UPPER AND LOWER FUNCTIONS FOR DIFFUSION PROCESSES

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The paper is concerned with the study of upper and lower functions for diffusion processes described by the non-linear time-dependent stochastic differential equation.

1. Introduction

Let \( X(t) \) be the solution of the non-linear time dependents stochastic differential equation

\[
dX(t) = a(X(t), t) \, dt + dW(t) \tag{1.1}
\]

with the initial condition

\[ X(0) = X_0 \tag{1.2} \]

where \( W(t) \) is a standard Wiener process and \( X_0 \) is independent of \( F\{W(t), t \geq 0\} \) with \( E(X_0^2) < \infty \).

Let \( A > 0 \). Let \( H_A \) be the class of non-negative, non-decreasing functions defined on \( [A, \infty) \) which increase to \( \infty \) with \( t \).

For \( h(t) \in H_A \), we say that \( h(t) \) belongs to the upper class or lower class according as

\[
P\{X(t) > h(t) \text{ i.o. as } t \to \infty\} = 0 \text{ or 1.}
\]

In this paper our aim is to develop the integral test criterion for the solution process of (1.1) to decide whether \( h(t) \) belongs to the upper class or lower class.

We give preliminary lemmas in section 2. In section 3, Theorem 3.1 gives a result analogous to Strassen's invariance principle. The integral test criterion for diffusion processes described by equation (1.1) have been developed in Theorem 3.2.

Problems of the above type have been considered by many authors. Let \( \{X_n\} \) be a sequence of independent random variables. Feller\(^2\) and Chung\(^1\) have studied the asymptotic growth rates of \( S_n = \sum_{i=1}^{n} X_i \) and \( M_n = \max_{1 \leq i \leq n} |S_i| \) respectively, which are
considered to be fundamental papers in this area. In case of Brownian motion \( W(t), t \geq 0 \), Kolmogorov has developed an integral test for non-decreasing function \( h(t) \) so that \( h(t) \) belongs to upper class or lower class according as the integral converges or diverges. The same problem has been considered by Strassen\(^8\) for a martingale difference sequence \( \{Y_n\} \) and by Jain et al.\(^6\) for the partial sum \( S_n = \sum_{i=1}^{n} Y_i \). Jain and Taylor\(^7\) have studied the asymptotic growth rate of \( M'(t) = \max_{0 \leq u \leq t} |W(u)| \). For the first time Mishra and Acharya\(^8\) have developed the integral test criterion to decide whether \( h(t) \) belongs to the upper class or lower class for diffusion processes described by the homogeneous stochastic differential equations of the Itô type.

2. Notations and Preliminaries

This section is devoted to the background materials which have been used in this paper. Let us consider the stochastic differential equation (1.1) where \( W(t) \) is a standard Wiener process. We assume that \( a(x, t) \) is real-valued, well defined, measurable for \( x \in (-\infty, \infty) \) and satisfy the following conditions,

\((A_1)\) for some constant \( K \) and \( 0 < \beta < 1 \),

\[ |a(x, t)| \leq K(1 + |x|)^{1+\beta} \]

\((A_2)\) for \( c > 0 \) and \( x, y \) in \((-\infty, \infty)\), there exists a constant \( L_c \) such that

\[ |a(x, t) - a(y, t)| \leq L_c |x - y| \]

where \( |x| \leq c \) and \( |y| \leq c \).

Under conditions \((A_1)\) and \((A_2)\) and the initial condition \( X(0) = X_0 \), Gikhman and Skorohod\(^4\) have shown that, there exists a unique solution \( X(t) \) of (1.1) in an arbitrary time interval \([0, T]\) and

\[ X(t) = X_0 + \int_0^t a(X(s), s) \, ds + W(t). \] \hspace{1cm} ...(2.1)

Moreover \( X(t) \) is a Markov process whose transition probability is given by

\[ P(X_0, t, A) = P_{X_0}(X(t) \in A). \] \hspace{1cm} ...(2.2)

To prove the main theorem we need the following lemmas.

Lemma 1 — Under condition \((A_1)\),

\[ E \left| \int_0^t a(X(s), s) \, ds \right|^2 = O(t^{1-\epsilon}). \]

For the proof of this Lemma refer to Friedman\(^8\) (p. 184).
Lemma 2—Let \( \phi \) (\( t \)) increase monotonically to infinity with \( t \) and \( \{ W(t), \ t \geq 0 \} \) be a Brownian motion process. Then
\[
P \{ W(t) > t^{1/2} \phi (t) \text{ i. o. as } t \to \infty \} = 0 \text{ or } 1.
\]
according as
\[
I(\phi) = \int_1^\infty \frac{\phi(t)}{t} e^{-t^2/2} \ dt
\]
is convergent or divergent.

The above result is due to Kolmogorov [see Ito and McKean, p. 165].

Lemma 3—Let \( g \) be an eventually non-increasing function from \([0, \infty)\) to \([0, \infty)\) and \( \psi \) be a measurable function from \([A, \infty)\) to \([0, \infty)\), for some fixed \( A > 0 \). For \( h \in H_A \), define
\[
F(h) = \int_A^\infty g(h(t)) \psi(t) \ dt
\]
which may be either finite or infinite. Assume that

\( a_1 \) for every \( h \in H_A \) and for every \( B \) such that \( B > A > 0 \),
\[
\int_A^B g(h(t)) \psi(t) \ dt < \infty.
\]

\( a_2 \) There exists \( h_1, h_2 \), two members of \( H_A \), such that \( h_1 \leq h_2 \), \( F(h_2) < \infty \), while \( F(h_1) = \infty \) and
\[
\lim_{B \to \infty} \int_A^B g(h_1(B)) \psi(t) \ dt = \infty.
\]

Define
\[
\hat{h} = \min [\max (h, h_1), h_2].
\]
Then for \( h \in H_A \),

\( b_1 \) \( F(h) < \infty \) implies \( \hat{h} \leq h \) near \( \infty \) and \( F(\hat{h}) < \infty \)

\( b_2 \) \( F(h) = \infty \) implies that \( F(\hat{h}) = \infty \).

We omit the proof as it is obvious analogous of the proof of Lemma 2.14 of Jain and Taylor.

In this paper we shall denote various positive constants by the same symbol \( C \).
3. Main Results

Theorem 3.1—Let

(i) \((A_1)\) and \((A_2)\) hold, (ii) \(X(t)\) be a solution of (1.1) with \(EX_0^2 < \infty\), (iii) \(a(x,t) \geq 0\) for all \(t\) and \(x\).

Then

\[ |X(t) - W(t)| = o(t^{1/2} (\log \log t)^{-1/2}) \]

almost surely as \(t \to \infty\).

Proof: We have

\[
\frac{X(t) - W(t)}{t^{1/2} (\log \log t)^{-1/2}} = \frac{X_0}{t^{1/2} (\log \log t)^{-1/2}} + \frac{\int_0^t a(x(s), s) \, ds}{t^{1/2} (\log \log t)^{-1/2}}.
\]

So for \(t_m = m^4, \lambda = 4/\delta, m\) a positive integer,

\[
P \left\{ \sup_{t_m \leq t \leq t_{m+1}} \left| \frac{X(t) - W(t)}{t^{1/2} (\log \log t)^{-1/2}} \right| > \frac{1}{m} \right\} \leq P \left\{ \sup_{t_m \leq t \leq t_{m+1}} \left| \frac{X_0}{t^{1/2} (\log \log t)^{-1/2}} \right| > \frac{1}{2m} \right\} + P \left\{ \sup_{t_m \leq t \leq t_{m+1}} \left| \frac{\int_0^t a(X(s), s) \, ds}{t^{1/2} (\log \log t)^{-1/2}} \right| > \frac{1}{2m} \right\}
\]

\[
\leq P \left\{ \frac{1}{t_m^{1/2} (\log \log t_m)^{-1/2}} > \frac{1}{2m} \right\} + P \left\{ \frac{\int_0^{t_{m+1}} a(X(s), s) ds}{t_m^{1/2} (\log \log t_{m+1})^{-1/2}} > \frac{1}{2m} \right\}
\]

\[
\leq 4m^2 (\log \log t_{m+1}) \frac{EX_0^2}{t_m} + 4m^2 (\log \log t_{m+1}) \frac{E \int_0^{t_{m+1}} a(X(s), s) \, ds}{t_m}.
\]

(equation continued on p. 1039)
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\[
\frac{Cm^2 (\log \log (m + 1)^{1/8})}{m^{4/8}} + \frac{Cm^2 (\log \log (m + 1)^{4/8})}{m^{4/8}}
\times \left( \frac{1}{m_{m+1}} \right)
\]

(by Lemma 1)

\[
\leq \frac{C \log m}{m^2}.
\]

Now since \( \sum_{m=1}^{\infty} C (\log m)/m^2 < \infty \), we have by applying Borel-Cantelli Lemma,

\[
P \left\{ \sup_{t_m \leq t \leq t_{m+1}} \left| \frac{X(t) - W(t)}{t^{1/2} (\log \log t)^{-1/2}} \right| > \frac{1}{m} \text{ i.o.} \right\} = 0,
\]

and consequently

\[
P \left\{ \lim_{t \to \infty} \left| \frac{X(t) - W(t)}{t^{1/2} (\log \log t)^{-1/2}} \right| = 0 \right\} = 1.
\]

**Theorem 3.2**—Let

(i) \((A_1)\) and \((A_2)\) hold, (ii) \(X(t)\) be a solution of (1.1) with \(EX^2_0 < \infty\), (iii) \(a(x, t) \geq 0\) for all \(t\) and \(x\), (iv) \(h(t) > 0\) increase monotonically to infinity with \(t\).

Then

\[
P \{X(t) > t^{1/2} h(t) \text{ i.o. as } t \to \infty\} = 0 \text{ or } 1 \text{ according as}
\]

\[
l(h) = \int_1^\infty \frac{h(t)}{t} \exp \left\{ - \frac{h^2(t)}{2} \right\} dt < \infty \text{ or } \infty. \quad \ldots(3.1)
\]

**Proof**: Let us assume that

\[
h_1(t) \leq h(t) \leq h_2(t) \text{ for all } t \text{ sufficiently large} \quad \ldots(3.2)
\]

where \(h_1(t) = (\log \log t)^{1/2}\) and \(h_2(t) = 2 (\log \log t)^{1/2}\).

Let us first establish the theorem under assumption (3.2). We will then show that the theorem is true for any arbitrary increasing nonnegative function \(h(t)\).

By Theorem 3.1, for any \(\beta > 0\)

\[
|X(t) - W(t)| < \beta t^{1/2} (\log \log t)^{-1/2} \text{ a.s. as } t \to \infty.
\]

i.e.

\[
W(t) - \beta t^{1/2} (\log \log t)^{-1/2} < X(t) < W(t) + \beta t^{1/2} (\log \log t)^{-1/2}
\]

almost surely as \(t \to \infty\). \quad \ldots(3.3)
Let us first consider the case when
\[ X (t) < W(t) + \beta t^{1/2} (\log \log t)^{-1/2} < W(t) + 2 \beta t^{1/2} h^{-1} (t) \text{ (by relation (3.2)).} \]

Therefore
\[ P \{ X (t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty \} \leq P \{ W(t) > t^{1/2} (h(t) - \frac{2\beta}{h(t)}) \text{ i. o. as } t \to \infty \}, \tag{3.4} \]

Since for \( h(t) \) increasing, \( h(t) - \frac{2\beta}{h(t)} \) is also increasing,

\[ I(h) < \infty \Rightarrow I\left(h - \frac{2\beta}{h}\right) < \infty. \]

So by Kolmogorov’s test criterion for Brownian motion, if \( I(h) < \infty \), then

\[ P \{ W(t) > t^{1/2} (h(t) - \frac{2\beta}{h(t)}) \text{ i. o. as } t \to \infty \} = 0. \]

Therefore,
\[ P \{ X(t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty \} = 0. \]

Next let us consider the case when
\[ W(t) - \beta t^{1/2} (\log \log t)^{-1/2} < X(t). \]

Since \( a \geq 0 \), by Theorem 3.1 the above expression can be written as
\[ X(t) < W(t) + \beta t^{1/2} (\log \log t)^{-1/2}. \]

So when \( I(h) < \infty \), we have
\[ P \{ X(t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty \} = 0 \]
as shown above.

The fact that \( I(h) = \infty \Rightarrow \)
\[ P \{ X(t) > t^{1/2} h(t) \text{ i. o. as } t \to \infty \} = 1 \]
is trivial in view of Kolmogorov’s test criterion for Brownian motion and the assumption that \( a(x, t) \geq 0 \) for all \( t \) and \( x \).

Now let us remove the restriction (3.2) and consider \( h(t) \) an arbitrary increasing nonnegative function.

Define,
\[ \hat{h}(t) = \min \{ \max (h(t), h_1(t)), h_2(t) \}. \tag{3.5} \]
Then by Lemma 3,

\[ I (h) < \infty \Rightarrow I \hat{h} < \infty \text{ and } \hat{h} \leq h \text{ near infinity.} \]

Again we have \( h_1 (t) \leq \hat{h} (t) \leq h_2 (t) \).

Therefore,

\[ P \{ X(t) > t^{1/2} \hat{h} (t) \text{ i. o. as } t \to \infty \} = 0 \]

when \( I (h) < \infty \).

But \( \hat{h} (t) \leq h (t) \) near infinity.

Hence,

\[ P \{ X(t) > t^{1/2} h (t) \text{ i. o. as } t \to \infty \} = 0 \]

when \( I (h) < \infty \).

Next again by Lemma 3,

\[ I (h) = \infty \Rightarrow I \hat{h} = \infty. \]

So

\[ P \{ X(t) > t^{1/2} \hat{h} (t) \text{ i. o. as } t \to \infty \} = 1. \]

This implies that there exists a sequence \( \{ t_n \} \uparrow \infty \) such that

\[ X(t_n) > t_n^{1/2} \hat{h} (t_n) \text{ a. s. for every positive integer } n. \] ...

Since \( I (h_2) < \infty \), we have

\[ P \{ X(t) > t^{1/2} h_2 (t) \text{ i. o. as } t \to \infty \} = 0. \]

So for \( \{ t_n \} \uparrow \infty \),

\[ X(t_n) \leq t_n^{1/2} h_2 (t_n) \text{ a. s. for every positive integer } n. \] ...

Now from (3.7) and (3.8), we get

\[ \hat{h} (t_n) \leq h_2 (t_n) \text{ for every positive integer } n \]

and hence,

\[ \hat{h} (t_n) = \max [h (t_n), h_1 (t_n)], \text{ by (3.5)} \]

i.e. \( \hat{h} (t_n) \geq h (t_n) \) for every positive integer \( n \).
Therefore by (3.7),

\[ X(t_n) \geq t_n^{1/2} h(t_n) \text{ a. s. for every positive integer } n. \]

Hence for \( I(h) = \infty \),

\[ P \{ X(t) \geq t^{1/2} h(t) \text{ i. o. as } t \to \infty \} = 1. \] \hspace{1cm} ...(3.9)

From (3.6) and (3.9), it is evident that we have removed the restriction (3.2). Hence without any loss of generality we can assume (3.2), (for the proof of this statement we have followed the technique adopted by Jain et al.\(^a\))

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REFERENCES