STUDIES ON A GENERALIZED UNIFIED TENSOR FIELD

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A generalized tensor method is developed here dealing with vectors and tensors of two types in standard forms involving hooked and unhooked indices obeying the usual transformation laws.

In a previous paper (Ghosh 1973) the geometry of a pair of four-dimensional varieties V_1 and V_2 embedded in a Minkowskian n-space M_n was studied by tensor methods. A non-symmetric tensor $g_{\mu\nu}$ was defined by the 16 scalar products $\left\{ \begin{matrix} x_\mu \\ y_\nu \end{matrix} \right\}$ of two sets of tangent vectors $x_\mu\, y_\nu$ at corresponding points P, Q of the varieties. It was shown that the symmetric part of $g_{\mu\nu}$ represents the gravitational tensor $S_{\mu\nu}$ and the anti-symmetric part the electromagnetic tensor $F_{\mu\nu}$. But, the connection coefficients $\left\{ \begin{matrix} x_\lambda \\ y_{\mu\nu} \end{matrix} \right\}$, $\left\{ \begin{matrix} y_\lambda \\ x_{\mu\nu} \end{matrix} \right\}$ are here symmetric in μ, ν.

The object of the present paper is to extend the tensor-field by introducing vectors and tensors of different types involving hooked and unhooked indices obeying the usual transformation laws. This generalization serves to bring in non-symmetric connection coefficients in the present analysis.

§1. Referring to the previous paper (Ghosh 1973) we first note the following changes in the notation.

$$\begin{array}{lll} C_{,\lambda} & \text{is changed to} & x_\lambda \\ D_{,\lambda} & \quad ,, & y_\lambda \\ C^{,\mu} & \quad ,, & x^\mu \\ D^{,\mu} & \quad ,, & y^\mu \end{array} \qquad \ldots(1.1)$$

Let us write down the relations existing among these M-vectors.

$$x^\mu = x_\lambda \left\{ \begin{matrix} y^\lambda \\ x^\mu \end{matrix} \right\},\quad y^\mu = y_\lambda \left\{ \begin{matrix} x^\lambda \\ y^\mu \end{matrix} \right\},\quad x_\lambda = x^\mu \left\{ \begin{matrix} y_\mu \\ x_\lambda \end{matrix} \right\},\quad y_\lambda = y^\mu \left\{ \begin{matrix} x_\mu \\ y_\lambda \end{matrix} \right\} \qquad \ldots(1.2)$$

where the symbol $\left\{ \quad \right\}$ stands for the scalar product of the vectors enclosed.

We now introduce new vectors with hooked indices

$$x^{\hat\mu} = x_\lambda \left\{ \begin{matrix} x^\lambda \\ y^\mu \end{matrix} \right\}, \; y^{\hat\mu} = y_\lambda \left\{ \begin{matrix} y^\lambda \\ x^\mu \end{matrix} \right\}, \; x_{\hat\lambda} = x^\mu \left\{ \begin{matrix} x_\mu \\ y_\lambda \end{matrix} \right\}, \; y_{\hat\lambda} = y^\mu \left\{ \begin{matrix} y_\mu \\ x_\lambda \end{matrix} \right\} \quad \text{...(1.3)}$$

It follows then that

$$\left\{ \begin{matrix} x_{\hat\mu} \\ y^{\hat\lambda} \end{matrix} \right\} = \left\{ \begin{matrix} x^\lambda \\ y^\mu \end{matrix} \right\}, \left\{ \begin{matrix} y_{\hat\mu} \\ x^{\hat\lambda} \end{matrix} \right\} = \left\{ \begin{matrix} y^\lambda \\ x^\mu \end{matrix} \right\}, \left\{ \begin{matrix} x^{\hat\lambda} \\ y_{\hat\mu} \end{matrix} \right\} = \left\{ \begin{matrix} x_\mu \\ y_\lambda \end{matrix} \right\}, \left\{ \begin{matrix} y^{\hat\lambda} \\ x_{\hat\mu} \end{matrix} \right\} = \left\{ \begin{matrix} y_\mu \\ x_\lambda \end{matrix} \right\} \text{...(1.4)}$$

Further one can show that

$$\left\{ \begin{matrix} x^{\hat\mu} \\ y_{\hat\lambda} \end{matrix} \right\} = \delta^\mu_\lambda = \left\{ \begin{matrix} y^{\hat\mu} \\ x_{\hat\lambda} \end{matrix} \right\} \quad \text{...(1.5)}$$

whence

$$x_{\hat\lambda} \left\{ \begin{matrix} y_{\hat\lambda} \\ x_{\hat\mu} \end{matrix} \right\} = x^{\hat\mu}, \; x^{\hat\lambda} \left\{ \begin{matrix} y^{\hat\lambda} \\ x_{\hat\mu} \end{matrix} \right\} = x_{\hat\mu} \quad \text{...(1.6)}$$

Raising the index of an arbitrary vectory of x-type is defined by the equations

$$\begin{aligned} A^{\mu} &= A_\lambda \left\{ \begin{matrix} y^\lambda \\ x^\mu \end{matrix} \right\} = A_{\hat\lambda} \left\{ \begin{matrix} x^\lambda \\ y^\mu \end{matrix} \right\} \\ A^{\hat\mu} &= A_\lambda \left\{ \begin{matrix} x^\lambda \\ y^\mu \end{matrix} \right\} = A_{\hat\lambda} \left\{ \begin{matrix} y^{\hat\lambda} \\ x^{\hat\mu} \end{matrix} \right\} \end{aligned} \quad \text{...(1.7)}$$

Lowering of an index is performed by the reverse procedure

$$\begin{aligned} A_\lambda &= A^\mu \left\{ \begin{matrix} y_\mu \\ x_\mu \end{matrix} \right\} = A^{\hat\mu} \left\{ \begin{matrix} x_\mu \\ y_\lambda \end{matrix} \right\} \\ A_{\hat\lambda} &= A^\mu \left\{ \begin{matrix} x_\mu \\ y_\lambda \end{matrix} \right\} = A^{\hat\mu} \left\{ \begin{matrix} y_{\hat\mu} \\ x_{\hat\lambda} \end{matrix} \right\} \end{aligned} \quad \text{...(1.8)}$$

Interchanging x, y in the above we obtain the formulae corresponding to an arbitrary vector of y-type. For raising and lowering of an index of a general tensor the same law prevails.

§2. Let us now consider a reversible transformation scheme from the coordinates x^λ to x'^λ given by the set of equations

$$X'^\lambda = \phi^\lambda (X^1, X^2, X^3, X^4) \quad \text{...(2.1)}$$

For any covariant vector hooked or unhooked the law of transformation will be given by

$$A'_\mu = A_\lambda \frac{\partial X^\lambda}{\partial X'^\mu} \qquad ...(2.2)$$

For any contravariant vector the transformation formula is given by

$$A'^\mu = A^\alpha \frac{\partial X'^\mu}{\partial X^\alpha} \qquad ...(2.3)$$

To find the covariant derivative of A_α we differentiate (2.2)

$$\partial'_\sigma A'_\mu = \partial_\beta A_\alpha \frac{\partial X^\alpha}{\partial X'^\mu} \frac{\partial X^\beta}{\partial X'^\sigma} + A_\alpha \frac{\partial^2 X^\alpha}{\partial X'^\mu \partial X'^\sigma} \qquad ...(2.4)$$

Differentiating

$$x'_\mu = x_\alpha \frac{\partial X^\alpha}{\partial X'^\mu}$$

we derive the formula

$$\frac{\partial^2 X^\rho}{X\partial'^\mu \partial X'^\sigma} = \left\{ \begin{matrix} \partial'_\sigma x'_\mu \\ y^\rho \end{matrix} \right\} - \left\{ \begin{matrix} x_{\alpha\beta} \\ y^\rho \end{matrix} \right\} \frac{\partial X^\alpha}{\partial X'^\mu} \cdot \frac{\partial X^\beta}{\partial X'^\sigma} \qquad ...(2.5)$$

Using (2·5) in (2·9) and rearranging terms we get

$$\partial'_\sigma A_\mu' - A'_\rho \left\{ \begin{matrix} y'^\rho \\ x'_{\mu\sigma} \end{matrix} \right\} = \left(\partial_\beta A_\alpha - A_\rho \left\{ \begin{matrix} y^\rho \\ x_{\alpha\beta} \end{matrix} \right\} \right) \frac{\partial X^\alpha}{\partial X'^\mu} \cdot \frac{\partial X^\beta}{\partial X'^\nu} \qquad ...(2.6)$$

The covariant derivatives of A_α, $A_{\hat{\alpha}}$, A^α, $A^{\hat{\alpha}}$, are given below:

$$A_{\alpha|\beta} = \partial_\beta A_\alpha = A_\rho \left\{ \begin{matrix} y^\rho \\ x_{\alpha\beta} \end{matrix} \right\}$$

$$A_{\hat{\alpha}|\beta} = \partial_\beta A_{\hat{\alpha}} - A_{\hat{\rho}} \left\{ \begin{matrix} y^{\hat{\rho}} \\ x_{\hat{\alpha}\beta} \end{matrix} \right\} \quad x_\alpha{}^\beta = x_{\hat{\beta}\alpha}$$

$$A^\alpha|_\sigma = \partial_\sigma A^\alpha + A^\rho \left\{ \begin{matrix} x^\alpha \\ y_{\rho\sigma} \end{matrix} \right\} \qquad ...(2.7)$$

$$A^{\hat{\alpha}}|_\sigma = \partial_\sigma A^{\hat{\alpha}} + A^{\hat{\rho}} \left\{ \begin{matrix} x^{\hat{\alpha}} \\ y_{\hat{\rho}\sigma} \end{matrix} \right\} \quad y^{\hat{\rho}\sigma} \neq y_{\sigma\hat{\rho}}$$

To obtain the covariant derivative of a mixed tensor of x–type $A^{\cdots\cdots\cdots}_{\cdots\cdots\cdots}$ with hooked and unhooked indices we take first the ordinary derivative

$$\partial_\sigma A^{\cdots\cdots\cdots}_{\cdots\cdots\cdots}$$

and for each unhooked lower index $A^{\cdots\cdots\cdots}_{\cdots\mu\cdots}$ we add a term

$$- \left\{ \begin{matrix} y_{\alpha} \\ x_{\mu\sigma} \end{matrix} \right\} A^{\cdots\cdots\cdots\cdots\cdots}_{\cdots\cdots\alpha\cdots\cdots}$$

and for each hooked lower index $A^{\cdots\cdots\cdots}_{\cdots\hat{\mu}\cdots}$ we add a term

$$- \left\{ \begin{matrix} y^{\hat{\alpha}} \\ x_{\hat{\mu}\sigma} \end{matrix} \right\} A^{\cdots\cdots\cdots\cdots\cdots}_{\cdots\cdots\hat{\alpha}\cdots\cdots}$$

also for each unhooked upper index $A^{\cdots\mu\cdots}_{\cdots\cdots\cdots}$ we add a term

$$+ \left\{ \begin{matrix} x^{\mu} \\ y_{\alpha\sigma} \end{matrix} \right\} A^{\cdots\cdots\alpha\cdots\cdots}_{\cdots\cdots\cdots\cdots\cdots}$$

and for each hooked upper index $A^{\cdots\hat{\mu}\cdots}_{\cdots\cdots\cdots}$ we add a term

$$+ \left\{ \begin{matrix} x^{\hat{\mu}} \\ y_{\hat{\alpha}\sigma} \end{matrix} \right\} A^{\cdots\cdots\hat{\alpha}\cdots\cdots}_{\cdots\cdots\cdots\cdots\cdots}$$

Consider now the identical relation

$$\mathfrak{D}_{\sigma} \left\{ \begin{matrix} x_{\mu} \\ y_{\nu} \end{matrix} \right\} = \left\{ \begin{matrix} x_{\mu\sigma} \\ y_{\nu} \end{matrix} \right\} + \left\{ \begin{matrix} x_{\mu} \\ y_{\nu\sigma} \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} x_{\alpha} \\ y_{\nu} \end{matrix} \right\} \left\{ \begin{matrix} y^{\alpha} \\ x_{\mu\sigma} \end{matrix} \right\} + \left\{ \begin{matrix} y_{\alpha} \\ x_{\mu} \end{matrix} \right\} \left\{ \begin{matrix} x^{\alpha} \\ y_{\nu\sigma} \end{matrix} \right\} \qquad \text{...(2.8)}$$

This shows that the covariant derivative

$$\left\{ \begin{matrix} x_{\mu} \\ y_{\nu} \end{matrix} \right\}_{|\sigma} = \mathfrak{D}_{\sigma} \left\{ \begin{matrix} x_{\mu} \\ y_{\nu} \end{matrix} \right\} - \left\{ \begin{matrix} x_{\alpha} \\ y_{\nu} \end{matrix} \right\} \left\{ \begin{matrix} y^{\alpha} \\ x_{\mu\sigma} \end{matrix} \right\} - \left\{ \begin{matrix} y_{\alpha} \\ x_{\mu} \end{matrix} \right\} \left\{ \begin{matrix} x^{\alpha} \\ y_{\nu\sigma} \end{matrix} \right\} = 0 \qquad \text{... (2.9)}$$

Changing over to hooked indices we get

$$\left\{ \begin{matrix} x_{\hat{\mu}} \\ y_{\hat{\nu}} \end{matrix} \right\}_{|\sigma} = \mathfrak{D}_{\sigma} \left\{ \begin{matrix} x_{\hat{\mu}} \\ y_{\hat{\nu}} \end{matrix} \right\} - \left\{ \begin{matrix} x_{\hat{\alpha}} \\ y_{\hat{\nu}} \end{matrix} \right\} \left\{ \begin{matrix} y^{\hat{\alpha}} \\ x_{\hat{\mu}\sigma} \end{matrix} \right\} - \left\{ \begin{matrix} y^{\hat{\alpha}} \\ x_{\hat{\mu}} \end{matrix} \right\} \left\{ \begin{matrix} x_{\hat{\alpha}} \\ y_{\hat{\nu}\sigma} \end{matrix} \right\} = 0 \qquad \text{...(2.10)}$$

Passing on to contravariant vectors we can show that

$$\left\{ \begin{matrix} x^{\mu} \\ y^{\nu} \end{matrix} \right\}_{|\sigma} = \mathfrak{D}_{\sigma} \left\{ \begin{matrix} x^{\mu} \\ y^{\nu} \end{matrix} \right\} + \left\{ \begin{matrix} x^{\alpha} \\ y^{\nu} \end{matrix} \right\} \left\{ \begin{matrix} x^{\mu} \\ y_{\alpha\sigma} \end{matrix} \right\}$$

$$+ \left\{ \begin{matrix} y^{\alpha} \\ x^{\mu} \end{matrix} \right\} \left\{ \begin{matrix} y^{\nu} \\ x_{\alpha\sigma} \end{matrix} \right\} = 0 \qquad \text{...(2.11)}$$

Also

$$\left\{ \begin{matrix} x^{\hat{\mu}} \\ y^{\hat{\nu}} \end{matrix} \right\}_{|\sigma} = \mathfrak{D}_\sigma \left\{ \begin{matrix} x^{\hat{\mu}} \\ y^{\hat{\nu}} \end{matrix} \right\} + \left\{ \begin{matrix} x^{\hat{\alpha}} \\ y^{\hat{\nu}} \end{matrix} \right\} \left\{ \begin{matrix} x^{\hat{\mu}} \\ y_{\hat{\alpha}\sigma} \end{matrix} \right\} + \left\{ \begin{matrix} y^{\hat{\alpha}} \\ x^{\hat{\mu}} \end{matrix} \right\} \left\{ \begin{matrix} y^{\hat{\nu}} \\ x_{\hat{\alpha}\sigma} \end{matrix} \right\} = 0 \qquad \ldots(2.12)$$

§3. The following theorem has been reproduced from our previous paper using the present notation.

Combining (2·8) with its conjugate (x, y interchanged) we obtain

$$\mathfrak{D}_\sigma S_{\mu\nu} = S_{\lambda\nu} P^\lambda{}_{\mu\sigma} - F_{\lambda\nu} V^\lambda{}_{\mu\sigma} + S_{\lambda\mu} P^\lambda{}_{\nu\sigma} - F_{\lambda\mu} V^\lambda{}_{\nu\sigma} \qquad \ldots(3.1)$$

$$\mathfrak{D}_\sigma F_{\mu\nu} = - S_{\lambda\nu} V^\lambda{}_{\mu\sigma} + F_{\lambda\nu} P^\lambda{}_{\mu\sigma} + S_{\lambda\mu} V^\lambda{}_{\nu\sigma} - F_{\lambda\mu} P^\lambda{}_{\nu\sigma} \qquad \ldots(3.2)$$

where

$$S_{\mu\nu} = \frac{1}{2}\left(\left\{ \begin{matrix} x_\mu \\ y_\nu \end{matrix} \right\} + \left\{ \begin{matrix} y_\mu \\ x_\nu \end{matrix} \right\} \right)$$

$$F_{\mu\nu} = \frac{1}{2}\left(\left\{ \begin{matrix} x \\ y_\nu \end{matrix} \right\} - \left\{ \begin{matrix} y_\mu \\ x_\nu \end{matrix} \right\} \right)$$

$$P^\lambda{}_{\mu\nu} = \frac{1}{2}\left(\left\{ \begin{matrix} x^\lambda \\ y_{\mu\nu} \end{matrix} \right\} + \left\{ \begin{matrix} y^\lambda \\ x_{\mu\nu} \end{matrix} \right\} \right)$$

$$V^\lambda{}_{\mu\nu} = \frac{1}{2}\left(\left\{ \begin{matrix} x^\lambda \\ y_{\mu\nu} \end{matrix} \right\} - \left\{ \begin{matrix} y^\lambda \\ x_{\mu\nu} \end{matrix} \right\} \right)$$

From eqn. (3.1) we get

$$\overset{\rho}{S}_{\mu\nu} = \overset{\rho}{P}_{\mu\nu} - S^{\rho\sigma} F_{\lambda\sigma} V_{\mu\nu}{}^{\lambda} \qquad \ldots(3.3)$$

where

$$\overset{\rho}{S}_{\mu\nu} = \frac{1}{2} S^{\rho\sigma} \left(\mathfrak{D}_\mu S_{\nu\sigma} + \mathfrak{D}_\nu S_{\mu\sigma} - \mathfrak{D}_\sigma S_{\mu\nu} \right)$$

Further, we obtain

$$\mathfrak{D}_\sigma S_{\mu\nu} = S_{\lambda\nu} S^\lambda{}_{\mu\sigma} + S_{\mu\lambda} S^\lambda{}_{\nu\sigma} \qquad \ldots(3.4)$$

$$\mathfrak{D}_\sigma F_{\mu\nu} + \mathfrak{D}_\mu S_{\nu\sigma} + \mathfrak{D}_\nu S_{\sigma\mu} = 0 \qquad \ldots(3.5)$$

$$\mathfrak{D}_\sigma F_{\mu\nu} - F_{\rho\nu} \overset{\rho}{S}_{\mu\sigma} + F_{\rho\mu} \overset{\rho}{S}_{\nu\sigma} = U_{\lambda\mu} V^\lambda{}_{\nu\sigma} - U_{\lambda\nu} V^\lambda{}_{\mu\sigma} \qquad \ldots(3.6)$$

where

$$U_{\lambda\mu} = U_{\mu\lambda} = S_{\lambda\mu} - F_{\rho\mu}\, S^{\rho\sigma}\, F_{\lambda\sigma}$$

We identify $S_{\mu\nu}$ with the gravitational tensor and $F_{\mu\nu}$ with the electromagnetic tensor.

Consider next the identical relation

$$\mathfrak{D}_\sigma \left\{ \begin{matrix} x^{\hat{\Lambda}}_{\mu} \\ y^{\hat{\Lambda}}_{\nu} \end{matrix} \right\} = \left\{ \begin{matrix} x^{\hat{\Lambda}}_{\mu\sigma} \\ y^{\hat{\Lambda}}_{\nu} \end{matrix} \right\} + \left\{ \begin{matrix} x^{\hat{\Lambda}}_{\mu} \\ y^{\hat{\Lambda}}_{\nu\sigma} \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} y^{\hat{\rho}} \\ x^{\hat{\Lambda}}_{\mu\sigma} \end{matrix} \right\} \left\{ \begin{matrix} x^{\hat{\Lambda}}_{\rho} \\ y^{\hat{\Lambda}}_{\nu} \end{matrix} \right\} + \left\{ \begin{matrix} x^{\hat{\rho}} \\ y^{\hat{\Lambda}}_{\nu\sigma} \end{matrix} \right\} \left\{ \begin{matrix} x^{\hat{\Lambda}}_{\mu} \\ y^{\hat{\Lambda}}_{\rho} \end{matrix} \right\} \qquad \ldots(3.7)$$

If we postulate

$$\left\{ \begin{matrix} x^{\hat{\rho}} \\ y^{\hat{\Lambda}}_{\nu} \end{matrix} \right\} = \left\{ \begin{matrix} y^{\hat{\rho}}_{\Lambda} \\ x_{\sigma\nu} \end{matrix} \right\} \qquad \ldots(3.8)$$

the above yields the non-symmetric equations of connection in Einstein's unified field theory.

Reference

Ghosh N. N. (1973). A new geometrical approach to non-symmetric unified field theory. *Indian J. pure appl. Math.*, **4**, 856.