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DIFFERENTIAL SUBORDINATION AND CONFORMAL MAPPINGS I

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It is well-known that if $N(z)$ and $D(z)$ are analytic in the unit disc, $N(0) = D(0) = 0$ and D maps the open unit disc onto a multi-sheeted domain that is starlike with respect to the origin then Re $(N'(z)/D'(z)) > 0$ implies Re $(N(z)/D(z)) > 0$. In this paper we give a sufficient condition on $D(z)$ which can guarantee that Re $(N'(z)/D'(z)) > 0$ implies Re $(N(z)/D(z)) > \alpha > 0$. A few interesting applications of this result are also given.

1. INTRODUCTION

Libera[4] showed that if $N(z)$ and $D(z)$ are analytic in the unit disc, $N(0) = D(0) = 0$ and D maps the open unit disc onto a multi-sheeted starlike region with respect to the origin then Re $(N'(z)/D'(z)) > 0$ implies Re $(N(z)/D(z)) > 0$. Various generalisations of this result can be found in the literature. However the examples $N(z) = zf'(z)$ where $f(z)$ is univalently convex and $D(z) = f(z)$ (which is necessarily univalently starlike of order 2^{-1}) suggest that there may exist some conditions on N and D so that Re $(N'(z)/D'(z)) > 0$ in the unit disc implies Re $(N(z)/D(z)) > \alpha > 0$ for $|z| < 1$. But just the stronger assumption that D is convex will not be sufficient for the required implication. This will be clear if we consider the following example. Let $D(z) = z/(1+z)$ (and hence D is convex) and $N(z)$ be determined by $(N'(z)/D'(z)) = (1+z)/(1-z)$. Then $(N(z)/D(z)) = ((1+z)/2z) \log ((1+z)/(1-z))$ and this function has a limit 0 as z goes to -1 through reals. Hence there cannot be a positive constant α such that Re $(N(z)/D(z)) > \alpha$ for $|z| < 1$. Thus it is interesting to ask whether there exists a condition on $D(z)$ for our required implication. The aim of this paper is to find at least one such condition and to give some interesting applications for this result.

2. PRELIMINARIES

Definition 1—A function $f(z)$ regular in the open unit disc U is said to be α convex for some real α if

$$(f(z)f'(z)/z) \neq 0 \text{ in } U \text{ and}$$

$$\text{Re}\,((1-\alpha)\, zf'(z)/f(z) + \alpha\,(1+(zf''(z)/f'(z)))) > 0 \text{ in } U \text{ (See Mocanu}^{6}).$$

We note that α-convex functions are always starlike and for $\alpha > 1$ these functions are in fact convex. We use the notation $K(\alpha)$ to denote the class of all α-convex functions in U.

Definition 2—A function $f(z)$ regular in U is said to be α-close-to-convex $(\alpha \geqslant 0)$ if

$$(f(z)\, f'(z)/z) \neq 0 \text{ in } U \text{ and}$$

there exists a starlike function $\varphi(z)$ in U such that

$$\text{Re}\,((1-\alpha)\,(zf'(z)/\varphi(z)) + \alpha\,((zf'(z))'/\varphi'(z))) > 0.$$

The class of all such functions will be denoted by $C(\alpha)$ and this class generalises the class $K(\alpha)$ for $\alpha \geqslant 0$.

Definition 3—A regular function $f(z)$ in U is said to be a Bazilevic function of type α (α real) if

$$(f(z)/z) \neq 0 \text{ in } U \text{ and}$$

$$\text{Re}\,\{f'(z)/(f(z))^{1-\alpha}\} > 0 \text{ in } U.$$

We choose a suitable branch for the power function used here. This class of functions will be denoted by $B(\alpha)$ (Bazilevic[1]).

Definition 4—Let $r = r_1 + ir_2$, $s = s_1 + is_2$. Let $b = e^{i\beta}$ with $|\beta| < \pi/2$. We shall say that a map $\psi : \mathbb{C}^2 \to \mathbb{C}$ belongs to $S(D)$ if

(1) $\psi(r, s)$ is continuous

(2) $(b, 0) \in D$ and $Re\,\psi(b, 0) > 0$.

(3) Re $\psi(r_2 i, s_1) \leqslant 0$ when $(r_2 i, s_1) \in D$ and

$$s_1 \leqslant -\tfrac{1}{2}(1 - 2r_2 \sin\beta + r_2^2)/\cos\beta$$

See Lewandowski[3].

Theorem A—Let $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ be regular in the open unit disc U of the complex plane. Suppose for each $z_0 \in U$ there exists a function $\psi_{z_0} \in S(\mathbb{C}^2)$ such that Re $\psi_{z_0}(p(z_0), z_0 p'(z_0)) > 0$ then Re $p(z) > 0$ for $z \in U$.

Proof: Essentially the proof is the same as that of Lemma 2.1 of Lewandowski[3]. But since we have slightly changed the hypothesis we shall complete the proof.

With the same notations as in the proof of Lemma 2.1 of Lewandowski[3] we assume that $p(z) = (1 + w(z))(1 - w(z))^{-1} \cos\beta + i \sin\beta$ and claim $|w(z)| < 1$ or Re $p(z) > 0$. If there exists a point $z_0 = e^{i\theta_0} \in U$ such that $\max_{|z| \leqslant r_0} |w(z)| = |w(z_0)| = 1$, we get as in Lewandowski[3] that

$$p(z_0) = \theta i,\ z_0\, p'(z_0) = d$$

where θ and d are real and $d \leqslant -\frac{1}{2}(1 - 2\theta \sin\beta + \theta^2)/\cos\beta$. But by hypothesis for this z_0 there exists a $\psi_{z_0} \in U(\mathbb{C}^2)$ with the property that $\mathrm{Re}\, \psi_{z_0}(p(z_0), z_0 p'(z_0)) > 0$ and $\mathrm{Re}\, \psi_{z_0}(\theta i, d) \leqslant 0$ with θ and d as above. Because of this contradiction $|w(z)| < 1$ and so $\mathrm{Re}\, p(z) > 0$ in U establishing our claim.

3. Main Results

Theorem B—Let α be real and non-negative, $M(z)$, $N(z)$ regular functions in U with $M(0) = N(0) = 0$ and $(M'(0)/N'(0)) = 1$. Let $N(z)$ satisfy

$$\mathrm{Re}\,(N(z)/zN'(z)) > \delta \ (0 \leqslant \delta < 1).$$

If

$$\mathrm{Re}\left[(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)}\right] > 0 \ (z \in U)$$

then

$$\mathrm{Re}\,(M(z)/N(z)) > \delta\alpha/(2 + \delta\alpha) \ (z \in U).$$

Proof: Let $\beta = \delta\alpha/(2 + \delta\alpha)$ and consider

$$p(z) = (1-\beta)^{-1}[(M(z)/N(z)) - \beta].$$

This $p(z)$ is regular in U and $p(0) = 1$. We set

$$\lambda(z) = N(z)/zN'(z)$$

and observe that by hypothesis $\mathrm{Re}\,(\lambda(z)) > \delta$.

A simple computation shows that

$$(1-\alpha)\frac{M(z)}{N(z)} + \alpha\frac{M'(z)}{N'(z)} = \beta + (1-\beta)\,p(z) + \alpha(1-\beta)\,\lambda(z)\,zp'(z)$$

$$= \psi_z(p(z), zp'(z))$$

where $\psi_z(r, s) = (1-\beta)\,r + \beta + \alpha(1-\beta)\,\lambda(z)\,s$. Now ψ_z is continuous in $\mathbb{C}^2$, $\mathrm{Re}\,\psi_z(1, 0) = 1 > 0$ and for all $(ir_2, s_1) \in \mathbb{C}^2$ with $s_1 \leqslant -(1 + r_2^2)/2$ we have

$$\mathrm{Re}\,\psi_z(ir_2, s_1) = \beta + \alpha(1-\beta)\,s_1\,\mathrm{Re}\,\lambda(z)$$

$$\leqslant \beta - \alpha(1-\beta)\frac{1 + r_2^2}{2}\,\mathrm{Re}\,\lambda(z)$$

$$\leqslant \beta - \alpha\frac{(1-\beta)\,\delta}{2} = 0.$$

Hence for each z, $\psi_z \in S(\mathbb{C}^2)$ and

$$\mathrm{Re}\,\psi_z(p(z), zp'(z)) > 0.$$

Hence by Theorem A, Re $p(z)) > 0$ and hence

$$\text{Re}\,(M(z)/N(z)) > \beta.$$

This proves our theorem.

Corollary 1—If M and N are regular in U, $M(0) = N(0) = 0$, $M'(0)/N'(0) = 1$ and $N(z)$ satisfies Re $(N(z)/zN'(z)) > \delta$ then

$$\text{Re}\,(M'/N') > 0 \text{ implies Re}\,(M/N) > \delta/(2+\delta).$$

PROOF : Take $\alpha = 1$ in Theorem B.

Corollary 2—If $f(z) \in B\,(m)$ where m is a positive integer then Re $(f(z)/z)^m > 1/(1+2m)$.

PROOF : Choose $M(z) = (f(z))^m$ and $N(z) = z^m$. Then $(M'(z)/N'(z)) = (f(z))^{m-1} f'(z)/z^{m-1}$ and $(M'(0)/N'(0)) = 1 > 0$. By Corollary 1 since $f \in B\,(m)$, Re $(M/N) > \delta/(2+\delta)$ whenever $(1/m) > \delta$. But δ can be chosen as near $1/m$ as we please and so we can allow $\delta \to 1/m$ from below. Thus $(\delta/(2+\delta) \to 1/(1+2m)$ and we have established our claim.

We can draw a very interesting conclusion from the following theorem.

Theorem C—If γ and c are positive integers, $f(z)$ is a normalized analytic function in U with the property that

$$\text{Re}\,\{f'(z)/(f(z)/z)^{1-\gamma}\} > -1/2\,(\gamma + c)$$

then the function $F(z)$ defined by

$$F^{\gamma}(z) = \frac{\gamma + c}{z^{c}} \int_{0}^{z} t^{c-1}\,(f(t))^{\gamma}\,dt$$

belongs to $B\,(\gamma)$.

PROOF : We shall write

$$p(z) = F'(z)/(F(z)/z)^{1-\gamma}$$

and after some simplification find

$$g = \frac{f'}{\left(\dfrac{f}{z}\right)^{1-\gamma}} = p + \frac{1}{\gamma + c}\,zp'.$$

Now we shall put $k = 1/2\,(\gamma + c)$ and consider

$$(g+k)/(1+k) = p/(1+k) + zp'/(\gamma + c)\,(1+k) + k/1 + k$$
$$= \psi\,(p, zp')$$

where

$$\psi(r, s) = r/(1 + k) + s/(\gamma + c)(1 + k) + k/(1 + k).$$

Clearly $\psi \in S(\mathbb{C}^2)$ because for $s_1 \leqslant \frac{-(1 + r_2^2)}{2}$ we have

$$\operatorname{Re} \psi(ir_2, s_1) = s_1/(\gamma + c)(1 + k) + k/(1 + k)$$
$$\leqslant \frac{-(1 + r_2^2)}{2(1+k)(\gamma+c)} + \frac{k}{1+k} \leqslant 0.$$
$$[\gamma + c = 1/2k \text{ and } -(1 + r_2^2) \leqslant -1].$$

Hence we can take $\psi_z = \psi$ for all $z \in U$ and find that $\operatorname{Re} \psi_z(p, zp') > 0$. Hence $\operatorname{Re}(p(z)) > 0$ and so $F(z) \in B(\gamma)$ as required.

Remark : Under the hypothesis f need not belong to $B(\gamma)$ and so Theorem C is an improvement of a theorem in Singh[7]. Taking $\gamma = c = 1$ we see that if a normalised analytic function f satisfies $\operatorname{Re}(f'(z)) > -\frac{1}{4}$ then the corresponding F satisfies $\operatorname{Re} F'(z) > 0$ and hence is univalently close-to-convex. This result is of special interest since it extends an earlier result due to Libera[4]. Incidentally this result provides an example of a non-univalent function for which the Libera transform is univalent. Using similar arguments we can improve several other interesting results available in the literature. We will illustrate this as follows. The proofs will be omitted.

Theorem D—If $f(z)$ normalised analytic function in U satisfying

$$\operatorname{Re}[(1 - \alpha)(f(z)/z) + \alpha f'(z)] > 0 \ (\alpha \geqslant 1)$$

then

$$\operatorname{Re}(f(z)/z) > \alpha/(\alpha + 2) \text{ and } \operatorname{Re}(f'(z)) > (\alpha - 1)/(\alpha + 2).$$

The above theorem improves some results found in Chichra[2].

Theorem E—If $f \in C(\alpha)$ and the corresponding φ satisfies

$$\operatorname{Re}(\varphi(z)/z\varphi'(z)) > \delta$$

then $\operatorname{Re}(zf'(z)/\varphi(z)) > \delta\alpha/(2 + \delta\alpha)$.

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