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ON THE EQUICONVERGENCE OF THE EIGENFUNCTION EXPANSION ASSOCIATED WITH CERTAIN 2ND ORDER DIFFERENTIAL EQUATIONS

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We consider the differential equation

$$L[y] = -\frac{d^2 y(x)}{dx^2} + q(x)\, y(x) = \lambda\, y(x),\ 0 \leqslant x < \infty$$

where λ is a complex parameter. It has been shown that the convergence of the expansion of any function $f(.)$, square integrable on $[0, \infty)$, in terms of the eigenfunctions of a boundary value problem associated with the differential equation is independent of the function $q(.)$, provided $q(.)$ is square integrable on $[0, \infty)$.

§ 1. We consider the differential equation

$$L[y] = -\frac{d^2 y(x)}{dx^2} + q(x)\, y(x) = \lambda\, y(x),\ 0 \leqslant x < \infty \qquad \ldots(1.1)$$

where $\lambda = u + iv$ is a complex parameter. One of the usual problems with such a differential equation is to consider the convergence of the expansion of an arbitary function $f(.)$, square-integrable on $[0, \infty)$, [we write $f \in L^2\ [0, \infty)$], in terms of the eigenfunctions of boundary value problem associated with the differential equation (1.1). The idea here is to prove the following :

Theorem—If $q(.) \in L^2\ [0, \infty)$, and the eigenfunction expansion of an arbitrary $f \in L^2\ [0, \infty)$ associated with a boundary value problem (II) consisting of the differential equation

$$L_0[y] = -\frac{d^2 y(x)}{dx^2} = \lambda\, y(x),\ 0 \leqslant x < \infty \qquad \ldots(1.2)$$

and the boundary condition

$$y(0) \cos\alpha + y'(0) \sin\alpha = 0\ (0 \leqslant \alpha < \pi) \qquad \ldots(1.3)$$

is convergent, then so is the eigenfunction expansion of f associated with the boundary value problem (I) consisting of (1.1) and the same boundary condition (1.3) as in the boundary value problem (II).

To establish this result, we obtain a relation between the Φ-functions (for the definition of the Φ-function we refer to section (2.6) of Titchmarsh[1] for these two boundary value problems and show that the contribution corresponding to the difference of these two Φ-functions towards the expansion of f is nil.

§ 2. Let $\phi \equiv \phi(x, \lambda)$ and $\theta \equiv \theta(x, \lambda)$ be the two linearly independent solutions of the differential equation (1.1) satisfying the following initial conditions :

$$\left.\begin{aligned} &\phi(0, \lambda) = \sin\alpha, \ \theta(0, \lambda) = \cos\alpha \\ &\phi^{(1)}(0, \lambda) = -\cos\alpha, \ \theta^{(1)}(0, \lambda) = \sin\alpha \end{aligned}\right\} -\pi < \alpha < \pi. \qquad ...(2.1)$$

It has been shown by Titchmash[1] (§ 2.1) that there exists a function $m(.)$, analytic in the two half planes im $\lambda > 0$ and im $\lambda < 0$ such that, for im $\lambda \neq 0$

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\,\phi(x, \lambda) \in L^2[0, \infty). \qquad ...(2.2)$$

With the help of these two solutions $\phi(x, \lambda)$ and $\psi(x, \lambda)$ for any $f \in L^2[0, \infty)$ let us define the function $\Phi(x, \lambda; f) = \Phi(x)$ as

$$\Phi(x) = \psi(x, \lambda) \int_0^x \phi(y, \lambda) f(y)\, dy + \phi(x, \lambda) \int_x^\infty \psi(y, \lambda) f(y)\, dy.$$

This is the so called Φ-function of Titchmarsh associated with the BVP (I). It can be easily proved that $\Phi(x)$ satisfies the following differential equation

$$L[y] = \lambda y(x) - f(x). \qquad ...(2.3)$$

Titchmarsh[1] (§ 3.1) has proved that the expansion of the arbitrary function $f \in L^2[0, \infty)$ associated with the boundary value problem (I) is obtained from

$$\frac{i}{\pi} \int_{-R+i\delta}^{R+i\delta} \Phi(x, \lambda; f)\, d\lambda \qquad ...(2.4)$$

by making $R \to \infty$ and $\delta \to 0$.

Let $\phi_0(x, \lambda)$, $\theta_0(x, \lambda)$ be the solutions of the differential equation (1.2) satisfying the following :

$$\left.\begin{aligned} &\phi_0(0, \lambda) = \sin\alpha, \ \phi_0^{(1)}(0, \lambda) = -\cos\alpha \\ &\theta_0(0, \lambda) = \cos\alpha, \ \theta_0^{(1)}(0, \lambda) = \sin\alpha \end{aligned}\right\} -\pi < \alpha < \pi.$$

It follows easily that

$$\theta_0(x, \lambda) = \cos\alpha \cos(x\sqrt{\lambda}) + \frac{1}{\sqrt{\lambda}} \sin\alpha \sin(x, \sqrt{\lambda})$$

$$\phi_0(x, \lambda) = \sin\alpha \cos(x\sqrt{\lambda}) - \frac{1}{\sqrt{\lambda}} \cos\alpha \sin(x\sqrt{\lambda}). \qquad ...(2.5)$$

So, the square integrable solution $\psi_0(x, \lambda)$ of this differential equation is given by (Titchmarsh[1], § 4.1)

$$\psi_0(x, \lambda) = \frac{e^{ix\sqrt{\lambda}}}{\cos\alpha + i\sqrt{\lambda}\sin\alpha}. \qquad \ldots(2.6)$$

As $q \in L^2[0, \infty)$, we know that the differential equation (1.1) is separated i.e. f and $L[f] \in L^2[0, \infty)$ should imply that $qf \in L^2[0, \infty)$.

Since for each $f \in L^2[0, \infty)$ we know that $\Phi(x, \lambda; f) \in L^2[0, \infty)$ it is follows that $\Phi(x, \lambda; f)$ and $L[\Phi(x, \lambda; f)] = \lambda\Phi(x, \lambda; f) - f \in L^2[0, \infty)$ and by seperability of the differential equation (1.1) we have $q\Phi(x, \lambda; f) \in L^2[0, \infty)$. Now

$$\Phi(x, \lambda, q\Phi(x)) = \psi_0(x, \lambda)\int_0^x \phi_0(y, \lambda)\, q(y)\, \Phi(y)\, dy$$

$$+ \phi_0(x, \lambda)\int_0^\infty \psi_0(y, \lambda)\, q(\lambda)\, \Phi(y)\, dy. \qquad \ldots(2.7)$$

Since Φ is a solution of the differential equation (2.3), we get

$$L[\Phi(x)] = \lambda\Phi(x) - f(x)$$

or

$$L_0[\Phi(x)] + q(x)\Phi(x) = \lambda\Phi(x) - f(x)$$

or

$$L_0[\Phi(x)] = \lambda\Phi(x) - g(x)$$

where $g(x) = f(x) + q(x)\Phi(x) \in L^2[0, \infty)$. This implies that $\Phi(x)$ is the Φ-function of the boundary value problem associated with the differential equation (1.2) corresponding to g (Note that both the boundary value problems have the same boundary conditions). So,

$$\Phi(x, \lambda; f) = \Phi_0(x, \lambda; g) = \Phi_0(x, \lambda; f + q\Phi)$$

$$= \Phi_0(x, \lambda; f) + \Phi_0(x, \lambda; q\Phi). \qquad \ldots(2.8)$$

In view of (2.4) and (2.8), it now follows clearly that in order to establish the theorem we need only to show that the contribution towards eigenfunction expansion arising from $\Phi_0(x, \lambda; q\Phi)$ is nil.

The following two lemmas will be useful for further discussion.

Lemma 1—For any $f \in L^2[0, \infty)$, $\lambda = u + iv$,

$$\int_0^\infty |\Phi(x, \lambda; f)|^2\, dx \leqslant \frac{1}{v^2}\int_0^\infty |f(x)|^2\, dx.$$

For proof we refer to § 2.8 of Titchmarsh[1].

Lemma 2—If $\sqrt{\lambda} = s = \sigma + it$, $\lambda = u + iv$ then

$$\Phi_0(x, \lambda; q\,\Phi) = O\left(\frac{1}{|v|\,|s|}\right).$$

PROOF :

$$\Phi_0(x, \lambda; q\,\Phi) = \frac{e^{ixs}}{\cos\alpha + is\sin\alpha}\int_0^x \left\{\sin\alpha\cos(ys) - \frac{1}{s}\cos\alpha\sin(ys)\right\} q(y)\,\Phi(y)\,dy + \frac{\left\{\sin\alpha\cos(xs) - \frac{1}{s}\cos\alpha\sin(xy)\right\}}{\cos\alpha + is\sin\alpha}\int_x^\infty e^{iys}.$$

Now $\sin\alpha\cos ys - \frac{1}{s}\cos\alpha\sin ys$

$$= \frac{\sin\alpha}{2}(e^{iys} + e^{-iys}) - \frac{\cos\alpha}{2is}(e^{iys} - e^{-iys})$$

$$= \left(\frac{\sin\alpha}{2} - \frac{\cos\alpha}{2is}\right)e^{iys} + \left(\frac{\sin\alpha}{2} + \frac{\cos\alpha}{2is}\right)e^{-iys}$$

$$= O\left(e^{-iy}\left|\frac{\sin\alpha}{2} - \frac{\cos\alpha}{2is}\right|\right) + O\left(e^{iy}\left|\frac{\sin\alpha}{2} + \frac{\cos\alpha}{2is}\right|\right)$$

$$= O(e^{iy}).$$

So,

$$e^{ixs}\left\{\sin\alpha\cos ys - \frac{\cos\alpha}{s}\sin ys\right\}$$

$$= O(e^{-ix})\,O(e^{iy}) = O(1) \text{ since } 0 \leqslant y \leqslant x.$$

Similarly

$$\sin\alpha\cos(xs) - \frac{\cos\alpha}{s}\sin(xs) = O(e^{ix})$$

and so

$$\left\{\sin\alpha\cos(xs) - \frac{\cos\alpha}{s}\sin(xs)\right\}e^{iys} = O(e^{ix}\,e^{-ix}) \text{ for all } y \geqslant x$$

$$= O(1).$$

Hence

$$\Phi_0(x, \lambda; q\,\Phi) = O\left(\frac{1}{(\cos\alpha + is\sin\alpha)}\int_0^\infty |q(y)\,\Phi(y)|\,dy\right)$$

(*equation continued on p.* 1058)

$$= O\left(\frac{1}{|s|} \left\{ \int_0^\infty |q(y)|^2 \, dy \right\}^{1/2} \left\{ \int_0^\infty |\Phi(y, \lambda; f)|^2 \, dy \right\}^{1/2} \right)$$

$$= O\left(\frac{1}{|v| \, |s|} \right), \text{ using Lemma 1, for all } \lambda. \qquad \ldots(2.9)$$

§ 3. Now consider the contour $\Gamma(R)$ formed by the segments of lines $(-R + i, -R + i\delta), (-R + i\delta, R + i\delta), (R + i\delta, R + i)$ joined by semicircles of radius R and centres $\pm i$. Using (2.8) we get,

$$\int_{\Gamma(R)} \Phi(y, \lambda; f) \, d\lambda = \int_{\Gamma(R)} \Phi_0(x, \lambda; f) \, d\lambda - \int_{\Gamma(R)} \Phi_0(x, \lambda; q\Phi) \, d\lambda. \qquad \ldots(3.1)$$

Using Lemma 2 we get

$$\int_{\Gamma(R)} \Phi_0(x, \lambda; q(x)\Phi(x)) \, d\lambda = O\left(\int_{\Gamma(R)} \frac{1}{|\sqrt{\lambda}| \, |v|} \, d\lambda \right). \qquad \ldots(3.2)$$

On the part of the upper semicircle in the 1st quadrant, we have

$$\lambda = i + Re^{i\phi} \left(0 \leqslant \phi \leqslant \frac{\pi}{2} \right).$$

Hence, $\int_{\Gamma(R)} \Phi_0(x, \lambda; q(x)\Phi(x, \lambda; f)) \, d\lambda$ integrated round this semicircle in the first quadrant, gives,

$$O\left\{ \int_0^{\pi/2} \frac{R \, d\phi}{R^{1/2}(1 + R \sin \phi)} \right\} = O\left\{ \int_0^{\pi/2} \frac{R^{1/2} \, d\phi}{1 + R \sin \varphi} \right\} = o(1) \text{ as } R \to \infty.$$

Hence finally it is a question of proving that

$$\lim_{\substack{\delta \to 0 \\ R \to \infty}} \int_{R+i\delta}^{R+i} \Phi_0(x, \lambda, q(x)\Phi(x, \lambda; f)) \, d\lambda = 0. \qquad \ldots(3.3)$$

Using Lemma 2 we get

$$\int_{R+i\delta}^{R+i} \Phi_0(x, \lambda; q(x)\Phi(x)) \, d\lambda = \int_{R+i\delta}^{R+i} O\left(\frac{1}{|\sqrt{\lambda}| \, |v|} \right) d\lambda$$

$$= \int_\delta^1 O\left(\frac{dv}{[R^2 + v^2]^{1/4} \, |v|} \right), \text{ since } \lambda = u + vi$$

(equation contiuned on p. 1059)

$$= O\left(\int_{\delta}^{1} \frac{dv}{|v|\,|R^2+v^2|^{1/4}}\right)$$

$$= O\left(\int_{(R^2+\delta^2)^{1/4}}^{(R^2+1)^{1/4}} \frac{2t^2\,dt}{(t^2-R)(t^2+R)}\right),$$

substituting $R^2 + v^2 = t^4$

$$= O\left(\int_{(R^2+\delta^2)^{1/4}}^{(R^2+1)^{1/2}} \frac{(t^2-R)+(t^2+R)}{(t^2-R)(t^2+R)}\,dt\right)$$

$$= O\left(\frac{1}{\sqrt{R}}\left\{\tan^{-1}\frac{(R^2+1)^{1/4}}{\sqrt{R}} - \tan^{-1}\frac{(R^2+\delta^2)^{1/4}}{\sqrt{R}}\right\}\right)$$

$$+ O\left(\frac{1}{\sqrt{R}}\log\frac{\{(R^2+1)^{1/4}-\sqrt{R}\}\{(R^2+\delta^2)^{1/4}+\sqrt{R}\}}{\{(R^2+1)^{1/4}+\sqrt{R}\}\{(R^2+\delta^2)^{1/4}-\sqrt{R}\}}\right)$$

$$= O\left(\frac{1}{\sqrt{R}}\tan^{-1}\frac{\left(1+\frac{1}{R^2}\right)^{1/4}-\left(1+\frac{\delta^2}{R^2}\right)^{1/4}}{1+\left(1+\frac{1}{R^2}\right)^{1/4}\left(1+\frac{\delta^2}{R^2}\right)^{1/4}}\right)$$

$$+ O\left(\frac{1}{\sqrt{R}}\log\frac{\left(1+\frac{1}{\sqrt{R}}\right)^{1/4}-1}{\left(1+\frac{1}{R^2}\right)^{1/4}+1}\cdot\frac{\left(1+\frac{\delta^2}{R^2}\right)^{1/4}+1}{\left(1+\frac{\delta^2}{R^2}\right)^{1/4}-1}\right)$$

$= o(1)$ as $R \to \infty$; $\delta \to 0$ by choosing δ suitably as dependent on R.

Hence (3.3) follows.

Reference

1. E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations*, Vol. I. Oxford University Press, 1962.