Indian J. pure appl. Math., 19 (11) : 1089-1100 November, 1988

THE MODIFIED DINI'S SERIES AND THE FINITE HANKEL-SCHWARTZ INTEGRAL TRANSFORMATION

J. M. MENDEZ

Departmento de Ecuaciones Funcionales, Facultad de Matemáticas, Universidad de La Laguna, La Laguna (Canary Islands—Spain)

(*Received* 5 *August* 1986; *after revision* 25 *April* 1988)

In this paper, an arbitrary function $f(x)$ defined on the interval $(0, a)$ is expressed as an expansion in Dini's series of the orthogonal family $\{\mathcal{F}_\nu(\rho_m x)\}$ of modified Bessel functions, where $\mathcal{F}_\nu(x) = x^{-\nu} J_\nu(x)$ and ρ_m denotes the mth postive root of the equation $x\,\mathcal{F}'_\nu(ax) + h\,\mathcal{F}_\nu(ax) = 0$. Next, the convergence theorem is rigorously established. The Dini's series suggest to consider a varaint of the finite Hankel transformation, which will be called the finite Hankel-Schwartz transformation of the second kind. This transformation is used in solving some partial differential equations which cannot be directly treated by applying the corresponding finite Hankel transformation. Finally, we remark that the initial term of our expansion depends only on the parameter h, whereas the classical Dini's expansion depends on $h + \nu$.

1. INTRODUCTION

Schwartz[9] investigated the following modified Hankel transformation

$$F(y) = \int_0^\infty \mathcal{F}(xy) f(x)\, dm(x) \qquad ...(1.1)$$

where $dm(x) = [2^\nu \Gamma(\nu + 1)]^{-1} x^{2\nu+1}\, dx$ and $\mathcal{F}(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu} J_\nu(x)$, $J_\nu(x)$ being the Bessel function of the first kind of order ν.

This transformation has been studied in spaces of distributions by several authors and Lee[7] calls it Hankel-Schwartz integral transformation.

To consider the expansion of an arbitrary function $f(x)$ defined in the interval $(0, a)$ as a Fourier-Bessel series, i.e., as a series of the type

$$f(x) = \sum_{n=1}^\infty a_n \mathcal{F}_\nu(j_n x) \qquad ...(1.2)$$

where $\mathcal{F}_\nu(x) = x^{-\nu} J_\nu(x)$, $\nu \geqq -\frac{1}{2}$ and j_n denote the positive zeros of the functions $\mathcal{F}_\nu(ax)$, i.e.,

$$\mathcal{F}_\nu(j_n a) = 0. \qquad ...(1.3)$$

Méndez[8] introduced the corresponding finite transformation through the equation

$$ {}_sh_{1,\nu}[f(x)] = F_{1,\nu}(n) = \int_0^a x^{2\nu+1}\, \mathcal{F}_\nu(j_n x) f(x)\, dx \qquad \ldots(1.4) $$

which is called Hankel-Schwartz transformation of the first kind of order ν. Its inversion theorem is stated as :

Theorem 1—Let $f(t)$ be a function defined in (0, 1) and assumed to be absolutely summable over the same interval. Let $\nu \geqq -\frac{1}{2}$ and

$$ a_n = \frac{2}{j_n^2\, \mathcal{F}_{\nu+1}^2(j_n)} \int_0^1 t^{2\nu+1}\, \mathcal{F}_\nu(j_n t)\, f(t)\, dt,\ n = 1, 2, \ldots $$

If $f(t)$ is of bounded variation in (a, b), $0 < a < b < 1$, and if $x \in (a, b)$, then the series (1.2) converges to

$$ \tfrac{1}{2}[f(x+0) + f(x-0)]. $$

In this paper we show how the modified Dini series expansion of an arbitrary function $f(x)$ leads naturally to the finite Hankel-Schwartz integral transformation of the second kind. The inverson theorem of this new transformation is rigorously established by studying the convergence of the series expansion. The operational calculus generated is used in the solution of several problems in Mathematical Physics.

Recall that the form of the Dini series is determined by the nature of the zeros of the equation

$$ z\, \mathcal{F}_\nu'(z) + h\, \mathcal{F}_\nu(z) = 0. $$

We emphasize that the first term of the expansions depend on the parameter h, and not on $h + \nu$, as it happens in the classical theory (cf. Watson[12], p. 597).

Another interesting feature of this transformation is its usefulness in the solution of partial differential equations which cannot be tackled by applying the corresponding finite Hankel transformation when $\nu \neq 1$ (cf. Colombo[4], p. 82).

Finally, we will denote in the sequel the partial sum of the series (1.2) by

$$ S_n(x) = \sum_{m=1}^{n} a_m\, \mathcal{F}_\nu(j_m x). \qquad \ldots(1.5) $$

Setting

$$ P_n(x, t) = \sum_{m=1}^{n} \frac{2\, \mathcal{F}_\nu(j_m x)\, \mathcal{F}_\nu(j_m t)}{j_m^2\, \mathcal{F}_{\nu+1}^2(j_m)} \qquad \ldots(1.6) $$

we have

$$S_n(x) = \int_0^1 t^{2\nu+1} P_n(x, t) f(t)\, dt. \qquad \ldots(1.7)$$

2. Preliminary Results

Let L denote the differential operator $x^{-2\nu-1} \frac{d}{dx} x^{2\nu+1} \frac{d}{dx}$. We begin by considering the following Sturm-Liouville problem (cf. Sneddon[11], p. 440)

$$(L + \lambda^2) y = 0, \quad 0 < a < b \qquad \ldots(2.1)$$

$$My(a) = a_1 y(a) + a_2 y'(a) = 0, \; Ny(b) = b_1 y(b) + b_2 y'(b) = 0 \qquad \ldots(2.2)$$

where a_1, a_2, b_1 and b_2 represent prescribed constants.

The general solution of eqn. (2.1) is

$$y = \phi(x, \lambda) = A(\lambda) \mathcal{F}_\nu(\lambda x) + B(\lambda) \mathcal{G}_\nu(\lambda x) \qquad \ldots(2.3)$$

where $\mathcal{F}_\nu(x) = x^{-\nu} J_\nu(x)$ and $\mathcal{G}_\nu(x) = x^{-\nu} Y_\nu(x)$, $J_\nu(x)$ being the Bessel function of the first kind of order ν and $Y_\nu(x)$ the one of second kind.

Let $y = \phi_n(x)$ be the eigenfunctions of the problem (2.1)—(2.2) which correspond to the nonzero eigenvalues λ_n. We have the general orthogonality condition

$$\int_a^b x^{2\nu+1} \phi_n(x) \phi_m(x)\, dx = \begin{cases} \frac{1}{2\lambda_n^2}\Big[x^{2\nu+1}\{x(\phi_n'(x))^2 + \lambda_n^2 x\phi_n^2(x) \\ \qquad + 2\nu \phi_n(x) \phi_n'(x)\}\Big], & \text{if } m = n \\ 0, & \text{if } m \neq n. \end{cases} \qquad \ldots(2.4)$$

Then, we deduce from (2.3) that the solution of the particular problem

$$(L + \lambda^2)\phi(x) = 0, \quad 0 \leqslant x \leqslant a \qquad \ldots(2.5)$$

$$N\phi(a) = \phi'(a) + h\phi(a) = 0, \; h > 0$$

is

$$\phi_n(x) = \mathcal{F}_\nu(\rho_n x) \qquad \ldots(2.6)$$

where $\rho_1, \rho_2, \ldots$ denote the positive zeros arranged in ascending order of magnitude of the transcendental equation

$$z\, \mathcal{F}_\nu'(az) + h\, \mathcal{F}_\nu(az) = 0 \qquad \ldots(2.7)$$

i.e., [cf. Gray *et al.*[6], p. 16, eqn. (24)],

$$-az^2 \mathcal{F}_{\nu+1}(az) + h\mathcal{F}_\nu(az) = 0.$$

The above orthogonality condition (2.4) now becomes

$$\int_0^a x^{2\nu+1}\,\mathcal{F}_\nu(\rho_m x)\,\mathcal{F}_\nu(\rho_n x)\,dx = \begin{cases} \dfrac{a^{2+\nu}}{2\rho_n^2}\,(ah^2 + a\rho_n^2 - 2\nu h)\,\mathcal{F}_\nu^2(\rho_n a), & \text{if } m=n \\ 0, & \text{if } m \neq n. \end{cases} \quad \ldots(2.8)$$

Given an arbitrary function $f(x)$ defined in the interval $(0, a)$, (2.8) allows one to formally express this function as a Dini expansion, as follows

$$f(x) = \sum_{m=1}^{\infty} b_m\,\mathcal{F}_\nu(\rho_m x) \quad \ldots(2.9)$$

where

$$b_m = \frac{2\rho_m^2}{a^{2\nu+1}\,\mathcal{F}_\nu^2(\rho_m a)\,(ah^2 + a\rho_m^2 - 2\nu h)} \int_0^a x^{2\nu+1}\,\mathcal{F}_\nu(\rho_m x)\,f(x)\,dx \quad \ldots(2.10)$$

$m = 1, 2, \ldots$, ρ_m being the positive roots of eqn. (2.7).

Note that we can extend the Dixon theorem (cf. Watson[12], p. 480) to the zeros of the function (2.7). Indeed, it can be proved that the zeros of the equations $Ax\,\mathcal{F}_\nu'(x) + B\mathcal{F}_\nu(x) = 0$ and $Cx\,\mathcal{F}_\nu'(x) + D\mathcal{F}_\nu(x) = 0$ are interlaced, whatever the real numbers A, B, C and D, provided they are such that $AD \neq BC$. Hence, the roots of eqns. (1.3) and (2.7) also are interlaced.

3. The Modified Dini Expansion—The Convergence Theorem

In this section it will be assumed that $a = 1$ for the sake of simplicity.

As it occurs in the classical theory of Dini expansions (cf. Watson[12], p. 597), we have to add an initial term to the series (2.9). In fact, the form of Dini expansion is based on the zeros of the function (2.7) and these depend upon the values of the parameter h. Thus, the expansion (2.9) only corresponds to the case $h > 0$.

When $h = 0$ it can easily be seen that the equation (2.7) has a zero at the origin. On the other hand, when $h < 0$ this function has two purely imaginary zeros.

Let ν be a real number such that $\nu \geqslant -\frac{1}{2}$. We write the modified Dini expansion of $f(x)$, as follows

$$f(x) = b_0 + \sum_{m=1}^{\infty} b_m\,\mathcal{F}_\nu(\rho_m x) \quad \ldots(3.1)$$

where b_0 denote the initial term which must be inserted in (2.9) as a consequence of the existence of these new roots.

If $h > 0$ the initial term $b_0 = 0$ and (3.1) coincides with (2.9). But when $h = 0$, taking into account that

$$\int_0^1 x^{2\nu+1} \mathcal{F}_\nu(\rho_m x)\, dx = \mathcal{F}_{\nu+1}(\rho_m) = 0,\ m = 1, 2, \ldots$$

[cf. Gray *et al.*[6], p. 16, eqn. (25)] and (2.7), we get

$$b_0 = (2\nu + 2) \int_0^1 x^{2\nu+1} f(x)\, dx. \qquad \ldots(3.2)$$

Finally, if $\pm \rho_0 i$ denote the imaginary zeros of (2.7) when $h < 0$, from (3.1) and (2.8) we infer that

$$b_0 = \frac{2\rho_0^2}{(\rho_0^2 + 2\nu h - h^2)\, \mathcal{F}_\nu^2(\rho_0 i)} \int_0^1 x^{2\nu+1} \mathcal{F}_\nu(i\rho_0 x) f(x)\, dx. \qquad \ldots(3.3)$$

Now, consider the function

$$\frac{2w\, \mathcal{F}_\nu(xw)\, \mathcal{F}_\nu(tw)}{\mathcal{F}_\nu(w)\, \{w\mathcal{F}'_\nu(w) + h\, \mathcal{F}_\nu(w)\}} \qquad \ldots(3.4)$$

whose poles are the zeros $j_1, j_2, \ldots$ of $\mathcal{F}_\nu(z)$ and the zeros $\rho_1, \rho_2 \ldots$ of $z\, \mathcal{F}'_\nu(z) + h\, \mathcal{F}_\nu(z)$.

The residues of this function at the first poles are

$$\frac{2\, \mathcal{F}_\nu(j_m x)\, \mathcal{F}_\nu(j_m t)}{j_m^2\, \mathcal{F}^2_{\nu+2}(j_m)}.$$

If $h > 0$ the residues at the poles $\rho_1, \rho_2, \ldots$ are

$$-\frac{2\rho_n^2\, \mathcal{F}_\nu(\rho_n x)\, \mathcal{F}_\nu(\rho_n t)}{(\rho_n^2 - 2\nu h + h^2)\, \mathcal{F}_\nu^2(\rho_n)}.$$

When $h = 0$ we must moreover consider the residue at the origin, whose value is $-4(\nu + 1)$.

When $h < 0$ the residues at $\pm i\rho_0$ are both equal to

$$-\frac{2\rho_0^2\, \mathcal{F}_\nu(\rho_0 xi)\, \mathcal{F}_\nu(\rho_0 ti)}{(\rho_0^2 + 2\nu h - h^2)\, \mathcal{F}_\nu^2(\rho_0 i)}.$$

By denoting the partial sum of the series (3.1)

$$\sigma_n(x) = b_0 + \sum_{m=1}^{\infty} b_m \mathcal{F}_\nu(\rho_m x) \qquad \ldots(3.5)$$

and

$$P_n(x, t; h) = A_0(x, t) + \sum_{m=1}^{n} \frac{2\rho_m^2 \mathcal{F}_\nu(\rho_m x) f_\nu(\rho_m t)}{(\rho_m^2 - 2\nu h + h^2) \mathcal{F}_\nu^2(\rho_m)} \qquad \ldots(3.6)$$

where

$$A_0(x, t) = \begin{cases} 0, \text{ if } h > 0 \\ 2(\nu + 1), \text{ if } h = 0 \\ \dfrac{2\rho_0^2 \mathcal{F}_\nu(\rho_0 xi) \mathcal{F}_\nu(\rho_0 ti)}{(\rho_0^2 + 2\nu h - h^2) \mathcal{F}_0^2(\rho_0 i)}, \text{ if } h < 0 \end{cases} \qquad \ldots(3.7)$$

we can express (3.5) as

$$\sigma_n(x) = \int_0^1 t^{2\nu+1} P_n(x, t; h) f(t)\, dt. \qquad \ldots(3.8)$$

Now choose D_n such that it is not equal to any of the number j_m and $\rho_n < D_n < \rho_{n+1}$ and let j_N be the greatest of the numbers j_m which does not exceed D_n (cf. Watson[12], p. 598).

The following expression

$$S_n(x, t; h) = \sum_{m=1}^{n} \frac{2\mathcal{F}_\nu(j_m x) \mathcal{F}_\nu(j_m t)}{j_m^2 \mathcal{F}_{\nu+1}^2(j_m)} - A_0(x, t)$$

$$- \sum_{m=1}^{n} \frac{2\rho_m^2 \mathcal{F}_\nu(\rho_m x) \mathcal{F}_\nu(\rho_m t)}{(\rho_m^2 + h^2 - 2\nu h) \mathcal{F}_\nu^2(\rho_m)} \qquad \ldots(3.9)$$

where $A_0(x, t)$ is given by (3.7), permits to connect the partial sums of the modified series of Fourier-Bessel (1.2) and Dini (3.1). Clearly, from (1.5), (1.6), (1.7), (3.5), (3.8) and (3.9) it can be deduced that

$$\int_0^1 t^{2\nu+1} S_n(x, t; h) f(t)\, dt = \sum_{m=1}^{N} a_m \mathcal{F}_\nu(j_m x) - b_0 - \sum_{m=1}^{n} b_m \mathcal{F}_\nu(\rho_m x)$$

$$= S_N(x) - \sigma_n(x). \qquad \ldots(3.10)$$

From Cauchy's theory of residues we find the following integral representation of (3.9)

$$S_n(x, t; h) = \frac{1}{2\pi i} \int_{D_n - \infty i}^{D_n + \infty i} \frac{2w\, \mathcal{F}_\nu(xw)\, \mathcal{F}_\nu(tw)}{\mathcal{F}_\nu(w)\{w\, \mathcal{F}'_\nu(w) + h\, \mathcal{F}_\nu(w)\}}\, dw. \quad \ldots(3.11)$$

Since

$$\int_0^t wt^{2\nu+1}\, \mathcal{F}_\nu(tw)\, dt = t^{\nu+2}\, \mathcal{F}_{\nu+1}(tw)$$

[cf. Gray *et al.*[6], p. 16, eqn. (25)], it can be inferred from (3.11) that

$$\int_0^t t^{2\nu+1} S_n(x, t; h)\, dt = \frac{t^{2\nu+2}}{\pi i} \int_{D_n - \infty i}^{D_n + \infty i} \frac{w\, \mathcal{F}_\nu(xw)\, \mathcal{F}_{\nu+1}(tw)}{\mathcal{F}_\nu(w)\{w\, \mathcal{F}'_\nu(w) + h\, \mathcal{F}_\nu(w)\}} dw. \quad \ldots(3.12)$$

As an immediate consequence of (3.11) and (3.12) we have

$$|S_n(x, t, h)| < \frac{c_3}{(xt)^{\nu+1/2}(2 - x - t)} \quad \ldots(3.13)$$

and

$$\left| \int_0^t t^{2\nu+1} S_n(x, t; h)\, dt \right| < \frac{c_4}{D_n} \left(\frac{t}{x}\right)^{\nu+1/2} \frac{1}{(2 - x - t)} \quad \ldots(3.14)$$

where c_3 and c_4 are constants independent of n, x and t.

Next, it can be proved with an argument similar to the one used in Watson[12] (p. 599) that if $f(t)$ is absolutely summable in the interval (a, b), $0 \leqslant a < b \leqslant 1$, then

$$\int_a^b t^{2\nu+1} S_n(x, t; h) f(t)\, dt \to 0, \text{ as } n \to \infty \quad \ldots(3.15)$$

provided $0 < x < 1$.

Theorem 2—Let $f(t)$ be a function defined and absolutely summable in the interval $(0, 1)$. If $f(t)$ is of bounded variation in (a, b) where $0 \leqslant a < b \leqslant 1$, then the series (3.1) converges to the sum $\frac{1}{2}[f(x + 0) + f(x - 0)]$ at all points x such that $a + \Delta \leqslant x \leqslant b - \Delta$, $\Delta > 0$ being arbitrarily small.

PROOF : By virture of Theorem 1 the series $\sum_{n=1}^{\infty} a_n\, \mathcal{F}_\nu(j_n x)$ converges to the sum $\frac{1}{2}[f(x + 0) + f(x - 0)]$. Our assertion follows directly from (3.10) and (3.15) to pass to the limit as $n \to \infty$.

Remark 1 : Note that the initial term b_0 of our expansion (3.1) only depends on the value of h, whereas this term depends on $h + \nu$ in the classical theory (cf. Watson[12], p. 598). Moreover, the roots ρ_n of the equation $z\,\mathscr{F}_\nu'(z) + h\,\mathscr{F}_\nu(z) = 0$ are not equal to the roots λ_n' s of $z\,J_\nu'(z) + hJ_\nu(z) = (z) = 0$.

4. The Finite Hankel—Schwartz Integral Transformation of the Second Kind—Applications

According to (2.9) and (2.10), we define the finite Hankel-Schwartz integral transformation of the second kind of order $\nu > -\frac{1}{2}$ by the equation

$$ {}_sh_{2,\nu}[f(x)] = F_{2,\nu}(n) = \int_0^a x^{2\nu+1}\,\mathscr{F}_\nu(\rho_n x)\,f(x)\,dx \qquad \ldots(4.1) $$

whose kernel is the modified Bessel function (2.6) and where ρ_n denote the roots of eqn. (2.7).

The corresponding inversion formula is

$$ {}_sh_{2,\nu}^{-1}[F_{2,\nu}(n)] = f(x) = b_0 + \frac{2}{a^{2\nu+1}} \sum_{n=1}^{\infty} \frac{\rho_n^2\,F_{2,\nu}(n)\,\mathscr{F}_\nu(\rho_n x)}{(ah^2 + a\rho_n^2 - 2\nu h)\,\mathscr{F}_\nu^2(\rho_n a)} \,. \qquad \ldots(4.2) $$

Theorem 2 not only guarantees existence of (4.1) but also ensures that inversion formula (4.2) holds.

If we assume that the $f \in C^2(0, a), f'(a) + h f(a) = 0$ and $h > 0$, we obtain the main operational formula of this transformation, i. e.,

$$ {}_sh_{2,\nu}\left[f''(x) + \frac{2\nu+1}{x} f'(x)\right] = -\rho_n^2\,{}_sh_{2,\nu}[f(x)] \qquad \ldots(4.3) $$

whatever the values of $f(0)$, and $f'(0)$, provided they are finite.

If $f'(a) + hf(a) \neq 0$ and $h > 0$, we get

$$ {}_sh_{2,\nu}\left[f''(x) + \frac{2\nu+1}{x} f'(x)\right] = a^{2\nu+1}\,\mathscr{F}_\nu(\rho_n a)\,(f'(a) + h f(a)) - \rho_n^2\,{}_sh_{2,\nu}[f(x)]. \qquad \ldots(4.4) $$

Remark 2 : Recall that the function $y = \mathscr{F}_\nu(x)$ is a solution of the equation $Ly \equiv y'' + \frac{1+2\nu}{x} y' + y = 0$. Its multiplication by $x^{2\nu}$ has only repercussions on the sign of the parameter ν, that is, the function

$$ y = \mathscr{F}_\nu^*(x) = x^{2\nu}\,\mathscr{F}_\nu(x) $$

is a solution of the equation

$$L^* y \equiv y'' + \frac{1 - 2\nu}{x} y' + y = 0.$$

Consequently, the solution of Sturm-Liouville problem

$$(L^* + \lambda^2)\, \phi(x) = 0$$

$$N^* \phi(a) \equiv \phi'(a) + \left(h - \frac{2\nu}{a}\right) \phi(a) = 0$$

is

$$\phi_n^*(x) = \mathcal{F}_\nu^* \rho_n(x) \qquad \ldots(4.5)$$

where $\rho_1, \rho_2, \ldots$ denote the positive zeros of the equation (2.7). The solutions (4.5) form a orthogonal system on the interval $(0, a)$ with respect to the weight function $x^{1-2\nu}$. Proceeding as before, we can now introduce the integral transform

$${}_sh_{\nu,2}^*[f(x)] = F_{2,\nu}^*(n) = \int_0^a x^{1-2\nu} \mathcal{F}_\nu^*(\rho_n x) f(x)\, dx \qquad \ldots(4.6)$$

whose inversion formula, in the case $h > 0$, is given by

$${}_sh_{2,\nu}^{*-1}[F_{2,\nu}^*(n)] = f(x) = \sum_{n=1}^{\infty} \frac{2\rho_n^2 F_{2,\nu}^*(n) \mathcal{F}_\nu^*(\rho_n x)}{a^{1-2\nu}(ah^2 + a\rho_n^2 - 2\nu h) \mathcal{F}_\nu^{*2}(\rho_n a)}. \qquad \ldots(4.7)$$

A similar result to that proven in Theorem 2 can be stated in relation with the convergence of the series (4.7), whenever $\nu \geqq -\frac{1}{2}$.

The main operational rule of the transform (4 6) is

$${}_sh_{2,\nu}^*\left[f''(x) + \frac{1 - 2\nu}{x} f'(x)\right] = -\rho_n^2\, {}_sh_{2,\nu}^*[f(x)] \qquad \ldots(4.8)$$

provided that $f'(a) + \left(h - \frac{2\nu}{a}\right) f(a) = 0$ and $h > 0$.

In the sequel we shall give a few examples to illustrate the use of the above transformations in solving some important problems.

(a) Let ν be any real number. We wish to find the solution of the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2\nu + 1}{r} \frac{\partial u}{\partial r} = \frac{1}{k} \frac{\partial u}{\partial t} \quad (t > 0,\ 0 < r < a,\ k > 0) \qquad \ldots(4.9)$$

satisfying the initial condition

$$u(r, 0) = f(r) \quad (0 \leqq r \leqq a)$$

and the boundary conditions

$$\frac{\partial u(a,t)}{\partial r} + h\,u(a,t) = 0\,, \text{ for every } \nu \geqq 0,$$

or

$$\frac{\partial u(a,t)}{\partial r} + \left(h - \frac{2\nu}{a}\right) u(a,t) = 0, \text{ for every } \nu \geqq 0.$$

By virtue of (4.3) and (4.8), we convert formally (4.9) into

$$\left(\frac{\partial}{\partial t} + k\,\rho_n^2\right) U_{n,\nu}(t) = 0,$$

where

$$U_{n,\nu}(t) = \begin{cases} {}_sh_{2,\nu}\,[t\,(r,u)]\,, & \nu \geqq 0 \\ {}_sh^*_{2,-\nu}\,[u\,(r,t)], & \nu \leqq 0. \end{cases}$$

Hence,

$$U_{n,\nu}(t) = F_{2,\nu}(n)\; e^{-k\,\rho_n^2 t} \qquad \ldots(4.10)$$

where

$$F_{2,\nu}(n) = \begin{cases} {}_sh_{2,\nu}\,[f(r)]\,, & \nu \geqq 0 \\ {}_sh^*_{2,-\nu}\,[f(r)] & \nu \leqq 0. \end{cases}$$

By applying the inversion formulas (4.2) and (4.7) to (4.10), we get the required solution

$$u(r,t) = \begin{cases} \dfrac{2}{a^{2\nu+1}} \displaystyle\sum_{n=1}^{\infty} \dfrac{\rho_n^2\; F_{2,\nu}(n)\; \mathcal{F}_\nu(\rho_n r)\; e^{-k\,\rho_n^2 t}}{(ah^2 + a\rho_n^2 - 2\nu h) + \mathcal{F}_\nu^2(\rho_n a)}, & \nu \geqq 0 \\ \dfrac{2}{a^{2\nu+1}} \displaystyle\sum_{n=1}^{\infty} \dfrac{\rho_n^2\; F_{2,\nu}(n)\; \mathcal{F}^*_{-\nu}(\rho_n r)\; e^{-k\rho_n^2 t}}{(ah^2 + a\rho_n^2 + 2\nu h)\,\mathcal{F}^{*2}_{-\nu}(\rho_n a)}, & \nu \leqq 0. \end{cases} \qquad \ldots(4.11)$$

Note that when $\nu = 0$ the prablem (4.9) reduces to the one considered by Sneddon[11] (eqns. 8-4-20, 33, 34) on the diffusion equation, since in this case $\mathcal{F}_0(x) = J_0(x)$ and $\rho_n = \xi_n$ are the roots of $x\,J_0'(ax) + h\,J_0(ax) = 0$. Then, both of formulas in (4.11) yield the sum solution and this coincides with the one achieved in the reference mentioned obove.

A procedure similar to the one used by Churchill[3] (p. 191), allows one to establish (4.11) as a rigorous solution of our problem.

Remark 3 : Note that the equation (4.9) cannot be solved directly by means of the finite Hankel transformation, except when $\nu = 0$. Nevertheless, the simultaneous application of the finite Hankel-Schwartz transformations (4.1) and (4.6) provides a simple method to solve immediately the problem (a), no matter what the real value of ν may be.

(b) Many partial differential equations involving the n-dimensional laplacian operator can also be solved by using the transformation (4.1). Indeed, the n-dimensional potential equation is

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_{n-1}^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \ldots (4.12)$$

where $u = u(x_1, x_2, \ldots, x_{n-1}, z)$. If we seek solntions which only depend on $r = x^2 + (x_2^2 + \ldots + x_{n-1}^2)^{1/2}$ and z, (4.12) reduces to the from (cf. Sneddon[11], p. 342)

$$\frac{\partial^2 u}{\partial r^2} + \frac{n-2}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad \ldots (4.13)$$

We find now the solution of (4.13) that satisfies the conditions

$$\begin{aligned} &\frac{\partial u(a, z)}{\partial r} + hu(a\, z) = 0 \quad (z \geqslant 0, h > 0) \\ &u(r, 0) = f(r) \qquad \ldots (4.14) \\ &u(r, z) \to 0, \text{ as } z \to \infty, \end{aligned}$$

by directly applying to (4.13) the finite Hankel-Schwartz transformation of the second kind of order $\nu = (n-3)/2$. Now denote $U_n(z) = {}_sh_{2,\nu}[u(r, z)]$. From (4.3) we see that $U_n(z)$ satisfies the equation

$$-\rho_n^2 U_n(z) + \frac{\partial^2 U_n(z)}{\partial z^2} = 0$$

whose solution is, in view of conditions (4.14),

$$U_n(z) = F(n)\, e^{-\rho_n z}$$

where

$$F(n) = {}_sh_{2,\nu}[f(r)].$$

Again making use of (4.2), the formal solution of the problem posed by equations (4.13)—(4.14) is

$$u(r, z) = \frac{2}{a^{2\nu+1}} \sum_{n=1}^{\infty} \frac{\rho_n^2\, e^{-\rho_n z}\, \mathscr{F}_\nu(\rho_n r)\, F(n)}{(ah^2 + a\rho_n^2 - 2\nu h)\, \mathscr{F}_\nu^2(\rho_n a)}. \qquad \ldots (4.15)$$

That (4.15) is truly a solution of our problem can be proved assuming that the function $f(r)$ is such that the above series and the series obtained by applying the operator L and $\frac{\partial^2}{\partial z^2}$ converge adequately.

When $\nu = 0$ (i.e., $n = 3$) the problem (4.13) consists of finding the bounded steady temperatures $u(r, z)$ in the cylinder $r \leqslant a$, $z \geqslant 0$, if it is assumed that heat transfer into surroundings at temperature zero takes place through the surface $r = a$, according to the linear law $u_r(a, z) = -hu(a, z)$.

Remark 4 : The problem (4.13) is usually solved by means of the finite Hankel transform only in the case $n = 3$ (Colombo[4], p. 82). Now, by combining the finite transforms (4.1) and (4.6), it is feasible to solve this problem for each $n \geqq 3$, even more, for an arbitrary integer n.

References

1. T. M. Apostol, *Mathematical Analysis*, Addision-Wesley, Mass., 1957.
2. H. S. Carslaw, *An Introduction to the Theory of Fourier's Series and Integrals*, Dover Publication, New York, 1950.
3. R. V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill Book Co., Inc., New York, 1963.
4. S. Colombo. *Les transformations de Mellin et Hankel*, C. N. R. S., Paris, 1959.
5. L. S. Dube and J. N. Pandey, *Tohoku Math. J.* 27 (1975), 337–54.
6. A. Gray, G. B. Mathews and T. M. MacRobert, *A Treatise on Bessel Functions and their Applications to Physics*, McMillan and Co. Ltd., London, 1952.
7. W. Y. Lee, *SIAM J. Math. Anal.* 6 (2) (1975), 427–32.
8. J. M. Méndez, The finite Hankel-Schwartz transform (To appear) .
9. A. L. Schwartz, *Proc. Am. Math. Soc.* 22 (1969), 713–17.
10. I. N. Snoddon, *Phill. Mag. Ser.* 7, 37 (264) (1964), 17–25.
11. I. N. Sneddon, *The use of Integral Transforms*, Tata-McGraw-Hill, New Delhi, 1979.
12. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, London, 1966.