(r)\alpha} + \sum_{m=2}^{r-1} R_{(m)}\, Q_{(r+1-m)\alpha},\ 3 \leqslant r \leqslant n-1 \qquad \text{...(5.12b)}$$

PROOF : (i) If X_i is tongential to the hypersurface to (M^{n-1}, L) then $\mu = 0$ and hecce

$$m_i\, N^i = 0.$$

Thus from (2.13, (3.10), (5.3), (5.7) it follows that

$$M_{\beta\gamma} = \sigma^{1/2}\ M_{\beta\gamma}. \qquad \text{...(5.13)}$$

This proves (i).

(ii) A direction t^α for which $M_{\alpha\beta}\, t^\alpha\, t^\beta = 0$ is said to be an asymptotic direction. In view of this dafinition and (5.13) we get (ii).

(iii) The validity of relation (5.9) is established by induction.

Since C_{ijk} is an indicatory tensor from (3.2b) and (3.11) it follows that $M_{\beta\gamma}$ $1 = 0$. Hence from (3.12) we get

$$C_{(r)\beta\gamma}\, l^\gamma = 0 = C_{(r)\beta\gamma}\, l^\beta,\ 2 \leqslant \gamma \leqslant n. \qquad \text{...(5.14)}$$

Hence from (5.10), (5.11), (5.12) we get

$$P_{(r)\alpha}\, l^\alpha = 0,\ Q_{(r)\alpha}\, l^\alpha = 0, \quad 2 \leqslant r \leqslant n. \qquad \text{...(5.15)}$$

From (5.2), (5.13) and (5.4), we get

$$M^{*\alpha}_{\beta} = \sigma^{-1/2}\left[M^{\alpha}_{\beta} - \frac{(1-\sigma)\,\beta\,\sigma\, l^\alpha}{L\,\lambda} - \frac{\sigma}{\lambda}\, X^\alpha \right] M_{\beta\gamma}\, X^\gamma. \qquad \text{...(5.16)}$$

The relations (3.12), (5.10), (5.14), (5.15) and (5.16) yield

$$C^{*}_{(3)\alpha\beta} = C_{(3)\alpha\beta} + \sqrt{\frac{\sigma}{\lambda}}\ P_{(2)\alpha}\, P_{(2)\beta}. \qquad \text{...(5.17)}$$

From (5.12a) and (5.17), it is evident that (5.9) holds for $r = 3$. For a given fixed value of the integer s with $3 \leqslant s \leqslant n-1$, we have

$$C^{*}_{(s+1)\alpha\beta} = C^{*}_{(s)\alpha\varepsilon}\ M^{*\varepsilon}_{\beta}. \qquad \text{...(5.18)}$$

Now let us suppose that (5.9) is valid for $s = 3, 4, 5, \ldots r$, so thai we can write (5.18) in the form

$$C^*_{(s+1)\alpha\beta} = \sigma^{(3-s)/2}\left[C_{(s)\alpha\beta} + \sum_{m=2}^{s-1} P_{(m)\beta}\, Q_{(s+1-m)}\right] M^{*s}_{\beta}$$

which in view of (5.10), (5.11), (5.12a), (5.14), (5.15) and (5.16) gives

$$C^*_{(s+1)\alpha\beta} = \sigma^{(2-s)/2}\Big[C_{(s-1)\alpha\beta} + \sum_{m=2}^{s-1} P_{(m+1)\beta}\, Q_{(s+1-m)\alpha}$$

$$+ P_{(2)\beta}\,\{P_{(2)s} + \sum_{m=2}^{s-1} R_{(m)}\; Q_{(s+1-m)\alpha}\}\Big]$$

$$= \sigma^{(2-s)/2}\,[C_{(s+2)\alpha\beta} + \sum_{m-2}^{s} P_{(m)\beta}\, Q_{(s+2-m)\alpha}].$$

This show that (5.9) is valid for $r = s + 1$, which completes is proof of (iii).

Acknowledgement

The second author is grateful to Dr R. S. Sinha for his kindly guidance and advice in the preparation of this paper.

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