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ON AN ERROR TERM INVOLVING THE TOTIENT FUNCTION

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This paper provides a mean-square result on the error term $E(t)$ in the asymptotic formula

$$\sum_{n \leqslant t} \frac{n}{\phi(n)} = \frac{315}{2\pi^4} \zeta(3)\, t - \tfrac{1}{2} \log t - c_1 + E(t)$$

where $\phi(n)$ is Euler's totient function and c_1 an effective constant. It is proven that

$$\int_0^x (E(t))^2\, dt = Cx + O\left(x^{4/5} (\log x)^{3/5} (\log \log x)^{6/5}\right)$$

with $C \approx 0{,}546$.

1. INTRODUCTION

For a positive integer n, let as usual $\phi(n)$ denote the number of positive integers $m \leqq n$ which are coprime to n. The rough observation that '$\phi(n)$ is on average of the order n' was supported in classic times by the asymptotics

$$\sum_{n \leqslant x} \frac{\phi(n)}{n} = \frac{6}{\pi^2} x + H(x),\ H(x) = O(\log x) \quad ...(1)$$

and

$$\sum_{n \leqslant x} \frac{n}{\phi(n)} = \frac{315}{2\pi^4} \zeta(3)\, x + E^*(x),\ E^*(x) = O(\log x) \quad ...(2)$$

see e. g. Landau[2] and Pillai and Chowla[4]. The error term $H(x)$ was subject of a detailed study in the sequel: It was shown that $H(x) = O((\log x)^{2/3} (\log \log x)^{4/3})$ (see Walfisz[11], p. 114), and that $H(x)$ has a significant oscillation behaviour, displayed by the mean-value results

$$\int_0^x H(t)\, dt = O\left(x \exp\left(- c (\log x)^{3/5} (\log \log x)^{-1/5}\right)\right),\ c > 0 \quad ...(3)$$

(see Suryanarayana and Sitaramachandrarao[8,9]) and

$$\int_0^x H^2(t)\,dt \sim \frac{1}{2\pi^2}\,x \qquad \ldots(4)$$

(see Chowla[1] and, more recently, Pétermann[3] for a quantitative version).

A thorough investigation of $E^*(x)$ was initiated only a few years ago by Sitaramachandrarao[6,7]. He proved that

$$E^*(x) = -\tfrac{1}{4}\log x - \tfrac{1}{2}\left(\gamma + \log 2\pi + \sum_p \frac{\log p}{p(p-1)}\right) + E(x) \qquad \ldots(5)$$

(here γ denotes the Euler-Mascheroni constant and the sum extends over all primes p), where $E(x) = O((\log x)^{2/3})$ and

$$\int_0^x E(t)\,dt = O(x^{4/5}). \qquad \ldots(6)$$

In the present note we study the asymptotic behaviour of the mean-square of $E(t)$, showing that

$$\int_0^x E^2(t)\,dt \sim Cx$$

(with a constant $C > 0$) which, together with (6), sheds some light on the oscillating nature of $E(t)$. More precisely, we prove the following.

Throrem—Let $E(t)$ be defined by (2) and (5), then we have the asymptotic

$$\int_0^x E^2(t)\,dt = Cx + O(x^{4/5}(\log x)^{3/5}(\log\log x)^{6/5}). \qquad \ldots(7)$$

where

$$C = \frac{1}{12}\prod_p \left(1 + \left(\frac{3}{p^2} - \frac{2}{p^3}\right)\left(1 - \frac{1}{p}\right)^{-2}\right) \approx 0{,}546. \qquad \ldots(8)$$

2. Some Technical Tools (see Walfisz[10])

Lemma A—For a large real parameter y,

$$\sum_{1\leq m,n\leq y}\ \sum_{\substack{u,v=1\\ um\neq vn}}^{\infty} (uv)^{-1}\,|um - vn|^{-1} = O(y\log y)$$

$$\sum_{1\leq m,n\leq y}\ \sum_{u,v=1}^{\infty} (uv)^{-1}(um + vn)^{-1} = O(y).$$

Lemma B—For arbitrary positive integers m, n

$$\sum_{\substack{u,v=1 \\ um=vn}}^{\infty} (uv)^{-1} = \frac{\pi^2}{6} \frac{(m, n)^2}{mn}$$

where (. , .) denotes the greatest common divisor.

Lemma C (cf. Chowla[1])—Let c_n denote some arithmetic function satisfying $c_n = O(n^{\epsilon})$ for every $\epsilon > 0$, and put $g_n = \sum_{d|n} c_d$. Then we have

$$\sum_{u,v=1}^{\infty} c_u \, c_v \, (u, v)^2 \, (uv)^{-2} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} g_n^2 \, n^{-2}.$$

3. Proof of the Theorem

For $t \geqq 3$, we put $y(t) = t^{4/5} (\log t)^{-2/5} (\log\log t)^{-4/5}$, then it has been proved in Sitaramachandrarao[7], formula (4.5), that (in our notation)

$$E(t) = -S(t) + O(t^{-1/5} (\log t)^{3/5} (\log\log t)^{6/5}) \qquad \ldots(9)$$

where

$$S(t) := \sum_{1 \leqq n \leqq y(t)} \frac{\mu^2(n)}{\phi(n)} P\left(\frac{t}{n}\right)$$

μ is the Moebius function and $P(w) = w - [w] - \frac{1}{2}$. (Properly speaking, this result was given in Sitaramachandrarao[8] for $\bar{y}(t) = t\,[t^{1/5} (\log t)^{2/5} (\log\log t)^{4/5}]^{-1}$ instead of $y(t)$, but we easily infer from $(\phi(n))^{-1} = O(n^{-1} \log\log n)$ (see e. g. Prachar[5], p. 24) that

$$\sum_{y(t) < n \leqq \bar{y}(t)} \frac{\mu^2(n)}{\phi(n)} P\left(\frac{t}{n}\right) = O(t^{-1/5}).$$

Now choose a constant a such that $y(t)$ increases for $t \geqq a$ and let z denote the inverse function of y; furthermore, put $M = M(m, n) = \max\{z(m), z(n), a\}$. Then we have for $x > a$

$$Q_0(x) := \int_a^x S^2(t)\, dt = \int_a^x \sum_{m,n \leqq y(t)} \frac{\mu^2(m)\,\mu^2(n)}{\phi(m)\,\phi(n)} P\left(\frac{t}{m}\right) P\left(\frac{t}{n}\right) dt$$

$$= \sum_{m,n < y(x)} \frac{\mu^2(m)\,\mu^2(n)}{\phi(m)\,\phi(n)} I(m, n) \qquad \ldots(10)$$

where

$$I(m,n) = \int_M^x P\left(\frac{t}{m}\right) P\left(\frac{t}{n}\right) dt.$$

By classic results on Fourier series,

$$I(m,n) = \frac{1}{\pi^2} \sum_{u'v=1}^{\infty} (uv)^{-1} \int_M^x \sin\left(2\pi u \frac{t}{n}\right) \sin\left(2\pi v \frac{t}{m}\right) dt$$

$$= \frac{1}{2\pi^2} \sum_{u,v=1}^{\infty} (uv)^{-1} \left(\int_M^x \cos\left(2\pi t\left(\frac{u}{n} - \frac{v}{m}\right)\right) dt \right.$$

$$\left. - \int_M^x \cos\left(2\pi t\left(\frac{u}{n} + \frac{v}{m}\right)\right) dt \right) = \frac{1}{2\pi^2} \sum_{\substack{u'v=1 \\ um=vn}}^{\infty} (uv)^{-1} (x - M)$$

$$+ O\left(\sum_{\substack{u,v=1 \\ um \neq vn}}^{\infty} (uv)^{-1} \left| \frac{u}{n} - \frac{v}{m} \right|^{-1} + \sum_{u,v=1}^{\infty} (uv)^{-1} \right.$$

$$\left. \times \left(\frac{u}{n} + \frac{v}{m} \right)^{-1} \right). \qquad \ldots(11)$$

Using again that $(\phi(n))^{-1} = O(n^{-1} \log\log n)$ and appealing to Lemma A, we thus get

$$Q_0(x) = \frac{1}{2\pi^2} \sum_{m,n \leqq y(x)} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} (x - M(m,n)) \sum_{\substack{u,v=1 \\ um=vn}}^{\infty} (uv)^{-1}$$

$$+ O(y(x) \log x (\log\log x)^2).$$

By Lemma B,

$$Q_0(x) = \frac{1}{12} \sum_{m,n \leqq y(x)} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} (x - M(m,n)) \frac{(m,n)^2}{mn}$$

$$+ O(y(x) \log x (\log\log x)^2). \qquad \ldots(12)$$

Furthermore, we easily deduce the estimates

$$\sum_{m,n \leqq y(x)} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} M(m,n) \frac{(m,n)^2}{mn} \ll \log\log x)^2 \sum_{m \leqq n \leqq y(x)}$$

(equation continued on p. 541)

$$\times (m, n)^2 m^{-2} n^{-3/4+\epsilon} \ll (\log \log x)^2 \sum_{d \leqq y(x)} d^{-3/4+\epsilon}$$

$$\sum_{m \leqq n \leqq \frac{y(x)}{d}} m^{-2} n^{-3/4+\epsilon} \ll (y(x))^{1/4+2\epsilon} \qquad \ldots(13)$$

and

$$x \sum_{\max(m,n) > y(x)} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} \frac{(m, n)^2}{mn} \ll x \sum_{\max(m,n) > y(x)}$$

$$\times (m, n)^2 (m n)^{-7/4} \ll x \sum_{d=1}^{\infty} d^{-3/2} \sum_{m > y/d} m^{-7/4}$$

$$\ll x \sum_{d=1}^{\infty} d^{-3/2} \min\left(\left(\frac{y}{d}\right)^{-3/4}, 1\right) \ll x (y(x))^{-1/2}$$

$$\ll x^{3/5+\epsilon}. \qquad \ldots(14)$$

Thus (12) may be simplified to

$$Q_0(x) = Cx + O(y(x) \log x (\log \log x)^2) \qquad \ldots(15)$$

where

$$C = \frac{1}{12} \sum_{m,n=1}^{\infty} \frac{\mu^2(m)\mu^2(n)}{\phi(m)\phi(n)} \frac{(m, n)^2}{m n}.$$

Going back to (9) and applying the Cauchy-Schwarz inequality for integrals, we obtain

$$\int_0^x E^2(t)\, dt = C x + O(x^{4/5} (\log x)^{3/5} (\log \log x)^{6/5})$$

which is just the assertion of our theorem, apart from the representation of the constant C. To evalute the latter, we apply Lemma C with $c_n = \mu^2(n) \dfrac{n}{\phi(n)}$.
By multiplicativity,

$$g_n = \sum_{m|n} \mu^2(m) \frac{m}{\phi(m)} = \prod_{p|n} \frac{2p-1}{p-1}$$

and

$$C = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} g_n^2 n^{-2} = \frac{1}{2\pi^2} \prod_{p} \left(1 + \left(\frac{2p-1}{p-1}\right)^2 \sum_{k=1}^{\infty} p^{-2k}\right)$$

(equation continued on p. 542)

$$= \frac{1}{2\pi^2} \prod_p \frac{p^4 - 2p^3 + 4p^2 - 2p}{(p-1)^3(p+1)} = \frac{\zeta(2)}{2\pi^2} \prod_p \left(1 - \frac{2}{p} + \frac{4}{p^2} - \frac{2}{p^3}\right)\left(1 - \frac{1}{p}\right)^{-2} = \frac{1}{12} \prod_p \left(1 + \left(\frac{3}{p^2} - \frac{2}{p^3}\right) \times \left(1 - \frac{1}{p}\right)^{-2}\right)$$

which completes the proof of our theorem.

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