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HODOGRAPH TRANSFORMATION IN CONSTANTLY INCLINED TWO-PHASE MFD FLOWS

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Hodograph transformation is employed for steady, plane, viscous, incompressible constantly inclined two-phase MFD flows and a partial differential equation of second order obtained which is used to find the solution for vortex flow.

1. INTRODUCTION

Multiphase fluid phenomena are of extreme importance in various fields of science and technology such as geophysics, nuclear engineering, chemical engineering etc. In recent years, considerable attention has been paid to the study of the multiphase fluid flow system in non-rotating or rotating frames of reference. Multiphase fluid systems are concerned with the motion of a liquid or gas containing immiscible inert particles. Of all multiphase fluid systems observed in nature, blood flow, flow in rocket chamber, dust in gas cooling systems to enhance heat transfer process, movement of inert particles in atmosphere and sand or other suspended particles in sea beaches are the most common examples Naturally, studies of these systems are mathematically interesting and physically useful. The presence of particles in a homogeneous fluid makes the dynamical study of flow problems quite complicated. However, these problems are usually investigated under various simplifying assumptions.

Saffman[1] has formulated the equations of motion of a dusty fluid which is represented in terms of large number density $N(x, t)$ of very small spherical inert particles whose volume concentration is small enough to be neglected. It is assumed that the density of the dust particles is large when compared with the fluid density so that the mass concentration of the particles is an appreciable fraction of unity. In this formulation, Saffman also assumed that the individual particles of dust are so small that stokes' law of resistance between the particles and the fluid remains valid. Using the model of Saffman, several authors including Michael and Miller[2], Liu[3], Debnath and Basu[4] and S. N. Singh *et al.*[5], have investigated various aspects of hydrodynamics and hydromagnetic two-phase fluid flows in non-rotating system.

Transformation techniques have become some of the powerful methods for solving non-linear partial differential equations. Amongst many, the hodograph transformation has gained considerable success on fluid dynamics problems. Ames[6] has given an excellent survey of this method together with its application in various other fields. Chandna *et al.*[8] have used the hodograph transformation for steady MFD flows. Also Singh *et al.*[10] have used hodograph transformation in steady rotating MHD flows and obtained some solutions.

In this paper, hodograph transformation is employed for steady, plane, viscous incompressible constantly inclined two-phase MFD flows and a partial differential equation of second order obtained which is used to find the solution for vortex flow.

2. Basic Equations

The basic equations of motion governing the steady flow of a dusty, incompressible, viscous fluid with infinite electrical conductivity in the presence of magnetic field are

$$\text{div}\, \bar{u} = 0 \qquad \text{...(2.1)}$$

$$\rho\,[(\bar{u}\,\text{grad})\,\bar{u}] = -\,\text{grad}\, p + \mu\,\text{curl}\,\bar{H} \times \bar{H} + KN(\bar{v} - \bar{u}) + \eta\,\nabla^2 \bar{u} \qquad \text{...(2.2)}$$

$$\text{curl}\,(\bar{u} \times \bar{H}) = \bar{0} \qquad \text{...(2.3)}$$

$$\text{div}\,(N\bar{v}) = 0 \qquad \text{...(2.4)}$$

$$m\,(\bar{v}.\,\text{grad})\,\bar{v} = K(\bar{u} - \bar{v}) \qquad \text{...(2.5)}$$

$$\text{div}\,\bar{H} = 0 \qquad \text{...(2.6)}$$

where $\bar{u}$, $\bar{v}$, $\bar{H}$, p, ρ, η, μ are fluid velocity vector, dust velocity vector, magnetic field vector, fluid pressure, fluid density, kinematic coefficient of viscosity and magnetic permeability respectively; m is the mass of each dust particle, N the number density of dust particles and K the Stokes' resistance coefficient for the particles.

The situation for which the velocity of fluid and dust particles are everywhere parallel, is defined as[11]

$$\bar{v} = \frac{\alpha}{N}\,\bar{u} \qquad \text{...(2.7)}$$

where α is some scalar satisfying

$$\bar{u}.\,\text{grad}\,\alpha = 0 \qquad \text{...(2.8)}$$

which implies that α is a constant on the fluid streamlines.

Introducing vorticity function, current density function and Bernoulli function defined, respectively, by

$$\xi = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \qquad ...(2.9)$$

$$\Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \qquad ...(2.10)$$

$$B = p + \tfrac{1}{2} \rho U^2 \qquad ...(2.11)$$

where $U^2 = u_1^2 + u_2^2$, the system of eqns. (2.1) – (2.6) can be replaced by the following system

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \qquad ...(2.12)$$

$$\eta \frac{\partial \xi}{\partial y} - \rho \xi u_2 + \mu \Omega H_2 - K(\alpha - N) u_1 = - \frac{\partial B}{\partial x} \qquad ...(2.13)$$

$$\eta \frac{\partial \xi}{\partial x} - \rho \xi u_1 + \mu \Omega H_1 + K(\alpha - N) u_2 = \frac{\partial B}{\partial y} \qquad ...(2.14)$$

$$u_1 H_2 - u_2 H_1 = f \text{(arbitrary constant)} \qquad ...(2.15)$$

$$\frac{m\alpha}{N} \left[\frac{\alpha}{N} \left(u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \right) + u_1 \left(u_1 \frac{\partial}{\partial x} \left(\frac{\alpha}{N} \right) + u_2 \frac{\partial}{\partial y} \left(\frac{\alpha}{N} \right) \right) \right]$$

$$= K \left(\frac{\alpha}{N} - 1 \right) u_1 \qquad ...(2.16)$$

$$\frac{m\alpha}{N} \left[\frac{\alpha}{N} \left(u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \right) + u_2 \left(u_1 \frac{\partial}{\partial x} \left(\frac{\alpha}{N} \right) + u_2 \frac{\partial}{\partial y} \left(\frac{\alpha}{N} \right) \right) \right]$$

$$= K \left(\frac{\alpha}{N} - 1 \right) u_2 \qquad ...(2.17)$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0. \qquad ...(2.18)$$

The advantage of this system over the original system is that the order of the partial differential equation is decreased.

We now consider constantly inclined plane flows and let α_0 denote the constant non-zero angle between $\bar{u}$ and $\bar{H}$. The vector and scalar products of $\bar{u}$ and $\bar{H}$, using the diffusion equation (2.15), yield

$$u_1 H_2 - u_2 H_1 = UH \sin \alpha_0 = f \qquad ...(2.19)$$

$$u_1 H_1 + u_2 H_2 = UH \cos \alpha_0 = f \cot \alpha_0$$

where

$$H = \sqrt{H_1^2 + H_2^2}\,.$$

Solving (2.19), we get

$$H_1 = \frac{f}{U^2}(C\,u_1 - u_2),\ H_2 = \frac{f}{U^2}(C\,u_2 + u_1) \qquad ...(2.20)$$

where $C = \cot \alpha_0$ is a known constant for a prescribed constantly inclined non-aligned flow.

Using (2.20) in the system of equations (2.9) – (2.18), we have

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \qquad ...(2.21)$$

$$\eta\ \frac{\partial \xi}{\partial x} - \rho\xi\, u_1 + \frac{\mu\Omega f}{U^2}(C\,u_2 + u_1) - K(\alpha - N)\,u_1 = -\frac{\partial B}{\partial x} \qquad ...(2.22)$$

$$\eta\ \frac{\partial \xi}{\partial x_1} - \rho\xi\, u_2 + \frac{\mu\Omega f}{U^2}(C\,u_1 - u_2) + K(\alpha - N)\,u_2 = \frac{\partial B}{\partial y} \qquad ...(2.23)$$

$$\frac{m\alpha}{N}\left[\frac{\alpha}{N}\left(u_1\,\frac{\partial\, u_1}{\partial x} + u_2\,\frac{\partial\, u_1}{\partial y}\right) + u_1\left(u_1\,\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right) + u_2\,\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\right] = K\left(\frac{\alpha}{N} - 1\right)u_1 \qquad ...(2.24)$$

$$\frac{m\alpha}{N}\left[\frac{\alpha}{N}\left(u_1\,\frac{\partial\, u_2}{\partial x} + u_2\,\frac{\partial\, u_2}{\partial y}\right) + u_2\left(u_1\,\frac{\partial}{\partial x}\left(\frac{\alpha}{N}\right) + u_2\,\frac{\partial}{\partial y}\left(\frac{\alpha}{N}\right)\right)\right] = K\left(\frac{\alpha}{N} - 1\right)u_2 \qquad ...(2.25)$$

$$(u_2^2 - u_1^2 + 2u_1\,u_2)\left(\frac{\partial\, u_1}{\partial y} + \frac{\partial\, u_2}{\partial x}\right) + (Cu_2^2 - Cu_1^2 + 2u_1\,u_2)\left(\frac{\partial u_1}{\partial x} - \frac{\partial\, u_2}{\partial y}\right) = 0 \qquad ...(2.26)$$

$$\frac{\partial\, u_2}{\partial x} - \frac{\partial\, u_1}{\partial y} = \xi \qquad ...(2.27)$$

$$\frac{\partial}{\partial x}\left(\frac{C\,u_2 + u_1}{U^2}\right) - \frac{\partial}{\partial y}\left(\frac{C\,u_1 - u_2}{U^2}\right) = \frac{\Omega}{f}. \qquad ...(2.28)$$

Let the flow variables $u_1\,(x, y)$, $u_2\,(x, y)$ be such that, in the flow region under consideration, the Jacobian

$$J = \frac{\partial\,(u_1, u_2)}{\partial\,(x, y)} \text{ satisfies } 0 < |\,J\,| < \infty.$$

In such a case, we consider x and y as functions of u_1 and u_2 such that the following relations hold true :

$$\frac{\partial u_1}{\partial x} = J \frac{\partial y}{\partial u_2}, \quad \frac{\partial u_1}{\partial y} = -J \frac{\partial x}{\partial u_1}$$

$$\frac{\partial u_2}{\partial x} = -J \frac{\partial y}{\partial u_1}, \quad \frac{\partial u_2}{\partial y} = J \frac{\partial x}{\partial u_1}. \qquad \text{...(2.29)}$$

Employing transformation equation (2.29) in (2 21) and (2.26), we get

$$\frac{\partial x}{\partial u_1} + \frac{\partial y}{\partial u_2} = 0 \qquad \text{...(2.30)}$$

$$(C u_1^2 - C u_2^2 - 2u_1 u_2)\left(\frac{\partial x}{\partial u_1} - \frac{\partial y}{\partial u_2}\right)$$

$$+ (u_1^2 - u_2^2 + 2C u_1 u_2)\left(\frac{\partial x}{\partial u_2} + \frac{\partial y}{\partial u_1}\right) = 0. \qquad \text{...(2.31)}$$

The equation of continuity implies the existence of stream function $\psi(x, y)$ so that

$$\frac{\partial \psi}{\partial x} = -u_2, \frac{\partial \psi}{\partial y} = u_1. \qquad \text{...(2.32)}$$

Likewise, equation (2.30) implies the existence of a function $L(u_1, u_2)$ called the Legendre transform of the stream function $\psi(x, y)$ such that

$$\frac{\partial L}{\partial u_1} = -y, \frac{\partial L}{\partial u_2} = x. \qquad \text{...(2.33)}$$

Employing (2.33) in (2.31), we have

$$(u_2^2 - u_1^2 - 2C u_1 u_2)\frac{\partial^2 L}{\partial u_1^2} + (2C u_1^2 - 2C u_2^2 - 4u_1 u_2)$$

$$\times \frac{\partial^2 L}{\partial u_1 \partial u_2} + (2C u_1 u_2 + u_1^2 - u_2^2) \frac{\partial^2 L}{\partial u_2^2} = 0.$$

$$\text{...(2.34)}$$

Now introducing the polar coordinate (U, θ) in the hodograph plane i.e. the (u_1, u_2) plane through the relation :

$$u_1 = U \cos \theta, \quad u_2 = U \sin \theta$$

equation (2.34) gets transformed into

$$\frac{\partial^2 L}{\partial U^2} - \frac{2C}{U}\frac{\partial^2 L}{\partial U \partial \theta} - \frac{1}{U^2}\frac{\partial^2 L}{\partial \theta^2} - \frac{1}{U}\frac{\partial L}{\partial U} + \frac{2C}{U^2}\frac{\partial L}{\partial \theta} = 0$$

$$\text{...(2.35)}$$

where θ is the inclination of vector field $\bar{u}$.

3. Vortex Flow

A solution of (2.35) is given by

$$L = B_2 U^2 + (A_1 \cos\theta + B_1 \sin\theta) U$$

$$= B_2 (u_1^2 + u_2^2) + A_1 u_1 + B_1 u_2 \qquad \ldots(3.1)$$

where A_1, B_1 and B_2 are arbitrary constant and $B_2 \neq 0$. In this case,

$$x = \frac{\partial L}{\partial u_2} = 2B_2 u_2 + B_1, \; y = -\frac{\partial L}{\partial u_1} = -(2B_2 u_1 + A_1) \qquad \ldots(3.2)$$

and therefore the velocity field is given by

$$u_1 = -\frac{y + A_1}{2B_2}, \; u_2 = \frac{x - B_1}{2B_2}. \qquad \ldots(3.3)$$

These relations represent a circulatory flow.

From (2.20), we get

$$H_1 = \frac{-2B_2 f[(x - B_1) + C(y + A_1)]}{(X - B_1)^2 + (y + A_1)^2}$$

and

$$H_2 = \frac{2B_2 f[C(x - B_1) - (y + A_1)]}{(X - B_1)^2 + (y + A_1)^2}. \qquad \ldots(3.4)$$

The vorticity ξ and current density Ω can be expressed as

$$\xi = \frac{1}{B_2}, \; \Omega = 0. \qquad \ldots(3.5)$$

From the integrability condition for B with the use of (2.13) and (2.14) and [(3.3)—(3.5)], we obtain

$$(x - B_1)\frac{\partial}{\partial x}(N - \alpha) + (y + A_1)\frac{\partial}{\partial y}(N - \alpha) + 2(N - \alpha) = 0. \qquad \ldots(3.6)$$

Solving (3.6), the number density of dust particles $N(x, y)$ is given by

$$N = \frac{C_1}{(x - B_1)(y + A_1)} + \alpha \qquad \ldots(3.7)$$

where C_1 is an arbitrary constant. From equation (2.8) and (3.3), we obtain

$$\alpha = C_2[(x - B_1)^2 + (y + A_1)^2] \qquad \ldots(3.8)$$

where C_2 is an arbitrary constant.

Hence

$$N = \frac{C_1}{(x - B_1) + (y + A_1)} + C_2[(x - B_1)^2 + (y + A_1)^2]. \qquad \ldots(3.9)$$

Using (3.3) – (3.5) and (3.7) in (2.22) and (2.23) and solving, we get

$$B = \frac{\rho}{4B_2^2}[(x - B_1)^2 + (y + A_1)^2] + \frac{K C_1}{2B_2}\ln\frac{x - B_1}{y + A_1} + C_3 \quad ...(3.10)$$

where C_3 is an arbitrary constant. The pressure P is given by

$$P = C_3 + \frac{\rho}{8B_2^2}[(x - B_1)^2 + (y + A_1)^2] + \frac{K C_1}{2B_2}\ln\frac{x - B_1}{y + A_1}. \quad ...(3.11)$$

In this case the streamlines are given by

$$(x - B_1)^2 + (y + A_1)^2 = \text{constant}$$

which are concentric circles. Summing up, we have :

Theorem 1—If the dust particle is everywhere parallel to the fluid velocity in the steady, plane, constantly inclined MFD flow of an incompressible, viscous, two phase fluid, then the streamlines are concentric circles and the dust particle number density is given by (3.9). Also the velocity, the magnetic field, the vorticity, the current density and the pressure are given by (3.3), (3.4), (3.5) and (3.11) respectively.

4. Conclusion

There are very few exact solutions of two-phase MFD flows. The mathematical complexity of the equations governing the flow of an electrically conducting has prohibited a thorough analysis. To reduce some of the complexity, it becomes necessary to make certain assumptions about the inherent properties of the two-phase fluid. Furthermore, all the methods of analysis to this date require that we impose some restrictions on the angle between velocity field vector and magnetic field vector. In the present work, Saffman model for infinitely conducting two-phase fluid flow considering constant angle between u and H, called constantly inclined flow, is taken and exact solutions of physical importance are obtained applying hodograph transformation. Although the scope of the present work is limited, it is believed that by using the approach of this paper and the exact solution obtained, work towards boundary value problem of practical importance can be pursued.

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