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SOME RESULTS ON ALMOST SEMI-INVARIANT SUBMANIFOLD OF AN *SP*-SASAKIAN MANIFOLD

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In the present paper I have obtained some of the properties of an almost semi-invariant submanifold of an *SP*-Sasakian manifold. The intergrability conditions of the distributions D, $D^{\perp}$, $\tilde{D}$, $D \oplus \{\xi\}$, $D^{\perp} \oplus \{\xi\}$ and $\tilde{D} \oplus \{\xi\}$ have also been discussed.

1. INTRODUCTION

Let $\tilde{M}$ be an n dimensional C^{∞}-manifold. If there exists in $\tilde{M}$ a tensor field F of type (1, 1) a vector field ξ and a 1-form η satisfying

$$F^2 X = X - \eta(X)\,\xi, \quad \eta(\xi) = 1. \qquad \ldots(1.1)$$

Then $\tilde{M}$ is called an almost paracontact manifold.

Let g be a Riemannian metric satisfying

$$\eta(X) = g(X, \xi) \qquad \ldots(1.2)$$

$$\eta(FX) = 0, \; F\xi = 0, \; \text{rank}\,(F) = n - 1 \qquad \ldots(1.3)$$

$$g(FX, FY) = g(X, Y) - \eta(X)\,\eta(Y). \qquad \ldots(1.4)$$

Then the set (F, ξ, η, g) is called an almost paracontact Riemannian structure and the manifold is called an almost paracontact Riemannian manifold[2].

Moreover if we define

$$'F(X, Y) = g(FX, Y) \qquad \ldots(1.5)$$

then

$$'F(X, Y) = {}'F(Y, X) \qquad \ldots(1.6a)$$

$$'F(FX, FY) = {}'F(X, Y). \qquad \ldots(1.6b)$$

Now, let us consider a manifold in which the 1-form η satisfies

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0 \qquad \ldots(1.7)$$

$$(\nabla_X \nabla_Y \eta)(Z) = (-g(X, Z) + \eta(X)\,\eta(Z))\,\eta(Y) + (-g(X, Y) + \eta(X)\,\eta(Y))\,\eta(Z) \qquad \ldots(1.8)$$

where ∇ denotes the covariant differentiation with respect to g. Furthermore, if we put

$$\eta(X) = g(X, \xi), \ \nabla_X \xi = FX \qquad ...(1.9)$$

then it is easily verified that the manifold in consideration becomes an almost para-contact manifold. Such a manifold is called a p-Sasakian manifold[4].

If the 1-form η in $\tilde{M}$ satisfies

$$(\nabla_X \eta)(Y) = -g(X, Y) + \eta(X)\eta(Y) \qquad ...(1.10)$$

we can easily show by putting $\eta(X) = g(X, \xi)$ and $(\nabla_X \eta)(Y) = {}'F(X, Y)$, that the manifold is p-Sasakian. Such a manifold is called an sp-Sasaksan manifold[4]. Thus in an sp-Sasakian manifold, we have

$${}'F(Y, Z) = -g(Y, Z) + \eta(Y)\eta(Z) \qquad ...(1.11)$$

and

$$(\nabla_X {}'F)(Y, Z) = (-g(X, Y) + \eta(X)\eta(Y))\eta(Z) + (-g(X, Z) + \eta(X)\eta(Z))\eta(Y). \qquad ...(1.12)$$

Let M be an m-dimensional submanifold immersed in an sp-Sasakian manifold $\tilde{M}$. Let TM and $TM^{\perp}$ be respectively the tangent and the normal bundle to M. Suppose the structure vector field ξ is tangent to M and denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M and $\{\xi\}^{\perp}$ the complementary orthogonal distribution to $\{\xi\}$ in TM. For each $X \in \Gamma(TM)$, put

$$FX = bX + cX \qquad ...(1.13)$$

where $bX \in \Gamma(\{\xi\}^{\perp})$ and $cX \in \Gamma(TM)^{\perp}$. Thus b is an endomorphism of the tangent bundle TM and c is a normal bundle valued 1-form on M.

Definition 1.1—The submanifold M of the sp-Sasakian manifold $\tilde{M}$ is said to be an almost semi-invariant submanifold if its tangent bundle TM has the decomposition

$$TM = D \oplus D^{\perp} \oplus \tilde{D} \oplus \{\xi\} \qquad ..(1.14)$$

where

(1) D is invariant distribution on M, i.e.

$$F(D_x) = D_x$$

(2) $D^{\perp}$ is an anti-invariant distribution on M, i.e.

$$F(D_x^{\perp}) \subset T_x M^{\perp} \quad \text{for } x \in M.$$

(3) $\tilde{D}$ is neither invariant nor an anti-invariant distribution on M, i.e. $b X_x \neq 0$ and $C X_x \neq 0$ for any $x \in M$ and $X_x \in D_x$.

(4) $\{\xi\}$ is the distribution spanned in M by the vector field ξ.

2. Basic Results

Let M be an almost semi-invariant submanifold of an *sp*-Sasakian manifold $\tilde{M}$. We denote the Riemannian metrics of $\tilde{M}$ and M both by g. Let P, Q and L be the projection morphisms of TM to the distributions D, $D^{\perp}$ and $\tilde{D}$ respectively. Then for $X \in T(M)$, we have

$$X = PX + QX + LX + \eta(X)\xi. \quad \text{...(2.1)}$$

Now, we take $X \in \Gamma(\tilde{D})$. Then $bX \neq 0$, $CX \neq 0$. Thus C defines a vector sub-bundle $C\tilde{D} : x \to C\tilde{D}_x$ of $TM^{\perp}$.

For any $N \in \Gamma(TM^{\perp})$, we put

$$FN = tN + fN \quad \text{..(2.2)}$$

where tN and fN are respectively the tangential and the normal components of FN. Then we have,

$$g(FD^{\perp}, C\tilde{D}) = 0. \quad \text{...(2.3)}$$

Next, we denote by ν the orthogonal complementary vector bundle to $FD^{\perp} \oplus C\tilde{D}$ in $TM^{\perp}$. By (1.4), we have

$$g(FX, CY) = g(FX, FY) = g(X, Y) = 0 \ \forall\ X \in \Gamma(D^{\perp}), \ Y \in \Gamma(\tilde{D}). \quad \text{...(2.4)}$$

Thus

$$TM^{\perp} = FD^{\perp} \oplus C\tilde{D} \oplus \nu. \quad \text{...(2.5)}$$

Lemma 2.1—The morphisms t and f satisfy

$$t(TM^{\perp}) = D^{\perp} \oplus \tilde{D} \quad \text{...(2.6)}$$

$$t(FD^{\perp}) = D^{\perp} \quad \text{..(2.7)}$$

$$t(C\tilde{D}) = \tilde{D} \quad \text{...(2.8)}$$

$$f(C\tilde{D}) = C\tilde{D}. \quad \text{..(2.9)}$$

PROOF : Let $N \in \Gamma(TM^{\perp})$, then

$$g(tN, X) = g(FN, X) = g(N, FX) = 0 \;\forall\; X \in \Gamma(D)$$

and

$$g(tN, \xi) = g(FN, \xi) = g(N, F\xi) = 0.$$

Thus $tN \in \Gamma(D^{\perp} \oplus \tilde{D})$ and we get (2.6). Next for each $X \in \Gamma(D^{\perp})$, we have

$$X = F^2 X = tFX + CFX = tFX$$

which implies (2.7). The proof of (2.8) and (2.9) is same as given in Bejancu and Papaghiuc[1].

Definition 2.1—An almost semi-invariant submanifold M in an *sp*-Sasakian manifold $\tilde{M}$ is said to be a semi-invariant submanifold if we have $\tilde{D} = \{0\}$.

Proposition 2.1—Let M be an almost semi-invariant submanifold of the *sp*-Sasakian manifold $\tilde{M}$. Then the endomorphism $b : TM \to TM$ is a para-f structure on M, that is, $b^3 - b = 0$ if and only if M is a semi-invariant submanifold.

PROOF : From (1.13), we see that

$$(b^3 - b) X = 0 \quad \text{for any} \quad X \in \Gamma(D \oplus D^{\perp} \oplus \{\xi\}).$$

Since $b\,\tilde{D}_x = \tilde{D}_x$, we see that

$$(b^3 - b)(\tilde{D}) = \{0\}$$

if and only if

$$(b^2 - I)(\tilde{D}) = \{0\}.$$

Applying F to (1.13) and using (1.13), (2.2) and (1.1) we get

$$X = F^2 X = b^2 X + CbX + tCX + fCX \text{ for } X \in \Gamma(D).$$

Equating the tangent part, we get

$$(b^2 - I) = -tC.$$

Therefore,

$$tC(\tilde{D}) = \{0\}$$

which with the help of (2.8) gives $\tilde{D} = \{0\}$. Hence the submanifold is semi-invariant.

Proposition 2.2—Let M be an almost semi-invariant sub-manifold of the *sp*-Sasakian manifold $\tilde{M}$. Then M is a semi-invariant submanifold if and only if

$$f^3 - f = 0. \qquad \text{...(2.10)}$$

PROOF : We see that if $N \in (FD^{\perp})$, we have $fN = 0$ and for $N \in \Gamma(\nu)$, $fN = FN$. By (2.9) we see that f is an automorphism on CD. Hence $(f^3 - f)(\widetilde{CD}) = \{0\}$ if and only if

$$(f^2 - I)(\widetilde{CD}) = \{0\}. \quad \text{...(2.11)}$$

By means of (1.13), (2.2) and (1.1), we get

$$CX = F^2(CX) = bfCX + tfCX + C + CX + f^2 CX \quad \text{...(2.12)}$$

for any $X \in (\widetilde{CD})$.

Thus (2.11) is equivalent to

$$Ct(\widetilde{CD}) = \{0\} \quad \text{...(2.13)}$$

which with the help of (2.8) gives (2.16).

Lemma 2.2—Let M be an almost semi-invariant submanifold of an sp-Sasakian manifold $\tilde{M}$. Then we have

$$(b^2 + tC)X = X - \eta(X)\xi, \ (Cb + fC)X = 0 \quad \text{...(2.14)}$$

$$(bt + tf)N = 0, \ (f^2 - I + Ct)N = 0 \quad \text{...(2.15)}$$

$$(f^3 - f + Ctf)N = 0 \quad \text{...(2.16)}$$

$$(b^3 - b + tCb)X = 0 \quad \text{...(2.17)}$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$.

PROOF : It follows directly from (1.1), (1.13) and (2.2).

Let $\tilde{\nabla}$ (resp ∇) be the Riemannian connection on $\tilde{M}$ (resp M) with respect to the Riemannian metric g. The linear connection induced by $\tilde{\nabla}$ on the normal bundle $TM^{\perp}$ is denoted by $\nabla^{\perp}$, then the equations of Gauss and Weingarten are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad \text{...(2.18)}$$

and

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \quad \text{...(2.19)}$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, h is the second fundamental form of M and A_N is the fundamental tensor with respect to the normal section N and

$$g(h(X, Y), N) = g(A_N X, Y). \quad \text{...(2.20)}$$

Lemma 2.3—Let M be an almost semi-invariant submanifold of the sp-Sasakian manifold $\tilde{M}$. Then we have

$$P(u(X, Y)) = FP(\nabla_X Y) - \eta(Y) PX \qquad ...(2.21)$$

$$Q(u(X, Y)) = Qt(h(X, Y)) - \eta(Y) Qx \qquad ...(2.22)$$

$$L(u(X, Y)) = bL(\nabla_X Y) + Lt(h(X, Y)) - \eta(Y) LX \qquad ...(2.23)$$

$$\eta(u(X, Y)) = -g(FX, FY) \qquad ...(2.24)$$

$$h(X, FPY) + h(X, bLY) + \nabla^{\perp}_X FQY + \nabla^{\perp}_X CLY$$

$$= FQ \nabla_X Y + CL \nabla_X Y + f(h(X, Y)) \qquad ...(2.25)$$

where

$$u(X, Y) = \nabla_X FPY + \nabla_X bLY - A_{FQY} X - A_{CLY} X \qquad ...(2.26)$$

for all $X, Y \in \Gamma(TM)$.

The proof follows from Bejancu and Papaghiuc[1].

Lemma 2.4—Let M be an almost semi-invariant submanifold of an sp-Sasakian manifold M. Then we have

$$\nabla_X \xi = FX, \; h(X, \xi) = 0; \quad \text{for any } X \in \Gamma(D) \qquad ...(2.27)$$

$$\nabla_Y \xi = 0, \; h(Y, \xi) = FY; \quad \text{for any } Y \in \Gamma(D^{\perp}) \qquad ...(2.28)$$

$$\nabla_Z \xi = bZ, \; h(Z, \xi) = CZ, \quad \text{for any } Z \in \Gamma(\tilde{D}) \qquad ...(2.29)$$

$$\nabla_{\xi} \xi = 0, \; h(\xi, \xi) = 0. \qquad ..(2.30)$$

PROOF : We have

$$\tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi) \qquad ...(2.31)$$

which with the help of (1.9) and (2.1) gives

$$\nabla_X \xi + h(X, \xi) = FPX + FQX + bLX + CLX \qquad ...(2.32)$$

for any $X \in \Gamma(TM)$.

Now, (2.27)–(2.30) follows from (2.32).

Lemma 2.5—Let M be an almost semi-invariant submanifold of the sp-Sasakian manifold $\tilde{M}$, then we have

$$A_{FX} Y + A_{FY} X = 0. \qquad ...(2.33)$$

for all $X, Y \in \Gamma(D^{\perp})$.

PROOF : With the help of (1.4), (2.18), (2.20), we have

$$g(A_{FX}Y, Z) = g(h(Y, Z), FX) = g(\tilde{\nabla}_Z Y, FX) = g(F\tilde{\nabla}_Z Y, X)$$

$$= g(\tilde{\nabla}_Z FY, X) = -g(FY, \tilde{\nabla}_Z X) = -g(h(X, Z), FY)$$

$$= -g(A_{FY}X, Z)$$

for all $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(TM)$, which implies (2.33).

Lemma 2.6—Let M be an almost semi-invariant submanifold of the *sp*-Sasakian manifold $\tilde{M}$. Then we have

$$\nabla_\xi U \in \Gamma(D) \quad \text{for any } U \in \Gamma(D) \qquad ...(2.34)$$

$$\nabla_\xi V \in \Gamma(D^{\perp}) \quad \text{for any } V \in \Gamma(D^{\perp}) \qquad ...(2.35)$$

$$\nabla_\xi W \in \Gamma(\tilde{D}) \quad \text{for any } W \in \Gamma(\tilde{D}). \qquad ...(2.36)$$

The proof follows from Bejancu and Papaghiuc[1].

Corollary 2.1—Let M be an almost semi-invariant submanifold of the *sp*-Sasakian manifold $\tilde{M}$. Then we have

$$[X, \xi] \in \Gamma(D) \quad \text{for any } X \in \Gamma(D) \qquad ...(2.37)$$

$$[Y, \xi] \in \Gamma(D^{\perp}) \quad \text{for any } Y \in \Gamma(D^{\perp}) \qquad ...(2.38)$$

$$[Z, \xi] \in \Gamma(\tilde{D}) \quad \text{for any } Z \in \Gamma(\tilde{D}) \qquad ...(2.39)$$

The proof follows from Lemmas (2.4) and (2.6).

3. INTEGRABILITY OF DISTRIBUTIONS

Theorem 3.1—Let M be an almost semi-invariant submanifold of an *sp*-Sasakian manifold $\tilde{M}$. Then the distribution D is integrable if and only if

$$h(X, FY) = h(Y, FX). \qquad ...(3.1)$$

PROOF : By using (2.27), we have

$$g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi)$$

$$= -g(\nabla_X \xi, Y) + ... + g(\nabla_Y \xi, X)$$

$$= -g(FX, Y) + g(FY, X)$$

$$= 0$$

for all $X, Y \in \Gamma(D)$.

Next, from (2.25), we have

$$h(X, FY) = FQ\,\nabla_X Y + CL\,\nabla_X Y + f(h(X, Y)) \quad \ldots(3.2)$$

for any $X, Y \in \Gamma(D)$. Hence we have further

$$h(X, FY) - h(Y, FX) = FQ([X, Y]) + CL([X, Y]) \quad \ldots(3.3)$$

which proves the theorem.

From Theorem 3.1 and (2.37) follows :

Corollary 3.1—The distribution $D \oplus \{\xi\}$ is integrable if and only if (3.1) is satisfied.

Theorem 3.2—The distribution $D^\perp$ is never integrable.

PROOF : For $X, Y \in \Gamma(D^\perp)$, (2.26) gives

$$u(X, Y) = -A_{FY}X.$$

Applying F to (2.21) and using (1.1), we get

$$P(\nabla_X Y) = FP(A_{FY}X), \quad \text{for any} \quad X, Y \in \Gamma(D^\perp)$$

which with the help of Lemma (2.5) gives

$$P([X, Y]) = FP(A_{FY}X - A_{FX}Y) = 2FP(A_{FY}X)$$

showing the non-integrability of $D^\perp$.

From Theorem 3.2 and (2.38) follows :

Corollary 3.2—The distribution $D^\perp \oplus \{\xi\}$ is never integrable.

Theorem 3.3—The distribution $\tilde{D}$ is integrable if and only if

$$A_{CY}X - A_{CX}Y - \nabla_X bY + \nabla_Y bX \in \Gamma(D^\perp \oplus \tilde{D} \oplus \{\xi\}) \quad \ldots(3.4)$$

and

$$h(bX, Y) - h(X, bY) + \nabla^\perp_Y CX - \nabla^\perp_X CY \in \Gamma(C\tilde{D} + \nu) \quad \ldots(3.5)$$

for any $X, Y \in \Gamma(\tilde{D})$.

PROOF : For any $X, Y \in \Gamma(\tilde{D})$, using (2.29), we get

$$\begin{aligned} g([X, Y], \xi) = g(\nabla_X Y - \nabla_Y X, \xi) &= g(X, \nabla_Y \xi) - g(Y, \nabla_X \xi) \\ &= g(X, bY) - g(Y, bX) \\ &= 0. \end{aligned} \quad \ldots(3.6)$$

Now, for any $X, Y \in \Gamma(\tilde{D})$, (2.26) gives

$$u(X, Y) = \nabla_X bY - A_{CY} X. \qquad ...(3.7)$$

Applying F to (2.21) and using (1.1) and (3.7), we get

$$P(\nabla_X Y) = FP(A_{CY} X - \nabla_X bY) \qquad ...(3.8)$$

$$P([X, Y]) = FP(A_{CY} X - A_{CX} Y - \nabla_X bY + \nabla_Y bX) = 0 \qquad ...(3.9)$$

if and only if (3.4) is satisfied. Applying F to (2.25) and taking the components in $D^{\perp}$, we get

$$Q \nabla_X Y = Qt(h(X, bY) + \nabla^{\perp}_X CY - fh(X, Y))$$

which further yields

$$Q([X, Y]) = Qt(h(X, bY) - h(Y, bX) + \nabla^{\perp}_X CY - \nabla^{\perp}_Y CX).$$

Hence $\tilde{D}$ is integrable if and only if (3.5) is also satisfied.

From Theorem 3.3 and (2.39) follows :

Corollary 3.3—The distribution $\tilde{D} \oplus \{\xi\}$ is integrable if and only if (3.4) and (3.5) are satisfied simultaneously.

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