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EXTREME POINTS OF SOME FAMILIES OF ANALYTIC FUNCTIONS RELATED TO UNIVALENT FUNCTIONS

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The concept of a^*-families has been introduced by Kapoor and Mishra[1] and Mishra[2]. In this paper some new results for functions in a^*-families with negative Taylor coefficients have been obtained. These results include coefficient characterization theorem, coefficient estimates, identification of extreme points and characterization for connectedness of a^*-families.

1. Introduction

Let $\mathscr{H}$ denote the class of functions f analytic in the unit disc $E = \{z: | z | < 1\}$, safisfying $f(0) = 0$ and $f'(0) \neq 0$. For $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ the Hadamard product $f * g$ of f and g is defined by $(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k$. Note that $f * g$ is also in $\mathscr{H}$.

Definition 1—Let s and g in $\mathscr{H}$ be given by

$$\left.\begin{aligned} s(z) &= \sum_{k=1}^{\infty} c_k z^k,\ c_1 > 0,\ c_k \geqslant 0 \\ g(z) &= \sum_{k=1}^{\infty} d_k z^k,\ d_1 > 0,\ d_k \geqslant 0 \end{aligned}\right\} \quad k = 2, 3 \ldots \qquad \ldots(1) \quad \ldots(2)$$

with

$$(c_k/c_1) - (d_k/d_1) > 0 \quad k = 1, 2, 3 \ldots \qquad \ldots(3)$$

We denote by $F(s, g, \alpha)$, $0 \leqslant \alpha < (c_1/d_1)$, the class of functions f in $\mathscr{H}$ satisfying

$$(g * f)(z) \neq 0,\ 0 < | z | < 1 \qquad \ldots(4)$$

and

$$\text{Re}\left[\frac{(s * f)(z)}{(g * f)(z)}\right] > \alpha, z \in E. \quad ...(5)$$

Definition 2—Let q given by

$$q(z) = 1 + \sum_{k=2}^{\infty} e_k z^{k-1} \quad ...(6)$$

be analytic in E We denote by $F(s, g, \alpha, q, z_0)$, z_0 real, $0 < |z_0| < 1$, the class of functions f in $F(s, g, \alpha)$ satisfying

$$\left(\frac{f}{z} * q\right)(z_0) = 1. \quad ...(7)$$

If B is a subset of $-1 < z < 1$, $z \neq 0$, then $F(s, g, \alpha, q, B)$ stands for $\bigcup_{z_i \in B} F(s, g, \alpha, q, z_i)$. Further, we denote by $F[s, g, \alpha]$ (respectively $F[s, g, \alpha, q, B]$) the class of functions f in $F(s, g, \alpha)$ (respectively in $F(s, g, \alpha, q, B)$) given by the series

$$f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0, \ k = 1, 2, 3 \ldots \quad ...(8)$$

The class $F(s, g, \alpha)$ is introduced and studied in Kapoor and Mishra[2] and Mishra[3] where it is called an a^*-family. Suitable choices of the touple give us many familiar families related to univalent functions. For example :

(*a*) $F\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha\right)$ is the class of starlike function of order α.

(b) $F\left(\frac{z(1+z)}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha\right)$ is the class of convex functions of order α.

(c) $F\left(\frac{z}{(1-z)^2}, \frac{z}{1-z^2}, \alpha\right)$ is the class of functions starlike with respect to symmetric points and of order α (Sakaguchi[5]).

(d) $F\left(\frac{z}{(1-z)^2}, z, \alpha\right)$ is an extensively studied subclass of close-to-convex functions.

(e) $F\left(\frac{z}{(1-z)^{n+2}}, \frac{z}{(1-z)^{n+1}}, \frac{1}{2}\right)$ $n = 0, 1, 2, \ldots$, is a class of functions studied in Goel and Sohi[1].

(f) $F\left(\frac{z}{(1-z)^{2(1-\alpha)+1}}, \frac{z}{(1-z)^{2(1-\alpha)}}, \frac{1}{2}\right)$, $0 \leq \alpha < 1$, is the class of prestarlike functions of order α Ruscheweyh[4].

(g) $F\left(\frac{z}{1-z}, z, \alpha\right)$ is the class of functions for which $\operatorname{Re}\{f(z)/z\} > \alpha$.

In Definition, 2, if we take $q(z) = \frac{1}{1-z}\left(\text{respectively } q(z) = \frac{1}{(1-z)^2}\right)$ then $F(s, g, \alpha, q, z_0)$ is the subclass of functions f in $F(s, g, \alpha)$ satisfying $f(z_0) = z_0$ (respectively $f'(z_0) = 1$). Finally, let G be a family of functions analytic in E. A function f in G is said to be an extreme point of G if $f \neq tg_1 + (1-t)g_2$ for $0 < t < 1$ and any pair of distinct functions g_1 and g_2 in $\mathscr{H}$.

Silverman[6,7] first studied the particular a^*-families

$$F\left[\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha\right], F\left[\frac{z(1+z)}{(1+z)^3}, \frac{z}{(1-z)^2}, \alpha\right]$$

$$F\left[\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \frac{1}{1-z}, z_0\right], F\left[\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, \frac{1}{(1-z)^2}, z_0\right]$$

$$F\left[\frac{z(1+z)}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \frac{1}{1-z}, z_0\right] \text{ and}$$

$$F\left[\frac{z(1+z)}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \frac{1}{(1-z)^2}, z_0\right]$$

and determined their extreme points. Subsequently Silverman and Silvia[8] have also investigated the a^*-family

$$F\left[\frac{z}{(1-z)^{2(1-\alpha)+1}} \frac{z}{(1-z)^{2(1-\alpha)}}, \tfrac{1}{2}\right]$$

and have determined its extreme points. In recent years there has been extensive study in the determination of extreme points of families of functions with negative co-efficients. Kapoor and one of the present authors[2,3] have determined the extreme points of any a^*-family $F[s, g, \alpha]$ In the present paper we determine the extreme points of $F[s, g, \alpha, q, z_0]$ for arbitrary choice of the touple (s, g, α, q, z_0). We also show that if B is a subset of $(0, 1)$ then $F[s, g, \alpha, q, B]$ is a convex family if and only if B is connected. We determine the extreme points of $F[s, g, \alpha, q, B]$ for connected B.

2. Extreme Points of $F[s, g, \alpha, q, z_0]$

We need the following theorem from Kapoor and Mishra[2] (also see Mishra[3]).

Theorem A—Let $\{c_k\}$ and $\{d_k\}$ be sequences of non negative numbers with $c_1 > 0$, $d_1 > 0$, $(c_k/c_1) - (d_k/d_1) > 0$ and let $0 \leqslant \alpha \leqslant (c_1/d_1)$. Then a function f given by (8) is in $F[s, g, \alpha]$ where s and g are as in Definition 1 if and only if

$$\sum_{k=2}^{\infty} (c_k - \alpha d_k)\, a_k \leqslant a_1 (c_1 - \alpha d_1). \qquad ...(9)$$

The next theorem, a direct consequence of Theorem A, is useful for further investigations.

Theorem 1—Let s, g and α be as in Theorem A, z_0 be a real number such that $0 < |z_0| < 1$ and let $q(z)$ be as in Definition 2. Then the function $f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geqslant 0$, $k = 1, 2, 3, \ldots$ is in $F[s, g, \alpha, q, z_0]$ if and only if

$$\sum_{k=2}^{\infty} \left(\frac{c_k - \alpha d_k}{c_1 - \alpha d_1} - e_k\, z_0^{k-1} \right) a_k \leqslant 1. \qquad ...(10)$$

Proof : Observe that $f(z)$ satisfies the condition (7) if and only if $a_1 = 1 + \sum_{k=2}^{\infty} a_k\, e_k\, z_0^{k-1}$. Now, substituting this value of a_1 in (9) the result follows.

See that each term in the summation in (10) in nonnegative. This can be seen using (3) and the condition $0 \leqslant \alpha \leqslant (c_1/d_1)$.

Corollary 1—Let $f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geqslant 0$, be in $F[s, g, \alpha, q, z_0]$ where s, g, α, q and z_0 are as in Theorem 1. Then

$$a_k \leqslant \frac{c_1 - \alpha d_1}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}, \quad k = 2, 3, \ldots \qquad ...(11)$$

with equality for

$$f_k(z) = \frac{(c_k - \alpha d_k)\, z - (c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}.$$

$F[s, g, \alpha, q, z_0]$ is a convex family. For, if f and g are in $F[s, g, \alpha, q, z_0]$ and $0 < \lambda < 1$, then the function $\lambda f + (1 - \lambda) g$ satisfies the coefficient inequality (9) and the condition (7).

Theorem 2—Let the touple (s, g, α, q, z_0) be as in Definition 2.

Set

$$f_1(z) = z$$

and

$$f_k(z) = \frac{(c_k - \alpha d_k)\, z - (c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, z_0^{k-1}}, \quad k = 2, 3, \ldots \tag{12}$$

Then, a function f is in $F[s, g, \alpha, q, z_0]$ if, and only if it can be expressed in the form $\sum_{k=1}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof : Suppose $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geqslant 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Then,

$$f(z) = \left[\lambda_1 + \sum_{k=2}^{\infty} \frac{\lambda_k (c_k - \alpha d_k)}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}} \right] z$$

$$- \sum_{k=2}^{\infty} \lambda_k \frac{(c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}.$$

We note that $\left(\frac{f}{z} * q \right)(z_0) = \sum_{k=1}^{\infty} \lambda_k \left(\frac{f_k}{z} * q \right)(z_0) = \sum_{k=1}^{\infty} \lambda_k = 1.$

Also,

$$\sum_{k=2}^{\infty} \frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}{(c_1 - \alpha d_1)} \lambda_k$$

$$\frac{(c_1 - \alpha d_1)}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}}$$

$= \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leqslant 1.$ Hence f is in $F[s, g, \alpha, q, z_0]$.

Conversely, suppose that $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ is in $F[s, g, \alpha, q, z_0]$. For $k = 2, 3, \ldots$, write

$$\lambda_k = \frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_0^{k-1}\, a_k}{(c_1 - \alpha d_1)}$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.$$

Observe that by Theorem 1, $\lambda_k \geqslant 0, k = 2, 3, \ldots$ and $\sum_{k=2}^{\infty} \lambda_k \leqslant 1$. Now $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$.

From Theorem 2 it follows that the extreme points of $F[s, g, \alpha, q, z_0]$ are precisely, the set of functions $\{f_k\}_{k=1}^{\infty}$ where f_k are defined as in (12).

3. The Family $F[s, g, \alpha, q, B]$

In this section we determine the extreme points of $F[s, g, \alpha, q, B]$ for a connected B. We have the following

Lemma 1—If $f \in F[s, g, \alpha, q, z_0] \cap F[s, g, \alpha, q, z_1]$ where s, g, α, and q are as in Theorem 1 and z_0 and z_1 are distinct positive numbers then $f(z) = z$.

Proof : Let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$, $a_k \geqslant 0$. Then we must have

$$a_1 = 1 + \sum_{k=2}^{\infty} a_k e_k z_0^{k-1} = 1 + \sum_{k=2}^{\infty} a_k e_k z_1^{k-1}.$$

But this means that $a_k \equiv 0$ for $k \geqslant 0$.

Theorem 3—If B is contained in the interval (0, 1) then, $F[s, g, \alpha, q, B]$ is a convex family if and only if B is connected.

Proof : Let B be connected and let z_0 and z_1 be in B with $z_0 \leqslant z_1$. If $f(z) = a_1 z - \sum_{k=2}^{\infty} a_k z^k$ is in $F[s, g, \alpha, q, z_0]$ and $g(z) = b_1 z - \sum_{k=2}^{\infty} b_k z^k$ is in $F[s, g, \alpha, q, z_1]$ and $0 \leqslant \lambda \leqslant 1$, then we shall show that there exists a z_2 in B ($z_0 < z_2 < z_1$) such that $h(z) = \lambda f(z) + (1 - \lambda) g(z)$ is in $F[s, g, \alpha, q, z_2]$. Note that the coefficients of $h(z)$ satisfy (9). Now, write,

$$t(z) = \left(\frac{h}{z} * q\right)(z_0) = \lambda a_1 + (1 - \lambda) b_1 - \lambda \sum_{k=2}^{\infty} a_k e_k z^{k-1}$$

(*equation continued on p.* 826)

$$-(1-\lambda)\sum_{k=2}^{\infty} b_k e_k z^{k-1} = 1 + \lambda \sum_{k=2}^{\infty} a_k e_k \left(z_0^{k-1} - z^{k-1} \right)$$

$$+ (1-\lambda)\sum_{k=2}^{\infty} b_k e_k \left(z_1^{k-1} - z^{k-1} \right).$$

Observe that $t(z)$ is real for real z and $t(z_0) > 1$ and $t(z_1) < 1$. Hence there exists a z_2, $z_0 < z_2 < z_1$ such that $t(z_2) = 1$ and it follows that $h(z)$ is in $F[s, g, \alpha, q, z_2]$.

Conversely, if B is not connected we can choose z_0, z_1 in B and z_2 hot in B with $z_0 < z_2 < z_1$. Assume that f and g are not both identity functions. We write $t(\lambda) = t(z, \lambda) = ((\lambda f(z) + (1 - \lambda) g(z))/z) * q(z)$ as above where $0 \leqslant \lambda \leqslant 1$. Now with z_2 fixed, $t(z_2, 0) > 1$ and $t(z_2, 1) < 1$. Hence, there exists a λ_0, $0 < \lambda_0 < 1$ such that $t(z_2, \lambda_0) = 1$ and it follows that $h(z) = \lambda_0 f(z) + (1 - \lambda_0 g_0)$ is in $F[s, g, \alpha, q, z_2]$. But then, by Lemma 1, $F[s,g,\alpha,q,B]$ is not convex.

Theorem 4—Let $[z_0,z_1] \subset (0, 1)$. Then the extreme points of $F[s,g,\alpha, q, z_0]$ are z

$$f_k(z) = \frac{(c_k - \alpha d_k) z - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_0^{k-1}}$$

and $\qquad k = 2, 3, \ldots.$

$$g_k(z) = \frac{(e_k - \alpha d_k) z - (c_1 - \alpha d_1) z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_1^{k-1}}$$

PROOF : Let f be an extreme point of $F[s, g, \alpha, q, B]$. Then, f must be an extreme point of $F[s, g, \alpha, q, z_2]$ for some z_2, $z_0 \leqslant z_2 \leqslant z_1$. We first show that either $z_2 = z_0$ or $z_2 = z_1$.

This we do by showing that if $z_0 < z_2 < z_1$ then

$$h_k(z) = \frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1) z_k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_2^{k-1}}$$

can be written as a convex combination of f_k and g_k. Write $h_k(z, \lambda) = \lambda f_k(z) + (1 - \lambda) g_k(z)$. For z real and positive we have $h_k(z, 0) > h_k > h_k(z, 1)$. Hence, there exists a λ_0, $0 < \lambda_0 < 1$, for which $h_k(z, \lambda_0) = h_k(z)$. Infact, for λ_0 such that

$$\frac{1}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_2^{k-1}} = \frac{\lambda_0}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_0^{k-1}}$$

$$+ \frac{(1-\lambda_0)}{(c_k - \alpha d_k) - (c_1 - \alpha d_1) e_k z_1^{k-1}}$$

the coefficients of $h(z, \lambda_0)$ and $h_k(z)$ agree for all z. That is $h_k(z, \lambda_0) = h_k(z)$ throughout in the unit disc for

$$\lambda_0 = \frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, z_0^{k-1}}{(c_k-\alpha d_k) - (c_1-\alpha d_1)\, e_k\, z_1^{k\ 1}} \; \frac{z_1^{k-1} - z_2^{k-1}}{z_1^{k\ 1} - z_0^{k-1}}.$$

Thus, $h_k(z)$ can not be an extreme point of $F[s, g, \alpha, q, B]$. We next show that f_k and g_k can not be expressed as the convex combination of any two elements of $F(s, g, \alpha, q, B]$. Infact, for z real and positive and $0 \leqslant \lambda \leqslant 1$.

$$f_k(z) < \lambda\left(\frac{(c_k - \alpha d_k)\, z - (c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_3^{k-1}}\right)$$

$$+ (1 - \lambda)\left(\frac{(c_k - a d_k)\, z - (c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_4^{k-1}}\right)$$

$$z_0 < z_3 \leqslant z_1,\ z_0 < z_4 \leqslant z_1$$

and

$$g_k(z) > \lambda\left(\frac{(c_k - \alpha d_k)\, z - (c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_5^{k-1}}\right)$$

$$+ (1 - \lambda)\frac{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, z^k}{(c_k - \alpha d_k) - (c_1 - \alpha d_1)\, e_k\, z_6^{k-1}}$$

$$z_0 \leqslant z_5 < z_1 \text{ and } z_0 \leqslant z_6 < z_1.$$

The proof is now complete.

Corollary—If $0 < z_0 < z_1 < 1$, the closed conve hull of $F[s, g, \alpha, q, \{z_0, z_1\}]$ is $F[s, g, \alpha, q, [z_0, z_1]]$.

PROOF : Let f_k and g_k be defined as in the theorem. Adopting the method of proof of the theorem it can be shown that the extreme points of $F[s, g, \alpha, q, \{z_0, z_1\}]$ are z, f_k and g_k, $k = 2, 3, \ldots$ Hence the closed convex hull of $F[s, g, \alpha, q, \{z_0, z_1\}]$ is the closed convex hull of $\{z, f_k, g_k : k \geqslant 2\}$. However, it follows from the theorem that the above closed convex hull is $F[s, g, \alpha, q, [z_0, z_1]]$. The proof is complete.

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