Indian J. pure appl. Math., **20** (8) : 782-785, August 1989

ON DIGRAPH RECONSTRUCTION

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(*Received* 24 *February* 1988; *after revision* 16 *January* 1989)

We reconstruct a class of digraphs and improve a result on the reconstruction of graphs due to P. Z. Chinn.

1. INTRODUCTION

A digraph D consists of a finite set V of points and a set A of ordered pairs of distinct points. Any such pair (u, v) is called an arc from u to v. With each point v of a digraph D, we can associate a triple (r, s, k), called the degree triple of v, where $r = |\{u \mid (v, u) \in A \text{ and } (u, v) \notin A\}|$, $s = |\{u \mid (u, v) \in A \text{ and } (v, u) \notin A\}|$, and $k = |\{u \mid (u, v) \in A \text{ and } (v, u) \in A\}|$. We use the terminology as given in Harary[2].

Reconstruction Conjecture—Any graph with at least three points can be reconstructed up to isomorphism, from the collection of its point-deleted subgraphs.

Digraph Reconstruction Conjecture (DRC)—Any digraph with at least five points can be reconstructed, up to isomorphism, from the collection of its point-deleted subdigraphs.

If D is a digraph with points $v_1, v_2, \ldots v_n$ and d_i is the degree triple of v_i in D, then $(D - v_i, d_i)$ is called a degree triple associated point-deleted subdigraph of D.

New Digraph Reconstruction Conjecture (NDRC)[3]—Any digraph can be reconstructed, up to isomorphism, from the collection of its degree triple associated point-deleted subdigraphs.

A digraph is called reconstructible if it obeys DRC and N-reconstructible if it obeys NDRC. It is obvious that the truth of DRC implies the truth of NDRC, which in turn implies the truth of the reconstruction conjecture. Stockmeyer[5] has found six infinite families of counterexamples to the DRC, but the digraphs in all these counterexamples obey NDRC[4]. In fact, no counterexample has so far been unearthed for NDRC. Consequently, the problem now is to determine which classes of digraphs are reconstructible. Chinn[1] proves the following result for graphs :

Theorem A[1]—Let G_i, $i = 1$ to n, be the point-deleted subgraphs of a graph G. If each point-deleted subgraph of one of the G_i, say G_1, occurs exactly once as an induced subgraph of any G_i, $i \neq 1$, then G is reconstructible.

We prove that a related class of digraphs is reconstructible and, from this, reconstruct a larger class of graphs then that covered by Chinn's result.

Let D be a digraph with points $v_1, v_2, \ldots, v_n$, $n > 2$, and assume that v_1 can be located (position of v_1 can be deduced from the information furnished by $D - v_i$, $i = 1$ to n) in $D - v_i$ for each $i > 1$. Let B be the set of degree triples of the point v_1 in $D - v_i$, $i = 2, \ldots, n$. If B has only one element and it is $(n - 2, 0, 0)$ ($(0, n - 2, 0)$ or $(0, 0, n - 2)$), then the degree triple of v_1 in D is $(n - 1, 0, 0)$ ($(0, n - 1, 0)$ or $(0, 0, n - 1)$, respectively). Otherwise, (k, m, n) is the degree triple of v_1 in D where k, m, and n are, respectively, the maximum values of the first, second and third coordinates among the triples in B. Such an argument works in the case of graphs also. This observation is used in the proof of Theorem 1 below.

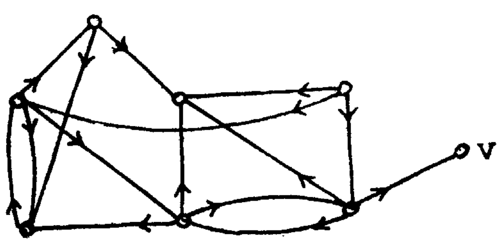
2. Reconstruction

Theorem 1—If D is a digraph having a point j such that $D - j$ has nonisomorphic point-deleted subdigraphs and j can be located (position of point j can be deduced) in each point-deleted subdigraph of D except $D - j$, then D is reconstructible.

Proof: Let $1, 2, \ldots, n$ be the points of D and let us denote $D - i$ by D_i. Without loss of generality, let $j = 1$. By hypothesis, the point 1 can be identified in each D_i, $i = 2$ to n. For each i, $i = 2$ to n, there is a unique point in D_1 whose removal gives a digraph isomorphic to $D_i - 1$. This point of D_1 corressponds to the point i in D and label it in D_1 accordingly. Thus D_1 is labeled with $2, \ldots, n$. From the set B of degree triples of the point 1 in $D_2, \ldots, D_n$, we can find the degree triple (k, m, n) of the point 1 in D as given in the previous section.

It is obvious that there is a single arc from 1 to i, there is a single arc from i to 1, there is a symmetric pair of arcs between 1 and i or there is no arc between 1 and i according as the degree triple of the point 1 in D_i is $(k - 1, m, n)$, $(k, m - 1, n)$, $(k, m, n - 1)$ or (k, m, n). Thus the arc (s) between 1 and any other point i is known. Since the arc (s) between any two points i and j, $1 \neq i \neq j \neq 1$ in D is same as that in the labeled digraph D_1, D is known.

The digraph D in Figure 1 satisfies the hypothesis of Theorem 1 with v in the place of point j as follows :



D

Fig. 1

Suppose the collection S of point-deleted subdigraphs of D are given. In S, there is a member, say D_1 (D_1 corresponds to $D - v$), having distinct point-deleted subdigraphs. From S, we can deduce that the point, say 1, whose deletion gives D_1 is incident with only one arc in D. Hence in any member of S except D_1, point 1 will be incident with at most one arc. However, each member of S except D_1 has exactly one such point and this point must be the point 1. Thus in each member of S except D_1, the point 1 can be located.

Corollary 1 · If G is a graph having a point j such that $G - j$ has nonisomorphic point-deleted subgraphs and j can be located (position of point j can be deduced) in each point-deleted subgraph of G except $G - j$, then G is reconstructible.

This is only the graph version of Theorem 1.

Corollary 2—Let D_i, $i = 1$ to n be the point-deleted subdigraphs of a digraph D. If each point-deleted subdigraph of one of the D_i, say D_1 occurs exactly once as an induced subdigraph of any D_i, $i \neq 1$, then D is reconstructible.

PROOF: Since D_i are the point-deleted subdigraphs of a digraph, points of each D_i can be labeled with $\{1, 2, \ldots, n\} - \{i\}$ such that $D_i - j \cong D_j - i$ for each $i \neq j$. If $D_1 - i \cong D_1 - j, i \neq j$, then the point-deleted subdigraph $D_1 - i$ of D_1 is isomorphic to the subdigraphs $D_i - 1$ and $D_j - 1$ of D_i and D_j respectively, contradicting the hypothesis. Hence the point-deleted subdigraphs of D_1 are distinct. Now forget the labeling assumed above. Take any point-deleted subdigraph H of D_1. H can be obtained by deleting a point corresponding to point 1 of D from D_j for some $j > 1$. Because of the hypothesis, there is a unique D_t, $t \neq 1$ and a unique point w in D_t such that $D_t - w$ is isomorphic to H. Clearly this point w in D_t is the point 1 of D. In this way, each point-deleted subdigraph of D_1 can be used to locate the point 1 in a D_i, $i \neq 1$. Since D_i and D_1 have a point deleted subdigraph in common for each $i > 1$, we are able to locate the point 1 in D_i for each $i > 1$. Hence D satisfies the hypothesis of Theorem 1. Hence D is reconstructible.

Theorem A is simply the graph version of the above corollary.

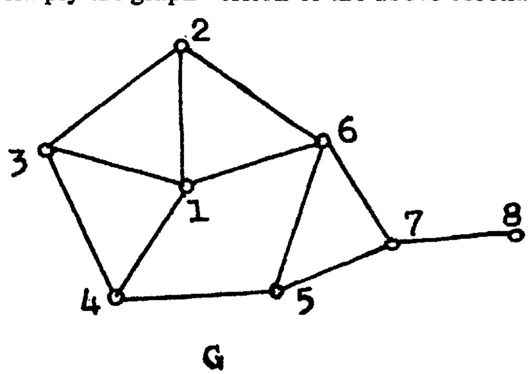


FIG. 2.

The proof of Corollary 2 shows that any digraph (graph) that satisfies the hypothesis of Corollary 2 (Theorem A) satisfies the hypothesis of Theorem 1 (Corollary 1). But the converse is not true. The graph G in Fig. 2 illustrates this. In G, G_8 (G_i denotes $G - i$) has nonisomorphic point-deleted subgraphs. Also, since 8 is a point of degree one in G, it will occur as a point of degree at most one in G_i, $j \neq 8$. However, each G_j, $j \neq 8$, has a unique point with degree at most one, and this point must be the point 8. Thus G satisfies the hypothesis of Corollary 1. But

$$G_1 - 2 \cong G_2 - 1 \cong G_4 - 2$$

$$G_2 - 7 \cong G_3 - 7 \cong G_7 - 2$$

and

$$G_6 - 8 \cong G_5 - 4 \cong G_8 - 6$$

so that there does not exist a G_i such that each point-deleted subgraph of G_i occur exactly once as an induced subgraph of any G_j, $j \neq i$. Hence G does not satisfy the hypothesis of Theorem A. Thus Corollary 1 reconstructs a larger class than that covered by Chinn's result.

Corollary 3—A graph (digraph) G with one point labeled is reconstructible if the subgraph (subdigraph) of G obtained by deleting the labeled point has distinct point-deleted subgraphs (subdigraphs).

The proof is obvious as the hypothesis of Theorem 1 holds.

Acknowledgememt

We are thankful to the referee for many valuable comments.

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