ON SOME RESULTS ON FIXED POINTS IN METRIC AND BANACH SPACES

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In this paper, a few theorems on fixed points have been presented, which give a few known results as corollaries. By taking a continuous function F in place of a metric d, more general results on fixed points have been obtained.

§ 1. Let (x, d) denote a metric space. A mapping $T : X \to X$ is said to be a contraction, if $d\,(Tx, Ty) \leqslant kd(x, y)$ for all $x, y \in X$, where k is a real number, such that $0 \leqslant k < 1$. The well-known Banach contraction principle states that a contraction mapping of a complete metric space has a unique fixed point.

A mapping $T : X \to X$ is said to be contractive, if $d\,(Tx, Ty) < d(x, y)$ for all $x, y\; (x \neq y)$ in X.

A contractive mapping on a complete metric space need not have a fixed point. However, if X is compact and T is contractive, then T has a unique fixed point.

The aim of this paper is to prove a few theorems on fixed points. A few known results can be obtained as particular cases of our theorems.

Theorem 1—Let X be a Hausdorff space and let T_1 and T_2 be two continuous mappings of X into itself. Let F be a continuous symmetric mapping of $X \times X$ into the set of non-negative reals, such that $F(x, y) = 0$ for $x = y$ and

$$F(T_1^p\, x, T_2^q\, y) < a_1 F(x, y) + a_2 F(x, T_1^p\, x) + a_3 F(y, T_2^q\, y) \qquad \text{...(1.1.1)}$$

for every two distinct $x, y \in X$, where $p > 0, q > 0$ are integers and a_1, a_2, a_3 are non-negative real numbers, such that $a_1 + a_2 + a_3 < 1$.

If for some $x_0 \in X$, the sequence $\{x_n\}$ consisting of points

$$x_{2n+1} = T_1^p\, x_{2n}, \quad x_{2n+2} = T_2^q\, x_{2n+1}, \quad n = 0, 1, 2, \ldots .$$

has a convergent subsequence, then T_1 and T_2 have a unique common fixed point.

PROOF: We have a monotone sequence of positive real numbers

$$F(x_0, x_1) > F(x_1, x_2) > \ldots > F(x_n, x_{n+1}) > \ldots$$

which must converge to some real number λ. Since $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in X, which converges to some $\xi \in X$, we may put $\xi = \lim x_{n_k}$.

Now,

$$
\begin{aligned}
&F(\xi, T_1^p \xi) \\
&\quad = F(\lim x_{2n_k}, T_1^p \lim x_{2n_k}) \\
&\quad = F(\lim x_{2n_k}, \lim\ T_1^p x_{2n_k}) \\
&\quad = F(\lim x_{2n_k}, \lim x_{2n_{k+1}}) \\
&\quad = \lim F(x_{2n_k}, x_{2n_{k+1}}) \\
&\quad = \lim\ F(x_{2n_{k+1}}, x_{2(n_{k+1})}) \\
&\quad = F(T_1^p \xi, T_2^q T_1^p \xi).
\end{aligned}
$$

But if $\xi \neq T_1^p \xi$, then from (1.1.1), it follows

$$F(\xi, T_1^p \xi) > F(T_1^q \xi, T_2^q T_1^p \xi).$$

Thus, if $\xi = T_1^p \xi$, then $F(\xi, T_1^p \xi) = F(T_1^p \xi, T_2^q T_1^p \xi) < F(\xi, T_1^p \xi)$, which gives a contradiction. Hence, $\xi = T_1^p \xi$.

Similarly, $\xi = T_2^q \xi$.

If η $(\eta \neq \xi)$ were another fixed point of T_1^p and T_2^q, then

$$
\begin{aligned}
F(\xi, \eta) &= F(T_1^p \xi, T_2^q \eta) \\
&< a_1 F(\xi, \eta) + a_2 F(\xi, T_1^p \xi) + a_3 F(\eta, T_2^q \eta) \\
&= a_1 F(\xi, \eta),
\end{aligned}
$$

a contradiction.

Hence, ξ is the unique common fixed point of T_1^p and T_2^q. Now, we show that ξ is a fixed point of T_1 and T_2.

We have $T_1^p T_1\ \xi = T_1^{p+1}\ \xi = T_1 T_1^p \xi = T_1 \xi$ and by the uniqueness of ξ, it follows that $T_1 \xi = \xi$. Similarly, $T_2 \xi = \xi$.

Moreover, ξ is the only fixed point of T_1 and T_2. Suppose, ζ is another common fixed point of T_1 and T_2. Then,

$$
\begin{aligned}
F(\xi, \zeta) &= F(T_1 \xi, T_2 \zeta) \\
&= F(T_1^p \xi, T_2^q \zeta) \\
&< a_1 F(\xi, \zeta) + a_2 F(\xi, T_1^p \xi) + a_3 F(\zeta, T_2^q \zeta),
\end{aligned}
$$

which implies $\xi = \zeta$.

This completes the proof of the theorem.

Corollary 1.—Let X a metric space with metric d and $T_1 : X \to X, T_2 : X \to X$ be such that for each $x, y \in X$ with $x \neq y$, $d(T_1^p x, T_2^q y) < a_1 d(x, y) + a_2^d (x, T_1^p x) + a_3 d(y, T_2^q y)$ where p, q, a_1, a_2, a_3 are the same as in Theorem 1. If for some

$x_0 \in X$, the sequence $\{x_n\}$ consisting of points $x_{2n+1} = T_1^p x_{2n}$, $x_{2(n+1)} = T_2^q x_{2n+1}$, $n = 0, 1, 2, \ldots$, has convergent subsequence, then T_1 and T_2 have a unique common fixed point. (T_1, T_2 are assumed to be continuous).

PROOF: All assumptions of Theorem 1 are satisfied with d playing the role of F.

Taking $p = q = 1$, $T_1 = T_2 = T$, $d = F$, $a_2 = a_3 = 0$, we get the following result due to Edelstein (1962) as a corollary to our Theorem 1.

Corollary 2—If X is a metric space, such that

$$d(Tx, Ty) < d(x, y) \text{ for } x \neq y \in X$$

and if for some $x_0 \in X$, the sequence $\{x_n\}$ where $x_n = T^n x_0$, has a subsequence $\{x_{nk}\}$ converging to a point $\xi \in X$, then ξ is the unique fixed point of T.

§2. Let X denote a metric space and A, a bounded subset of X. Following Kuratowski (1952), we denote by $\alpha(A)$ the infimum of all $r > 0$, such that A admits a finite converging with subsets of diameter less than r. We shall use the following properties of the number α :

1. $\alpha(A) = 0$, if only if A is precompact. For this reason, $\alpha(A)$ is called the measure of noncompactness of A.
2. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$, $A \subset X, B \subset X$,
3. $\alpha(A) = 0$ and X complete imply A is compact.
4. If A is compact, then $\alpha(A) = 0$.
5. If A is the closure of A, then

$$\alpha(\bar{A}) = 0 \Leftrightarrow \alpha(A) = 0 \text{ (Szufla 1968).}$$

Let $T : X \rightarrow X$ be continuous.

If $\alpha(TA) < \alpha(A)$ for any bounded and non-precompact subset $A \subset X$, then T is called condensing (Sadovaski 1972).

Theorem 2.1—Let (X, d) be a complete metric space and let $F : X \times X \rightarrow [0, \infty)$ be continuous. Let $T : X \rightarrow X$ be a condensing mapping of X into itself, such that

$$F(Tx, Ty) < a_1F(x, y) + a_2F(x, Tx) + a_3F(y, Ty)$$

for each pair of distinct points $x, y \in X$ and for non-negative real numbers a_1, a_2, a_3 with $a_1 + a_2 + a_3 < 1$.

If for some $x_0 \in X$, the sequence $\{T^n x_0\}$ is bounded, then T has a unique fixed point.

PROOF: Let $A = \bigcup_{n=0}^{\infty} T^n x_0$ and let $\bar{A}$ be the closure of A. We shall show that $\bar{A}$ is compact, which, by the completeness of X, will be true if $\alpha(\bar{A}) = 0$.

We suppose $\alpha(\bar{A}) > 0$ or equivalently $\alpha(A) > 0$.

Then, $\alpha(TA) < \alpha(A)$, since T is condensing.

Now, $TA = \bigcup_{n=1}^{\infty} x_n$. Hence, $A = \{x_0\} \cup TA$ and $TA \subset A$. So,

$$\alpha(A) = \max\{\alpha\{x_0\}, \alpha(TA)\}$$
$$= \alpha(TA).$$

This contradiction gives $\alpha(\bar{A}) = 0$ and hence $\bar{A}$ is compact.

Now, by the continuity of T, we get $T(\bar{A}) \subset \overline{T(A)} \subset \bar{A}$. So, the space $\bar{A}$ with $T: \bar{A} \to \bar{A}$ now satisfies all the axioms of Theorem 1 (with $T_1 = T_2 = T$ and $p = q = 1$), and therefore, there is a fixed point $\xi \in \bar{A}$ and from (1.1.1) [with $T_1 = T_2 = T, p = q = 1$] it follows that ξ is unique.

§ 3. Let X denote a Banach space. A mapping $T: X \to X$ is said to be asymptotically regular at $x_0 \in X$ if $\| T^n x_0 - T^{n+1} x_0 \| \to 0$ as $n \to \infty$ whenever $T^n x_0$ is defined for all n. Recently, Petryshyn (1973) proved the following theorem.

Theorem A—Let X be a uniformly convex Banach space; K, a subset of X; T, a non-expansive mapping (i.e., $\| Tx - Ty \| \leqslant \| x - y \| \ \forall\ x, y \in K$) of K into itself, such that the set of fixed points of T, viz., $F(T)$ is not empty. If there exists an x_0 in K and a λ in $(0, 1)$, such that $T^n{}_\lambda x_0$ is defined and lies in K for each $n \geqslant 1$ where $T_\lambda = \lambda I + (1 - \lambda) T$, then T_λ is asymptotically regular at x_0.

In this section, we extend the above theorems

Theorem 3.1—Let X be a uniformly convex Banach space; K, a subset of X, T, a mapping of K into itself, such that

$$\| Tx - Ty \| \leqslant a_1 \| x - y \| + a_2 [\| x - Tx \| + \| y - Ty \|]$$
$$+ a_3 [\| x - Ty \| + \| y - Tx \|] \qquad ..(3.1.1)$$

for all $x, y \in K$ and for non-negative real numbers a_1, a_2, a_3 with $a_1 + 2a_2 + 2a_3 \leqslant 1$ and such that $F(T) \neq \phi$. If there exists an x_0 in K and constants $\alpha_0, \alpha_1, \ldots, \alpha_n$, $\alpha_i \geqslant 0$, $\alpha_1 > 0$ and $\sum_{i=0}^{n} \alpha_i = 1$, such that $S^n(x_0)$ is defined and lies in K for each $n \geqslant 1$, where

$$S = \sum_{i=0}^{n} \alpha_i T^i, I = T^\circ, \text{ then } \| S^n x_0 - S^{n+1} x_0 \| \to 0, \text{ as } n \to \infty,$$

i.e., S is asymptotically regular at x_0.

Proof: Let $p \in F(T)$ and let x_0 be an element in K and let α_i $(i = 0, 1, \ldots, n)$, $\alpha_i \geqslant 0, \alpha_1 > 0$, $\sum_{i=0}^{n} \alpha_i = 1$ be such that $x_n = S^n x_0 \in K$ for each $n \geqslant 1$. It is obvious that $F(T) = F(S)$. So, $F(S) \neq F(T) \neq \phi$ and for all $x \in K$.

$$x_{n+1} - p = S(x_n) - p$$
$$= \sum_{i=0}^{n} \alpha_i T^i(x_n) - p, T^\circ = I$$
$$= \alpha_0 (x_n - p) + (1 - \alpha_0) z_n,$$

where

$$z_n = \frac{1}{1 - \alpha_0} \sum_{i=1}^{n} \alpha_i \left(T^i(x_n) - p\right).$$

Since $\| T^i(x_n) - p \| = \| T^i(x_n) - T^i p \|$, it follows from (3.1.1) that

$$\| T^i(x_n) - p \| \leqslant \| x_n - p \|,$$

and since

$$\sum_{i=0}^{n} \alpha_i = 1,$$

it follows that

$$\| x_{n+1} - p \| = \| S(x_n) - p \| \leqslant \| x_n - p \|$$

and, therefore, $\| x_n - p \| \to d_0$ for some $d_0 \geqslant 0$. If $d_0 = 0$, then $x_n \to p$, as $n \to \infty$ and so in this case

$$x_n - x_{n+1} = S^n x_0 - S^{n+1} x_0 \to 0, \quad \text{as} \quad n \to \infty,$$

i.e., S is asymptotically regular at x_0. Now, suppose that $d_0 > 0$. Since $\| x_n - p \| \to d_0$, $\| S(x_n) - p \| \leqslant \| x_n - p \|$ for each $n \geqslant 1$ and $\| S(x_n) - p \| = \| x_{n+1} - p \| \to d_0$, as $n \to \infty$; it follows from the uniform convexity of X that

$$\| (x_n - p) - \left(S(x_n) - p\right) \| \to 0, \quad \text{as } n \to \infty$$

i.e.,

$$\| x_n - S(x_n) \| = \| S^n x_0 - S^{n+1} x_0 \| \to 0, \quad \text{as } n \to \infty.$$

This completes the proof.

Remark—Theorem A of Petryshyn and Williamson (1973, p. 475) follows from our Theorem 3.1 by taking $a_2 = a_3 = 0$, $a_1 = 1$ and $\alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$, $\alpha_0 = \lambda$.

References

Edelstein, M. (1962). On fixed and periodic points under contractive mappings. *J. Lond. math. Soc.*, **37**, 74–79.

Furi, M., and Vignoli, A. (1969). Fixed points for densifying mappings. *Rend. Acad. Nag. Lincei*, **47** (6), 1–3.

Kirk, W. A. (1971). On successive approximations for non-expansive mappings in Banach spaces. *Glasg. math. J.*, **12**, 6–9.
Kuratowski, C. (1952). *Topology*, **1**, 318.
Petryshyn, W. V., and Williamson, T. E. (Jr.) (1973). Strong and weak convergence of the sequence of successive approximations for quasi-non-expansive mappings. *J. Math. anal. appl.*, **43** (2), 459–97.
Ray, B. (1976). Contraction mappings and fixed points. *Colloquium math.*, **35** (2) (to appear).
Sadovski, B. N. (1972). Limit compact and condensing mappings. *Usp. mat. Nauk.*, **27**, 81–145.
Szufla, A. (1968). On the existence of solutions of an ordinary differential equation in the case of Banach space. *Bull. Acad. Pol. Sci.*, **16** (14), 311–16.