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GRONWALL, BIHARI AND LANGENHOP TYPE INEQUALITIES FOR DISCRETE PFAFFIAN EQUATION

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The object of this paper is to establish some new discrete inequalities of the Gronwall, Bihari and Langenhop type for Pfaffian equation.

INTRODUCTION

It is well known that the discrete analogue of celebrated Gronwall's lemma[4] and its generalization, Bihari's lemma[3] established by Jones[5] and Sugiyama[9] (also see Agarwal and Thandapani[1,2] and Pachpatte[8]) play an important role in the theorey of difference equations and numerical analysis. They have been used, for example, to obtain upper bounds of solutions as in Langenhop[7] it is shown that lower bounds can be obtained similarly. The purpose of this note is to demonstrate how similar estimates can be derived for solutions of discrete Pfaffian equations and for the continuous case similar results are available in Grudo and Yarchuk[6].

Before giving the main result, we shall first introduce some notations which we shall use throughout the paper. N denotes the set $\{0, 1, \ldots\}$. The expression $\sum_{s=0}^{t-1} b(s)$ represents a solution of the linear difference equation $\Delta z(t) = b(t)$ for all $t \in N$ under the initial condition $z(0) = 0$ ...where Δ is the operator defined by $\Delta z(t) = z(t+1) - z(t)$. It is supposed that $\sum_{s=0}^{-1} b(s) = 0$. The expression $\prod_{s=0}^{t-1} c(s)$ represents a solution of the linear difference equation $z(t+1) = c(t)\, z(t)$ for all $t \in N$ with the initial condition $z(0) = 1$. It is supposed that $\prod_{s=0}^{-1} c(s) = 1$. We also define

$$\Delta z_{t_1}(t_1, t_2) = z(t_1 + 1, t_2) - z(t_1, t_2)$$

$\Delta z_{t_2}(t_1, t_2) = z(t_1, t_2 + 1) - z(t_1, t_2)$ where the letters t_1 and t_2 are used to denote the two independent variable which are members of N.

MAIN RESULT

Consider the discrete Pfaffian equation in two independent variables :

$$x(t_1, t_2) = x(0,0) + \sum_{s_1=0}^{t_1-1} f_1(s_1, t_2, x(s_1, t_2)) + \sum_{s_2=0}^{t_2-1} f_2(0, s_2, x(0, s_2)) \quad \ldots(1)$$

where

$f_j\ (j = 1, 2)$ are real-valued and defined for all $t_1 > 0,\ t_2 \geqslant 0$

and

$$|f_j(t_1, t_2, x)| \leqslant F_j(t_1, t_2)\, W_j(|x|)\ (j = 1, 2) \quad \ldots(2)$$

where the F_j are real-valued, non negative and defined for all $t_1 \geqslant 0,\ t_2 \geqslant 0$ and $W_j(u)$ are continuous and non decreasing for $u \geqslant 0$ and $W_j(u) > 0$ for $u > 0$.

Theorem —Let $|x(t_1, t_2)| > 0$ for $t_1 \geqslant 0$ and $t_2 \geqslant 0$. Then, if $t_1 \geqslant 0,\ t_2 \geqslant 0$, we have

$$|x(t_1, t_2)| \leqslant G_1^{-1}\left\{\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) + G_1\left[G_2^{-1}\left(\sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right)\right]\right\} \quad \ldots(3)$$

where,

$$G_i(u) = \int_{u_0}^{u} \frac{d\,u_1}{W_i(u_i)} (0 < u < \bar{u}_i,\ u_0 = |x(0,0)|) \quad \ldots(4)$$

and

$G_i^{-1}\ (i = 1, 2)$ are the inverses of G_i.

Furthermore,

$$|x(t_1, t_2)| > G_1^{-1}\left\{-\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) + G_1\left[G_2^{-1}\left(-\sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right)\right]\right\} \quad \ldots(5)$$

for all $t_1 \geqslant 0,\ t_2 > 0$ for which

$$-\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) + G_1\left[G_2^{-1}\left(-\sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right)\right] \in \operatorname{dom}(G_1^{-1})$$

and

$$-\sum_{s_2=0}^{t_2-1} F_2(o, s_2) \in \operatorname{dom}(G_2^{-1}).$$

PROOF : By putting $|x(t_1, t_2)| = u(t_1, t_2)$, we conclude from (2) that

$$u(t_1, t_2) \leqslant u_0 + \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) W_1(u(s_1, t_2)) + \sum_{s_2=0}^{t_2-1} F_2(0, s_2) W_2(u(0, s_2)). \quad \ldots(6)$$

Now put

$$R(t_1, t_2) = u_0 + \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) W_1(u(s_1, t_2)) + \sum_{s_2=0}^{t_2-1} F_2(0, s_2) W_2(u(0, s_2))$$

to obtain

$$\Delta R_{t_1} = F_1(t_1, t_2) W_1(u(t_1, t_2)) \leqslant F_1(t_1, t_2) W_1(R(t_1, t_2))$$

since $W_1(u)$ is non decreasing and $u(t_1, t_2) \leqslant R(t_1, t_2)$. Since $R(t_1, t_2) > 0$ and $W_1(u) > 0$ for $u > 0$ for $u > 0$, this inequality together with (4) yields

$$G_1(R(t_1, t_2)) - G_1(R(0, t_2)) \leqslant \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2). \quad \ldots(7)$$

From (6), we have, on putting $t_1 = 0$

$$u(0, t_2) \leqslant u_0 + \sum_{s_2=0}^{t_2-1} F_2(0, s_2) W_2(u(0, s_2)).$$

As before, we conclude that

$$G_2(R(0, t_2)) - G_2(R(0, 0)) \leqslant \sum_{s_2=0}^{t2-1} F_2(0, s_2).$$

Since $R(0,0) = u_0$ and $G_2(u_0) = 0$, we have

$$R(0, t_2) \leqslant G_2^{-1}\left[\sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right]. \quad \ldots(8)$$

Hence, since $G_1(R(t_1, t_2)) \geqslant G_1(u(t_1, t_2))$, we conclude from (7) and (8) that

$$G_1(u(t_1, t_2)) \leqslant G_1\left\{G_2^{-1}\left[\sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right]\right\} + \sum_{s_1=0}^{t_1-1} F_1(s_1, t_2)$$

and the result (3) follows.

We now prove (5). Again using the notation $|x(t_1, t_2)| = u(t_1, t_2)$, We find from (1) and (2)

$$u(t_1, t_2) \geqslant u(s_1, s_2) - \sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2) W_1(u(r_1, t_2)$$

$$- \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2) W_2(u(s_1, r_2)) \qquad \text{...(9)}$$

for $t_1 > 0$, $t_2 > 0$ and $0 \leqslant s_i \leqslant t_i$ $(i = 1, 2)$.

Let

$$R(s_1, s_2) = u(t_1, t_2) + \sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2) W_1(u(r_1, t_2))$$

$$+ \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2) W_2(u(s_1, r_2)).$$

Then

$$\Delta R_{s_2} = - F_2(s_1, s_2) W_2(u(s_1, s_2))$$

and since $R(s_1, s_2) \geqslant u(s_1, s_2)$

$$\Delta R_{s_2} \geqslant - F_2(s_1, s_2) W_2(R(s_1, s_2)).$$

Thus

$$G_2(R(s_1, t_2)) - G_2(R(s_1, s_2)) > - \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2)$$

and we conclude that

$$G_2(R(s_1, t_2)) \geqslant G_2(u(s_1, s_2)) - \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2).$$

This implies that

$$R(s_1, t_2) \geqslant G_2^{-1}[G_2(u\, s_1, s_2)) - \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2)]. \qquad \text{...(10)}$$

Putting $s_2 = t_2$ in (9), we obtain

$$u(t_1, t_2) \geqslant u(s_1, t_2) - \sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2) W_1(u(r_1, t_2)).$$

Hence

$$G_1(R(t_1, t_2)) - G_1(R(s_1, t_2)) > - \sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2).$$

Since $R(t_1, t_2) = u(t_1, t_2)$, (10) implies that

$$G_1(u(t_1, t_2)) \geqslant -\sum_{r_1=s_1}^{t_1-1} F_1(r_1, t_2) + G_1\left\{G_2^{-1}\left[G_2(u(s_1, s_2)) - \sum_{r_2=s_2}^{t_2-1} F_2(s_1, r_2)\right]\right\}$$

and putting $s_1 = 0$, $s_2 = 0$, we obtain

$$G_1(u(t_1, t_2)) \geqslant -\sum_{r_1=0}^{t_1-1} F_1(r_1, t_2) + G_1\left\{G_2^{-1}\left[-\sum_{r_2=0}^{t_2-1} F_2(0, r_2)\right]\right\}.$$

This proves (5).

Remark 1: If $W_1(u) = W_2(u)$, (3) and (5) simplify :

$$|x(t_1, t_2)| \leqslant G_1^{-1}\left[\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) + \sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right]$$

$$|x(t_1, t_2)| \geqslant G_1^{-1}\left[\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) - \sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right].$$

Remarks 2 : If $W_1(u) = W_2(u) = u$, we have

$$|x(t_1, t_2)| \leqslant |x(0,0)| \prod_{s_1=0}^{t_1-1} [1 + F_1(s_1, t_2)] \prod_{s_2=0}^{t_2-1} [1 + F_2(0, s_2)]$$

$$\leqslant |x(0,0)| \exp\left[\sum_{s_1=0}^{t_1-1} F_1(s_1 t_2) + \sum_{s_2=0}^{t_1-1} F_2(0, s_2)\right].$$

$$|x(t_1, t_2)| \geqslant |x(0,0)| \prod_{s_1=0}^{t_1-1} [1 - F_1(s_1, t_2)] \prod^{t_2-1} [1 - F_2(0, s_2])$$

$$\geqslant |x(0,0)| \exp\left[-\sum_{s_1=0}^{t_1-1} F_1(s_1, t_2) - \sum_{s_2=0}^{t_2-1} F_2(0, s_2)\right].$$

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