India J. pure appl. Math., **20** (8) : 786–789, August 1989

VARIANTS OF HOPFICITY IN MODULES

B. M. PANDEYA

Department of Applied Mathematics, I. T., B. H. U., Varanasi 221005

S. A. PARAMHANS

Department of Mathematics, Faculty of Science, B. H. U., Varanasi 221005

AND

S. P. KOIRALA**

Department of Applied Mathematics, I. T., B. H. U., Varanasi 221005

(*Received* 28 *March* 1988; *after revision* 27 *December* 1988)

It has been shown here that a direct sum of Hopfian modules need not be Hopfian. A class of modules whose strong Hopficity is preserved under taking injective hulls, has been provided. Also a characterization has been obtained for the super Hopficity of a quasi-injective module.

INTRODUCTION

As has been noted by Hiremath[1], the concept of Hopfian groups was introduced by G. Baumslag in 1943. It was generalized to Hopfian rings and modules by Hiremath. In fact, the study of endomorphism rings of various rings and modules has been a topic of keen interest since the end of the nineteen sixties when injectivity and its variants began to flourish. Answering several questions raised by Hiremath, it is shown here that a direct sum of Hopfian modules need not be Hopfian. A class of modules whose strong Hopficity is preserved under taking injective hulls, has also been provided (Proposition 4). Lastly, a characterization has been obtained for the super-Hopficity of a quasi-injective module (Theorem 6).

PRELIMINARIES

Throughout this paper, the basic ring R is associative with 1 and the R-modules M are left unitary. M is said to be uniform if it is essential extension of each of its nonzero submodules. The injective hull of a module M is denoted as $\hat{M}$.

An R module M is said to be Hopfian if every ependomorphism of M is an automorphism of M. Similarly, a ring R is Hopfian if every ring homomorphism of R onto R is an automorphism. Tiwary and Pandeya[2] have introduced the *-property as

***Current address* : Department of Mathematics, Kathamandu University, Nepal.

follows : M is said to satisfy* if every non-zero endomorphism of M is a monomorphism. This property implies Hopficity. But the converse is not true. Thus, we call the *-property "strong Hopficity". Similarly we call M super-Hopfian if its endomorphism ring is a division ring. This property implies strong Hopficity but the converse is not true. These concepts are illustrated by the following examples

(i) Z_4 as a Z-module being Noetherian, is Hopfian but it is not strongly Hopfian.

(ii) Z, considered as module over itself, is cyclic torsionfree and therefore strongly Hopfian but it is not super-Hopfian since $\mathrm{Hom}_Z(Z, Z) \cong Z$ which is not a division ring (cf. Proposition 4).

(iii) Any simple module e.g. Z_2 is a super Hopfian Z-module.

Note 1 : Hiremath[1] has asked whether the Hopficity of a ring R implies the Hopficity of the polynomial ring $R[X]$ in an indeterminate X. We note here that the class of Noetherian rings among the Hopfian ones obviously enjoy this property since R is Noetherian implies $R[X]$ is Noetherian[4] (Theorem 1, p. 201) and hence it is Hopfian.

Further, Hiremath has raised the question whether a direct sum (even finite) of Hopfian modules is Hopfian. It is pointed out here that this is not true at least in the case of infinite sums.

Example 2—The Z-module $\bigoplus_C Z$ is Hopfian if C is finite. But if C is infinite it is not Hopfian according to Hiremath[1], Proposition 12 and Theorem 14. But each Z is strongly Hopfian and so Hopfian.

This answers part of Hiremath's question in the line just before Remark 7 on p. 897 of Hiremath[1].

Another problem posed by Hiremath[1] (Remark 7) is as follows :

Is Hopficity a hereditary property? Here we note that strong Hopficity is inherited in the case of a quasi-injective module.

Remark 3 : Tiwary and Pandeya[2] (Proposition 2.6) have established that if M is a quasi injective module then the $1-1$-ness of all the nonzero endomorphisms in $\mathrm{Hom}(M, M)$ forces the $1-1$ ness of all the nonzero endomorphisms of any non-zero submodule N of M.

As is noted by Hiremath[1] (Remark 10), the Hopficity of a module does not guarantee the Hopficity of its injective hull. However, we give below a class of strongly Hopfian modules whose injective hulls are also strongly Hopfian.

Proposition 4—A cyclic torsion-free module and its injective hull are strongly Hopfian.

PROOF : Take a nonzero $f \in \text{Hom}(M, M)$ and a nonzero element $x \in \ker f$. Then $x = ry$ for some $r \in R$ and $f(x) = rf(y) = 0$. Because $f(y) \neq 0$, $r = 0$ showing that f is monic. Hence M is strongly Hopfian.

Let f be a nonzero endomorphism in $\text{Hom}(\hat{M}, \hat{M})$. Suppose $\ker f \neq 0$. Then $\text{Ker} f \cap M \neq 0$. So there will exist some nonzero $x \in M$ such that $f(x) = 0$. Since $M = Ry$ this will give $f(ry) = 0$ where $r \in R$. So $rf(y) = 0$. But the torsion-freeness of M gives $f(y) = 0$, a contradiction since $f(y) \neq 0$ as $\hat{M}$ is also the rational extension of M. Hence $\ker f = 0$ and consequently M is strongly Hopfian.

Lastly, we give a characterization for the super-Hopficity of a quasi-injective module. First we have the following :

Lemma 5—If M is quasi-injective and $\hat{M}$ is strongly Hopfian, then M is super-Hopfian.

PROOF : Let $K = \text{Hom}(\hat{M}, \hat{M})$ and $D = \text{Hom}(M, M)$. Take a nonzero $f \in D$. Due to the injectivity of $\hat{M}$ there exists a nonzero $k \in K$ with $f(M) = k(M)$. Because k is monic, so is f, whence we can define a map $g \in \text{Hom}_R(f(M), M)$ by $g(f(m)) = m$. Because M is quasi-injective there exists an $h \in \text{Hom}_R(M, M)$ such that $h \mid f(M) = g$. Clearly $hf = 1$ and so D is a division ring.

Theorem 6—Let M be a quasi-injective module. Consider the following conditions :

(i) $\hat{M}$ is super-Hopfian.

(ii) M is super-Hopfian.

(iii) M is uniform and $x, y \in M$, $(0 : x) > (0 : y)$ implies $x = 0$.

Then i) implies ii) whereas ii) and iii) are equivalent. Moreover, if the rational completion of M coincides with $\hat{M}$ then ii) implies i).

PROOF : Let $K = \text{Hom}(\hat{M}, \hat{M})$ and $D = \text{Hom}(M, M)$.

(i) implies (ii) : This follows from Lemma 5.

(ii) implies (i) : Suppose that the rational completion of M coincides with its injective hull. Let D be a division ring and $k \in K$. Under the given conditions, one-oneness of k is clear from Tiwary and Pandeya[2] (Theorem 1.5).

Since $\hat{M}$ is injective, the exact sequence $0 \to \hat{M} \to k(\hat{M})$ splits and so $k(\hat{M}) = \hat{M}$, whence k is an isomorphism.

(ii) implies (iii) and conversely : This follows from Wong[3].

Corollary—If we take M to be nonsingular in the above Theorem then i), ii) and iii) are all equivalent.

ACKNOWLEDGEMENT

The authors are thankful to the referee for his valuable suggestions, in particular, for the Example 2 and the statement alongwith proof of Lemma 5. The second author gratefully acknowledges financial support by U. G. C. India.

REFERENCES

1. V. A. Hiremath, *Indian J. pure appl. Math.* **17** (1986), 895-900.
2. A. K. Tiwary and B. M. Pandeya, *Proceedings of Seminar on Algebra and its Application* (Edited by H. L. Manocha and J. B. Srivastava), Marcel Dekker Lecture Notes No. 91 (1984), 199-203.
3. E. T. Wong, *Canad. Math. Bull.* **13** (1) (1970), 260-61.
4. O. Zariski and P. Samuel, *Commutative Algebra*, Vol. I. Von Nostrand Co. Inc. New York, 1958.