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STRESS DISTRIBUTION AROUND TWO EQUAL CIRCULAR ELASTIC INCLUSIONS IN AN INFINITE PLATE UNDER THE ACTION OF AN ISOLATED FORCE APPLIED AT THE ORIGIN

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This paper contains the two-dimensional solutions of the problem in bipolar coordinates. The inclusions are elastic and perfectly bonded to an infinite elastic plate under the action of an isolated force. (i) X applied at the origin in the positive direction of x-axis; and (ii) Y applied at the origin in the positive direction of y-axis.

The stresses on the boundary, in each case, are found numerically and presented graphically. It is found that the boundary $\alpha_1 = 0.8$, the maximum values of the stresses $\widehat{\alpha\alpha}$ and $\widehat{\alpha\beta}$ are (i) $15.10X$ units attained at $\beta = 109.5°$ and $10.47X$ units attained at $\beta = 180°$ respectively, in the first case; and (ii) $3.13Y$ units at $\beta = 0°$ and $25.64Y$ units at $\beta = 15°$ respectively, in the second case.

INTRODUCTIONS

The problem of stress distribution around elastic inclusions in an infinite plate has been studied by many authors. Saleme[1] has studied the stress distribution around a circular elastic inclusion in a semi infinite elastic plate under tension parallel to the straight edge. Mahata[2] has studied the stress distributions around two equal circular elastic inclusions in an infinite elastic plate under the action of (i) a centre of pressure radiating from a point, and (ii) a bi-axial or uni-axial tension.

In the present paper we have studied, in the absence of the body forces, the stress distribution around two equal circular elastic inclusions in an infinite elastic plate under the action of an isolated force (i) X applied at the origin in the positive direction of x-axis, in section 1, and (ii) Y applied at the origin in the position direction of y-axis, is secion 2.

In each section, we find two solutions, one of which is regular inside and the other is is regular outside the inclusions. Numerical calculations are performed and graphs of stresses are drawn.

The bipolar coordinates[3] are defined by

$$\alpha + i\beta = \log \frac{x + i(y + a)}{x + i(y - a)} \qquad \ldots(1.1)$$

where a is a positive length.

$$\therefore\ x = \sin\beta/h,\ y = \sinh\alpha/h,\ ah = \cosh\alpha - \cos\beta. \qquad \ldots(1.2)$$

Now, α = constant, represents a set of co-axial circles having the two poles $A(o, a)$ and $B(o, -a)$ of the above transformation, for limitimg points. The curves β = constant are a system of circles through A and B and intersecting the first set orthogonally. $\beta > 0$ on R. H. S. of y-axis and $\beta < 0$ on its L. H. S. while on the y-axis, $\beta = 0$, except on the segment AB where $\beta = \pm\pi$. At infinite, $\alpha = 0$, $\beta = 0$ and at A, B, $\alpha \to \infty$ and $\alpha \to -\infty$ respectively.

The stress function χ, in the absence of the body forces, satisfies the differential equation,

$$\left(\frac{\partial^4}{\partial\alpha^4} + 2\frac{\partial^4}{\partial\alpha^2\,\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - 2\frac{\partial^2}{\partial\alpha^2} + 2\frac{\partial^2}{\partial\beta^2} + 1\right)(h\chi) = 0. \qquad \ldots(1.3)$$

The stresses in terms of $h\chi$ are

$$\widehat{a\alpha\alpha} = [(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\beta^2} - \sinh\alpha\frac{\partial}{\partial\alpha} - \sin\beta\frac{\partial}{\partial\beta} + \cosh\alpha]\,(h\chi) \qquad \ldots(1.4)$$

$$\widehat{a\alpha\beta} = -(\cosh\alpha - \cos\beta)\frac{\partial^2}{\partial\alpha\,\partial\beta}(h\chi). \qquad \ldots(1.5)$$

The components of the displacements are

$$Eu_\alpha = (1 - \nu)\left(\frac{\partial}{\partial\alpha} - \frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\right)(h\chi) - \left(\frac{\partial}{\partial\beta} - \frac{\sinh}{\cosh\alpha - \cos\beta}\right)_{(hQ)} \qquad \ldots(1.6)$$

$$Eu_\beta = (1 - \nu)\left(\frac{\partial}{\partial\beta} - \frac{\sin\beta}{\cosh\alpha - \cos\beta}\right)(h\chi) + \left(\frac{\partial}{\partial\alpha} - \frac{\sinh\alpha}{\cosh\alpha - \cos\beta}\right)_{(hQ)} \qquad \ldots(1.7)$$

where hQ is the associated displacement function and satisfies the differential equation

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}(hQ) = \left(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\beta^2} - 1\right)(h\chi). \qquad \ldots(1.8)$$

Method of Solution

Let the inclusions be defined by

$$\alpha = \pm\alpha_1,\ \alpha_1 > 0. \qquad \ldots(2.1)$$

Section I

For an isolated force X applied at the origin in the positive direction of x-axis,

$$\chi_0 = -\frac{X}{2\pi}(y\theta - \nu x \log r) \qquad \ldots(2.2)$$

where (x, y) and (r, θ) are the cartesian and polar coordinates of any point in the plate.

$$\therefore h\chi_0 = -\frac{X}{2\pi}\left[\sinh\alpha \tan^{-1}\left(\frac{\sinh\alpha}{\sin\beta}\right) - \tfrac{1}{2}\nu \sin\beta \log a - \tfrac{1}{2}\nu \sin\beta \log\left(\frac{\cosh\alpha + \cos\beta}{\cosh\alpha - \cos\beta}\right)\right]$$

$$= \frac{X}{2\pi}\left[-\frac{\pi}{2}\sinh\alpha + \sum_1^\infty K_n(\alpha)\sin n\beta\right] \qquad \ldots(2.3)$$

where for $\alpha > 0$

$$\left.\begin{aligned} K_1(\alpha) &= -\left(\frac{\nu}{2} + 1\right)e^{-2\alpha} + 1 + \nu \log a \\ \text{and for} \quad n \geqslant 2,\ K_n(\alpha) &= -e^{-n\alpha}[\{(-1)^n - 1\}\frac{\sinh\alpha}{n} - \frac{2\nu}{n^2-1}(n\sinh\alpha \\ &\qquad + \cosh\alpha)]. \end{aligned}\right\} \qquad \ldots(2.4)$$

For the complete stress function we have to add to χ_0, a stress function χ_1 which will give no stress at infinity and satisfy eqn. (1.3). We assume,

$$h\chi_1 = \frac{X}{2\pi}\sum_1^\infty \phi_n(\alpha)\sin n\beta \qquad \ldots(2.5)$$

where

$$\left.\begin{aligned} \phi_1(\alpha) &= A_1 \cosh 2\alpha \\ \text{and} \quad \phi_n(\alpha) &= A_n \cosh(n+1)\alpha + B_n \cosh(n-1)\alpha,\ n \geqslant 2 \end{aligned}\right\}. \qquad \ldots(2.6)$$

The associated displacement function hQ_1 corresponding to $h\chi_1$ is given by

$$hQ_1 = \frac{X}{2\pi}\sum_1^\infty \psi_n(\alpha)\cos n\beta \qquad \ldots(2.7)$$

where

$$\left.\begin{aligned} &\psi_1(\alpha) = -2A_1 \sinh 2\alpha \\ \text{and}\quad &\psi_n(\alpha) = -2A_n \{\sinh(n+1)\alpha + B_n \sinh(n-1)\alpha,\}\ n \geqslant 2. \end{aligned}\right] \quad \ldots(2.8)$$

The complete stress function is given by

$$h\chi = h\chi_0 + h\chi_1. \quad \ldots(2.9)$$

The stresses $\widehat{\alpha\alpha}$, $\widehat{\alpha\beta}$ and displacements u_α, u_β corresponding to $h\chi$ are given by

$$\frac{a\widehat{\alpha\alpha}}{X/2\pi} = \sum_1^\infty [(1-n^2)\cosh\alpha\{\phi_n(\alpha) + K_n(\alpha)\} - \sinh\alpha\{\phi_n''(\alpha) + K_n'(\alpha)\} + \tfrac{1}{2}(n-1)(n-2)\{\phi_{n-1}(\alpha) + K_{n-1}(\alpha)\} + \tfrac{1}{2}(n+1)(n+2)\{\phi_{n+1}(\alpha) + K_{n+1}(\alpha)\}]\sin n\beta \quad \ldots(2.10)$$

$$\frac{a\widehat{\alpha\beta}}{X/2\pi} = \phi_1'(\alpha) + K_1'(\alpha) + [-\cosh\alpha\{\phi_1'(\alpha)\} + K_1'(\alpha)\} + 2\{\phi_2'(\alpha) + K_2'(\alpha)\}]\cos n\beta + \sum_2^\infty \Big[-n\cosh\alpha\{\phi_n'(\alpha) + K_n'(\alpha)\} + \tfrac{1}{2}(n-1)\{\phi_{n-1}'(\alpha) + K_{n-1}'(\alpha)\} + \tfrac{1}{2}(n+1)\{\phi_{n+1}'(\alpha) + K_{n+1}'(\alpha)\}\Big]\cos n\beta \quad \ldots(2.11)$$

$$\frac{Eh_{u\alpha}}{X/2\pi a} = (1+\nu)\left[\frac{\pi}{2}(1-\cosh\alpha\cos\beta) + \sum_1^\infty L_n(\alpha)\sin n\beta\right] + \sum_1^\infty [(1-\nu)\{\phi_n'(\alpha)\cosh\alpha - \tfrac{1}{2}\phi_{n-1}'(\alpha) - \tfrac{1}{2}\phi_{n+1}'(\alpha) - \sinh\alpha\,\phi_n(\alpha)\} + n\,\psi_n(\alpha)\cosh\alpha - \tfrac{1}{2}(n-2)\psi_{n-1}(\alpha) - \tfrac{1}{2}(n+2)\psi_{n+1}(\alpha)]\sin n\beta \quad \ldots(2.12)$$

$$\frac{Ehu_\beta}{X/2\pi} = (1+\nu)\left[M_0(\alpha) - \frac{\pi}{2}\sinh\alpha\sin\beta + \sum_1^\infty M_n(\alpha)\cos n\beta\right]$$

$$-\{(1-\nu)\phi_1(\alpha) + \tfrac{1}{2}\psi_1'(\alpha)\} + \sum_1^\infty [(1-\nu)\{n\cosh\alpha$$

$$\phi_n(\alpha) - \tfrac{1}{2}(n-2)\phi_{n-1}(\alpha) - \tfrac{1}{2}(n+2)\phi_{n+1}(\alpha)\}$$

$$+ \cosh\alpha\,\psi_n'(\alpha) - \tfrac{1}{2}\psi_{n-1}'(\alpha)\,\tfrac{1}{2}\psi_{n+1}'(\alpha) - \sinh\alpha\,\psi_n(\alpha)]$$

$$\cos n\beta \qquad \ldots(2.13)$$

where $L_1(\alpha) = 2e^{-\alpha}(\sinh 2\alpha - 1) - \frac{3\nu}{2}e^{-2\alpha}\sinh\alpha + \nu\sinh\alpha\log a$

for

$$n \geqslant 2,\ L_n(\alpha) = e^{-n\alpha}[\{(-1)^n - 1\}(\frac{1}{n} - \sinh\alpha) + (-1)^n\nu\sinh 2\alpha$$

$$+ \frac{e^{-n\alpha}}{n^2-1}[\{(-1)^n + 1\}\cosh\alpha(n\cosh\alpha + \sinh$$

$$\alpha) + 2\nu\sinh\alpha(n\sinh\alpha + \cosh\alpha)]$$

$$M_0(\alpha) = \nu(\log a - \tfrac{1}{2} - 2e^{-\alpha}\cosh\alpha) + (1 - e^{-2\alpha})$$

$$M_1(\alpha) = \nu(-\cosh\alpha\log a + 2e^{-\alpha} - \tfrac{3}{2}e^{-2\alpha}\cosh\alpha)$$

$$-\left(\frac{e^{-2\alpha}}{2},\sinh\alpha + e^{-\alpha}\sinh 2\alpha\right)$$

$$M_2(\alpha) = \frac{\nu}{2} + N_2(\alpha)$$

for $n \geqslant 2$,

$$N_n(\alpha) = \frac{2\nu}{n} + (-1)^n(1-\nu)\sinh 2\alpha\, e^{-n\alpha}$$

$$-[2\nu\cosh\alpha(n\cosh\alpha + \sinh\alpha)$$

$$+ \{(-1)^n + 1\}\sinh\alpha(n\sinh\alpha + \cosh\alpha)]\frac{e^{-n\alpha}}{n^2-1}$$

and for $n \geqslant 3$, $N_n(\alpha) = M_n(\alpha)$.

Consideration, analogous to those introduced earlier, regarding the single valuedness of the displacement field, leads us to take the stress function within the inclussions, in the form,

$$h\bar{\chi} = \frac{X}{2\pi}\left[\bar{A}_0 \sinh \alpha + \sum_1^\infty \bar{\phi}_n(\alpha) \sin n\beta\right] \qquad \ldots(2.14)$$

where

$$\left.\begin{aligned} &\bar{\phi}_1(\alpha) = A_1 e^{-2\alpha} \text{ for } \alpha > 0 \\ &\qquad = \bar{A}_1 e^{2\alpha} \text{ for } \alpha < 0 \\ &\text{and} \\ &\text{for } n \geqslant 2,\ \bar{\phi}_n = \bar{A}_n e^{-(n+1)\alpha} + \bar{B}_n e^{-(n-1)\alpha} \text{ for } \alpha > 0 \\ &\qquad = \bar{A}_n e^{(n+1)\alpha} + \bar{B}_n e^{(n-1)\alpha}, \text{ for } \alpha < 0. \end{aligned}\right\} \qquad \ldots(2.15)$$

The associated displacement function corresponding to $h\bar{\chi}$ is

$$h\bar{Q} = \frac{X}{2\pi}\sum_1^\infty \bar{\psi}_n(\alpha) \cos n\beta \qquad \ldots(2.16)$$

where

$$\left.\begin{aligned} &\bar{\psi}_1(\alpha) = 2\bar{\phi}_1(\alpha) \\ &\text{and for} \\ &n \geqslant 2,\ \bar{\psi}_n(\alpha) = 2\bar{\phi}_n(\alpha). \end{aligned}\right] \qquad \ldots(2.17)$$

The stresses within the inclusions are

$$\bar{\widehat{a\alpha\alpha}} = \frac{X}{2\pi}\sum_1^\infty [(1 - n^2)\cosh \alpha\, \bar{\phi}_n(\alpha) + \tfrac{1}{2}(n-1)(n-2)\bar{\phi}_{n-1}(\alpha)$$

$$+ \tfrac{1}{2}(n+1)(n+2)\bar{\phi}_{n+1}(\alpha) - \sinh \alpha\, \bar{\phi}'_n(\alpha)] \sin n\beta \qquad \ldots(2.18)$$

$$\bar{\widehat{a\alpha\beta}} = \frac{X}{2\pi}\left[\tfrac{1}{2}\bar{\varphi}'_1(\alpha) + \sum_1^\infty \{\tfrac{1}{2}(n-1)\bar{\phi}'_{n-1}(\alpha) + \tfrac{1}{2}(n+1)\bar{\phi}'_{n+1}(\alpha)\right.$$

$$\left. - n\cosh \alpha\, \bar{\phi}'_n(\alpha)\} \cos n\beta\right]. \qquad \ldots(2.19)$$

The displacements $\bar{u}_\alpha$ and $\bar{u}_\beta$ within the inclusions are for $\alpha > 0$,

$$\frac{\bar{E}h\bar{u}_\beta}{X/2\pi a} = -\bar{A}_0(1 - \bar{\nu})\sinh \alpha \sin \beta + \tfrac{1}{2}\bar{\psi}'_1(\alpha) - (1 - \bar{\nu})\bar{\phi}_1(\alpha)$$

(equation continued on p. 1121)

$$+ \sum_{1}^{\infty} [1 - \bar{\nu}) \{n \cosh \alpha \, \bar{\phi}_n (\alpha) - \tfrac{1}{2} (n - 2) \, \bar{\phi}_{n-1} (\alpha)$$

$$- \tfrac{1}{2} (n + 2) \, \bar{\phi}_{n+1} (\alpha)\}$$

$$- \cosh \alpha \, \bar{\psi}'_n (\alpha) + \tfrac{1}{2} \, \bar{\psi}'_{n-1} (\alpha) + \tfrac{1}{2} \, \bar{\psi}'_{n+1} (\alpha) + \sinh \alpha \, \bar{\psi}_n (\alpha)]$$

$$\times \cos n\beta \qquad \ldots(2.20)$$

$$\frac{\bar{E}h\bar{u}_\alpha}{X/2\pi a} = (1 - \bar{\nu}) \, \bar{A}_0 (1 - \cosh \alpha \cos \beta) + \sum_{1}^{\infty} [(1 - \bar{\nu}) \{\cosh \alpha \, \bar{\phi}'_n (\alpha)$$

$$- \sinh \alpha \, \bar{\phi}_n (\alpha) - \tfrac{1}{2} \, \bar{\phi}'_{n-1} (\alpha) - \tfrac{1}{2} \, \bar{\phi}'_{n+1} (\alpha)\}$$

$$- n \cosh \alpha \, \bar{\psi}_n (\alpha) + \tfrac{1}{2} (n - 2) \, \bar{\psi}_{n-1} (\alpha) + \tfrac{1}{2} (n + 2) \, \bar{\psi}_{n+1} (\alpha)] \sin n\beta.$$

Boundary Conditions

In case of perfect bond between the plate and the inclusions, we must have on $\alpha = \pm \alpha_1$,

$$\widehat{\alpha\alpha} = \widehat{\bar{\alpha}\alpha}, \; \widehat{\alpha\beta} = \widehat{\bar{\alpha}\beta} \qquad \ldots(3.1)$$

$$u_\alpha = \bar{u}_\alpha, \; u_\beta = \bar{u}_\beta. \qquad \ldots(3.2)$$

By eqns (2.10), (2.11), (2.18), (2.19) and (3.1), we have

$$\therefore \tfrac{1}{2} \, \bar{\psi}_n (\alpha_1) = \bar{\phi}_n (\alpha_1) = \phi_n (\alpha_1) + K_n (\alpha_1) \qquad \ldots(3.3)$$

$$\times \tfrac{1}{2} \, \bar{\psi}'_n (\alpha_1) = \bar{\phi}'_n (\alpha_1) = \phi'_n (\alpha_1) + K'_n (\alpha_1), \qquad \ldots(3.4)$$

for $n = 1, 2, 3, \ldots$.

By eqns. (2.12), (2.13), (2.20), (2.21) and (3.2) we have

$$(1 + \nu) \frac{\pi}{2} = \frac{E}{\bar{E}} (1 - \bar{\nu}) \, \bar{A}_0 \qquad \ldots(3.5)$$

$$A_1 = [(1 + \nu) \, M_0 (\alpha_1) + \frac{E}{\bar{E}} \{(1 - \bar{\nu}) \, K_1 (\alpha_1) - K'_1 (\alpha_1)\}] / V'_2 (\alpha_1) \qquad \ldots(3.6)$$

$$3A_2 V_3(\alpha_1) + B_2\{3V_1(\alpha_1) - a' \sinh \alpha_1\} - A_1\{2V_2(\alpha_1)\cosh \alpha_1 + a' \sinh \alpha_1\} - \frac{E}{\bar{E}}[(1-\bar{\nu})\{K_1'(\alpha_1)\cosh \alpha_1 + \tfrac{1}{2}K_2'(\alpha_1) + K_1(\alpha_1)\sinh \alpha_1\} + 2K_1(\alpha_1)\cosh \alpha_1 - 3K_2(\alpha_1)] - (1+\nu) \times L_1(\alpha_1) = 0 \quad \ldots(3.7)$$

$$A_2 V_3'(\alpha_1) + B_2\{V_1'(\alpha_1) + a' \cosh \alpha_1\} - A_1\{V_2'(\alpha_1)\cosh \alpha_1 + 2V_1' \times (\alpha_1) - a' \cosh \alpha_1\} + \frac{E}{\bar{E}}[(1-\bar{\nu})\{K_1(\alpha_1)\cosh \alpha_1 - \tfrac{3}{2}K_2(\alpha_1)\} - 2K_1'(\alpha_1)\cosh \alpha_1 + K_2'(\alpha_1) + 2K_1(\alpha_1)\sinh(\alpha_1] - (1+\nu)M_1(\alpha_1) = 0 \quad \ldots(3.8)$$

and for $n \geqslant 2$, we have,

$$(n+2)V_{n+2}(\alpha_1)A_{n+1} + \{(n+2)V_n(\alpha_1) - a' \sinh n\alpha_1\}B_{n+1} - A_n[2nV_{n+1}(\alpha_1)\cosh \alpha_1 + a' \sinh n\alpha_1] - B_n[2n \cosh \alpha_1 V_{n-1}(\alpha_1) - a' \sinh n\alpha_1] + A_{n-1}[(n-2)V_n(\alpha_1) + a' \sinh n\alpha_1] + (n-2) \times V_{n-2}(\alpha_1)B_{n-1} + \frac{E}{\bar{E}}[(1-\bar{\nu})\{K_n'(\alpha_1)\cosh \alpha_1 - \tfrac{1}{2}K_{n+1}'(\alpha_1) - \tfrac{1}{2}K_{n+1}'(\alpha_1) - \sinh \alpha_1 K_n(\alpha_1)\} - 2n \cosh \alpha_1 K_n(\alpha_1) + (n+2) \times K_{n-1}(\alpha_1)(n+2)K_{n+1}(\alpha_1)] - (1+\nu)L_n(\alpha_1) = 0 \quad \ldots(3.9)$$

$$V_{n+1}'(\alpha_1)A_{n+1} + \{V_n'(\alpha_1) + a' \cosh n\alpha_1\}B_{n+1} - [2\{\cosh \alpha_1 V_{n+1}'(\alpha_1) - \sinh \alpha_1 V_{n+1}(\alpha_1)\} - a' \cosh n\alpha_1]A_n - [2\{\cosh \alpha_1 V_{n-1}'(\alpha_1) - \sinh \alpha_1 V_{n-1}'(\alpha_1)\} + a' \cosh n\alpha_1]B_n + [V_n(\alpha_1) - a' \cosh n\alpha_1] \times A_{n-1} + V_{n-2}'(\alpha_1)B_{n-1} + \frac{E}{\bar{E}}[(1-\bar{\nu})\{n \cosh \alpha_1 K_n(\alpha_1) - \tfrac{1}{2}(n-2)K_{n-1}(\alpha_1) - \tfrac{1}{2}(n+2)K_{n+1}(\alpha_1)\} - 2\cosh \alpha_1 K_n'(\alpha_1) + K_{n-1}'(\alpha) + K_{n+1}'(\alpha_1) + 2K_n(\alpha_1)\sinh \alpha_1] - (1+\nu)M_n(\alpha_1) = 0 \quad \ldots(3.10)$$

where

$$a' = 1 - \nu - \frac{E}{\bar{E}}(1 - \bar{\nu})$$

$$b = 1 + \nu - \frac{E}{\bar{E}}(1 - \bar{\nu})$$

$$V_n(\alpha) = -\frac{b}{2}\sinh n\alpha + \frac{E}{\bar{E}}\cosh n\alpha$$

$$V_n'(\alpha) = n\left(-\frac{b}{2}\cosh n\alpha + \frac{E}{\bar{E}}\sinh n\alpha\right).$$

Thus eqn. (3.5) gives $\bar{A}_0$; eqn. (3.6) gives A_1; eqns. (3.7) and (3.8) give A_2, B_2; and eqns. (3.9), (3,10) give A_n, B_n for $n = 3, 4, \ldots$: Finally the equations (2.10) and (2.11) give the stresses outside while eqns. (2.12) and (2.19) give those inside the inclusions.

Section II

For an isolated force[3] Y applied at the origin in the positive direction of y-axis,

$$\chi_0 = \frac{Y}{2\pi}(x\theta + \nu y \log r). \qquad \ldots(4.1)$$

Since the stresses and displacements corresponding to the stress function $\cosh \alpha - \cos \beta$ are zero, so any constant multiple of it is omitted from the stress function, and finally we have for $\alpha > 0$

$$h\chi_0 = \frac{Y}{2\pi}\left[\sinh \alpha\,(1 + \nu \log a) + \frac{\pi}{2}\sin \beta + \sum_1^\infty G_n(\alpha)\cos n\beta\right] \qquad \ldots(4.2)$$

where

and for

$$\left.\begin{aligned} &G_1(\alpha) = \nu(1 - e^{-2\alpha}) - 1 \\ &n \geqslant 2,\ G_n(\alpha) = \tfrac{1}{2}\{(-1)^n + 1\}\left[\frac{e^{-(n-1)\alpha}}{n-1} - \frac{e^{-(n+1)\alpha}}{n+1}\right] \\ &\qquad\qquad + \frac{\nu}{n}\left[e^{-(n-1)\alpha} - e^{-(n+1)\alpha}\right]. \end{aligned}\right\} \qquad \ldots(4.3)$$

The associated displacement function hQ_0 is given by

$$\frac{hQ_0}{Y/2\pi} = -2\sum_1^\infty G_n(\alpha)\sin n\beta. \qquad \ldots(4.4)$$

Here we assume,

$$h\chi_1 = = \frac{Y}{2\pi}\left[B_0\,\alpha\,(\cosh\alpha - \cos\beta) + \sum_1^{\infty}\phi_n(\alpha)\cos n\beta\right] \qquad ...(4.5)$$

where

and for
$$\left.\begin{aligned}&\phi_1(\alpha) = C_1\sinh 2\alpha\\ &n \geqslant 2,\ \phi_n(\alpha) = C_n\sinh(n+1)\alpha + D_n\sinh(n-1)\alpha\end{aligned}\right] \quad ..(4.6)$$

$$\therefore\ hQ_1 = \frac{Y}{2\pi}[2B_0\,\beta\cosh\alpha + \sum_1^{\infty}\psi_n(\alpha)\sin n\beta] \qquad ...(4.7)$$

where

and for
$$\left.\begin{aligned}&\psi_1(\alpha) = 2C_1\cosh 2\alpha\\ &n > 2,\ \psi_n(\alpha) = 2\,[C_n\cosh(n+1)\alpha + D_n\cosh(n-1)\alpha]\end{aligned}\right]. \qquad ...(4.8)$$

The complete stress function is given by

$$h\chi = h\chi_0 + h\chi_1. \qquad ...(4.9)$$

The stresses $\widehat{\alpha\alpha}$, $\widehat{\alpha\beta}$ and the displacements u_α, u_β corresponding to $h\chi$ are given by

$$\begin{aligned}\frac{\widehat{a\alpha\alpha}}{Y/2\pi} = &-B_0\sinh\alpha\,(\cosh\alpha - \cos\beta) + \phi_1(\alpha) + G_1(\alpha)\\ &+ \sum_1^{\infty}[(1-n^2)\cosh\alpha\,\{\varphi_n(\alpha) + G_n(\alpha)\} + \tfrac{1}{2}(n-1)(n-2)\\ &\times\{\phi_{n-1}(\alpha) + G_{n-1}(\alpha)\} + \tfrac{1}{2}(n+1)(n+2)\{\phi_{n+1}(\alpha) + G_{n+1}(\alpha)\}\\ &- \sinh\alpha\,\{\phi_n'(\alpha) + G_n'(\alpha)\}]\cos n\beta \qquad ...(4.10)\end{aligned}$$

$$\begin{aligned}\frac{\widehat{a\alpha\beta}}{Y/2\pi} = &-B_0\cosh\alpha\sin\beta + \frac{B_0}{2}\sin 2\beta + \sum_1^{\infty}[n\cosh\alpha\,\{\phi_n'(\alpha)\\ &+ G_n'(\alpha)\} - \tfrac{1}{2}(n-1)\,\{\phi_{n-1}'(\alpha) + G_{n-1}'(\alpha)\} - \tfrac{1}{2}(n+1)\\ &\times\{\phi_{n+1}'(\alpha) + G_{n+1}'(\alpha)\}]\sin n\beta \qquad ...(4.11)\end{aligned}$$

$$\frac{aEhu_\alpha}{Y/2\pi} = B_0 [\tfrac{1}{2}(1-\nu) - (1+\nu)\cosh^2\alpha + 2\nu\cosh\alpha\cos\beta + \tfrac{1}{2}(1-\nu) \times \cos 2\beta + 2\beta\cosh\alpha\sin\beta] - \tfrac{1}{2}(1-\nu)\phi_1'(\alpha) + \psi_1(\alpha) + \frac{1+\nu}{2}G_1'(\alpha) + (1-\nu)(1+\nu\log a)(1-\cosh\alpha\cos\beta) - \frac{\pi}{2}(1-\nu)\sinh\alpha\sin\beta + \sum_1^\infty ((1-\nu)[\cosh\alpha\{\phi_n'(\alpha) + G_n'(\alpha)\} - \sinh\alpha\{\phi_n(\alpha) + G_n(\alpha)\} + \tfrac{1}{2}\{\phi_{n-1}'(\alpha) + G_{n\alpha 1}'(\alpha)\} - \tfrac{1}{2}\{\phi_{n+1}'(\alpha) + G_{n+1}'(\alpha)\}] + n\cosh\alpha\{-\psi_n(\alpha) + 2G_n(\alpha)\} + (n+2)\{\tfrac{1}{2}\psi_{n+1}(\alpha) - G_{n+1}(\alpha)\} + (n-2)\{\tfrac{1}{2}\psi_{n-1} - G_{n-1}(\alpha)\}) \times \cos n\beta \quad ...(4.12)$$

$$\frac{aEhu_\beta}{Y/2\pi} = -B_0\beta\sinh\alpha\cos\beta + (1-\nu)[-\frac{\pi}{2}(1+\cosh\alpha) + \frac{\pi}{2}\cosh\alpha\cos\beta - \frac{\pi}{4}(\cosh\alpha - 1)\cos 2\beta - \sinh\alpha(1+\nu\log a)\sin\beta] + \sum_1^\infty ((1-\nu)[-n\cosh\alpha\{\phi_n(\alpha) + G_n(\alpha)\} + \tfrac{1}{2}(n-2) \times \{\phi_{n-1}(\alpha) + G_{n-1}(\alpha)\} + \tfrac{1}{2}(n+2)\{\phi_{n+1}(\alpha) + G_{n+1}(\alpha)\}) + \cosh\alpha\{\psi_n'(\alpha) - 2G_n'(\alpha)\} + \sinh\alpha\{-\psi_n(\alpha) + 2G_n(\alpha)\} + \{-\tfrac{1}{2}\psi_{n+1}'(\alpha) + G_{n-1}'(\alpha)\} + \{-\tfrac{1}{2}\psi_{n+1}'(\alpha) + G_{n-1}'(\alpha)\} \sin n\beta. \quad ...(4.13)$$

Within the inclusions, we take

$$h\bar{\chi} = \frac{Y}{2\pi}\left[\bar{B}_0\sin\beta + \sum_1^\infty \bar{\phi}_n(\alpha)\cos n\beta\right] \quad ...(4.14)$$

where

$$\left.\begin{aligned}&\bar{\phi}_1(\alpha) = C_1 e^{-2\alpha}, \text{ for } \alpha > 0\\ &= \bar{C}_1 e^{2\alpha} \text{ for } \alpha < 0\end{aligned}\right\}$$

and for

$$\left.\begin{aligned}&n \geqslant 2,\ \bar{\phi}_n(\alpha) = \bar{C}_n e^{-(n+1)\alpha} + \bar{D}_n e^{-(n-1)\alpha}, \text{ for } \alpha > 0\\ &\qquad = \bar{C}_n e^{(n+1)\alpha} + \bar{D}_n e^{(n-1)\alpha}, \text{ for } \alpha < 0\end{aligned}\right\} \quad \ldots(4.15)$$

$$\therefore h\bar{Q} = \frac{Y}{2\pi} \sum_1^{\infty}{}' \bar{\psi}_n(\alpha) \sin n\beta \quad \ldots(4.16)$$

where

$$\left.\begin{aligned}&\bar{\psi}_1(\alpha) = -2\bar{\phi}_1(\alpha)\\ \text{and for}\quad &\\ &n \geqslant 2,\ \bar{\psi}_n(\alpha) = -2\bar{\phi}_n(\alpha)\end{aligned}\right]. \quad \ldots(4.17)$$

The stresses and displacements corresponding to $h\bar{\chi}$ are

$$\frac{\bar{\bar{a}}_{\alpha\alpha}}{Y/2\pi} = \bar{\phi}_1(\alpha) + \sum_1^{\infty} [1 - n^2) \cosh \alpha\, \bar{\phi}_n(\alpha) + \tfrac{1}{2}(n-1)(n-2)\bar{\phi}_{n-1}(\alpha)$$

$$+ \tfrac{1}{2}(n+1)(n+2)\bar{\phi}_{n+1}(\alpha) - \sinh \alpha \bar{\phi}_n'(\alpha)] \operatorname{coc} n\beta. \quad \ldots(4.18)$$

$$\frac{\bar{\bar{a}}_{\alpha\beta}}{Y/2\pi} = \sum_1^{\infty} (n \cosh \alpha\ \bar{\phi}_n'(\alpha) - \tfrac{1}{2}(n -)\bar{\phi}_{n-1}'(\alpha) - \tfrac{1}{2}(n+1)\bar{\phi}_{n+1}'$$

$$(\alpha)] \sin n\beta. \quad \ldots(4.19)$$

$$\frac{ah\bar{E}\bar{u}_{\alpha}}{Y/2\pi} = \frac{1+\bar{\nu}}{2}\bar{\phi}_1'(\alpha) + (1-\bar{\nu})[-\bar{B}_0 \sinh \alpha \sin \beta + \sum_1^{\infty} \{\cosh \alpha$$

$$\alpha\, \bar{\phi}_n'(\alpha) - \tfrac{1}{2}\bar{\phi}_{n-1}'(\alpha) - \tfrac{1}{2}\bar{\phi}_{n+1}'(\alpha) - \sinh \alpha\, \bar{\phi}_n(\alpha)\} \cos n\beta]$$

$$+ \sum_1^{\infty} [-\cosh \alpha\ \bar{\psi}_n(\alpha) + \tfrac{1}{2}(n-2)\ \bar{\psi}_{n-1}(\alpha)$$

$$+ \tfrac{1}{2}(n-2)\bar{\psi}_{n+1}(\alpha)] \cos n\beta \quad \ldots(4.20)$$

$$\frac{ah\bar{E}\bar{u}_{\beta}}{y/2\pi} = (1-\bar{\nu})[-\bar{B}_0(1 - \cosh \alpha \cos \beta) + \sum_1^{\infty} \{-n \cosh \alpha\, \bar{\phi}_n(\alpha)$$

$$+ \tfrac{1}{2}(n-1)\bar{\phi}_{n-1}(\alpha) + \tfrac{1}{2}(n+2)\bar{\phi}_{n+1}(\alpha)\} \sin n\beta]$$

(equation continued on p. 1127)

$$+ \sum_{1}^{\infty} [\cosh \alpha \, \bar{\psi}'_n (\alpha) - \tfrac{1}{2} \, \bar{\psi}'_{n-1} (\alpha) - \tfrac{1}{2} \, \bar{\psi}'_{n+1} (\alpha) - \sinh \alpha$$

$$\bar{\psi}_n (\alpha)] \sin n\beta \cdot \qquad \ldots(4.21)$$

Boundary Conditions

We have on the boundary $\alpha = \pm \alpha_1$,

$$\widehat{\alpha\alpha} = \bar{\widehat{\alpha\alpha}}; \quad \widehat{\alpha\beta} = \bar{\widehat{\alpha\beta}} \qquad \ldots(5.1)$$

$$u_\alpha = \bar{u}_\alpha, \; u_\beta = \bar{u}_\beta. \qquad \ldots(5.2)$$

Therefore from (4.10), (4.11), (4.18), 4.19) and (5.1) we have,

$$B_0 = 0 \qquad \ldots(5.3)$$

$$\left.\begin{aligned} \bar{\phi}_n (\alpha_1) &= \phi_n (\alpha_1) + G_n (\alpha_1) \\ \bar{\phi}'_n (\alpha_1) &= \phi'_n (\alpha) + G'_n (\alpha_1) \end{aligned}\right] \qquad \ldots(5.4)$$

for $n = 1, 2, 3, \ldots$

From eqns. (4.12), (4.13), (4.20), (4.21) and (5.2), and using (5.4), we have

$$\bar{B}_0 = (1 - \nu) \, \pi/2 \; (1 - \bar{\nu}) \, \frac{E}{\bar{E}} \qquad \ldots(5.5)$$

$$C_1 = - \left[\tfrac{1}{2} \, G'_1 (\alpha_1) + \frac{(1 - \nu)(1 + \nu \log a)}{1 + \nu - \dfrac{E}{\bar{E}} (1 + \bar{\nu})} \right] / \cos 2\alpha_1 \qquad \ldots(5.6)$$

$$3C_2 (C_0 - a') \cosh 3\alpha_1 + D_2 (3C_0 - a') \cosh \alpha_1 + 2C_1 (a' - C_0 \cosh 2\alpha_1)$$

$$\times \cosh \alpha_1 + (2a' - C_0) \cosh \alpha_1 \, G'_1 (\alpha_1) - 2 \sinh \alpha_1 \, G_1 (\alpha_1)$$

$$- 3C_0 \, G_2 (\alpha_1) - a' \, G'_2 (\alpha_1) - 2 (1 - \nu)(1 + \nu \log a) \cosh \alpha_1 = 0 \qquad \ldots(5.7)$$

$$3C_2 (a' - C_0) \sinh 3\alpha_1 + D_2 (3a' - C_0) \sinh \alpha_1 + 2C_1 [(C_0 - a') \cosh$$

$$\alpha_1 \sinh 2\alpha_1 + C_0 \sinh \alpha_1] + 3G_2 (\alpha_1) \left(a' - \frac{C_0}{4}\right)$$

$$- \tfrac{1}{2} \left(1 + \frac{C_0}{2}\right) G''_2 (\alpha_1) - 2a' \cosh \alpha_1 \, G_1 (\alpha_1) - C_0 \sinh \alpha_1 \, G'_1$$

(equation continued on p. 1128)

$$(\alpha_1) + C_0 \cosh \alpha_1 \, G_1^{''}(\alpha_1) - 2(1-\nu)(1+\nu \log a) \sinh \alpha_1 = 0 \quad \ldots (5.8)$$

and for $n \geqslant 2$, we have,

$$R'_{n+2}(\alpha_1) C_{n+1} + D_{n+1}\left[\frac{n+2}{n} R'_n(\alpha_1) + a' \operatorname{cosn} n\alpha_1\right]$$

$$- C_n\left[\frac{2n}{n+1} \cosh \alpha_1 \, R'_{n+1}(\alpha_1) + a' \cosh n\alpha_1\right]$$

$$- D_n\left[\frac{2n}{n-1} \cosh \alpha_1 \, R'_{n-1}(\alpha_1) + a' \cosh n\alpha_1\right]$$

$$+ C_{n-1}\left[\frac{n-2}{n} R'_n(\alpha_1) - a' \cosh n\alpha_1\right] + D_{n-1} R'_{n-2}(\alpha_1)$$

$$+ a'\left[\cosh \alpha_1 \, G'_n(\alpha_1) - \sinh \alpha_1 . \, G_n(\alpha_1) - \tfrac{1}{2} G'_{n-1}(\alpha_1)\right.$$

$$\left. - \tfrac{1}{2} G'_{n+1}(\alpha_1)\right] + C_0\left[n \cosh \alpha_1 \, G_n(\alpha_1) - \frac{n+2}{2} G_{n+1}(\alpha_1)\right.$$

$$\left. - \frac{n-2}{2} G_{n-1}(\alpha_1)\right] = 0 \quad \ldots (5.9)$$

$$(n+2) R_{n+2}(\alpha_1) C_{n+1} + D_{n+1}[nR_n(\alpha_1) - a' \sinh n\alpha_1] - C_n[a' \sinh n\alpha_1$$

$$+ 2(n+1) \cosh \alpha_1 \, R_{n+1}(\alpha_1) - \frac{2 \sinh \alpha_1}{n+1} R'_{n+1}(\alpha_1)]$$

$$- D_n[2(n-1) \cosh \alpha_1 \, R_{n-1}(\alpha_1) - \frac{2 \sinh \alpha_1}{n-1} R'_{n-1}(\alpha_1)$$

$$- a' \sinh n\alpha_1] + C_{n-1}[nR_n(\alpha_1) + a' \sinh n\alpha_1]$$

$$+ (n-2) R_{n-2}(\alpha_1) D_{n-1} + a'[n \cosh \alpha_1 \, G_n(\alpha_1) - \frac{n-2}{2}$$

$$\times G_{n-1}(\alpha_1) - \frac{n+2}{2} G_{n+1}(\alpha_1)] + C_0[\cosh \alpha_1 \, G'_n(\alpha_1)$$

$$- \sinh \alpha_1 \, G_n(\alpha_1) - \tfrac{1}{2} G'_{n-1}(\alpha) - \tfrac{1}{2} G'_{n+1}(\alpha_1))] = 0 \quad \ldots (5.10)$$

where $C_0 = 2(1 - E/\bar{E})$

$$R_n(\alpha) = \frac{b}{2} \sinh n\alpha + \frac{E}{\bar{E}} \cosh n\alpha$$

$$R'_n(\alpha) = n\left(\frac{b}{2} \cosh n\alpha + \frac{E}{\bar{E}} \sinh n\alpha\right).$$

Thus (5.3) gives B_0, the equation (5.5) gives $\bar{B}_0$, the equation (5.6) gives C_1, the equations (5.7), (5.8) give C_2, D_2; and equations (5.9), (5.10) give C_n, D_n for $n = 3, 4, \ldots$ Finally, the equations (4.10), (4.11) give the stresses outside and eqns. (4.18) and (4.19) give those inside the inclusions.

Numerical Calculation

Numerical Calculations are carried out with $a = 1$, $\nu = \nu = \frac{1}{3}$, $E/\bar{E}$ $\frac{1}{2}$, $\alpha_1 = 0.8$. The circumferential stresses are found Table I and are shown graphically for both the cases against β (Fig. 1, 2).

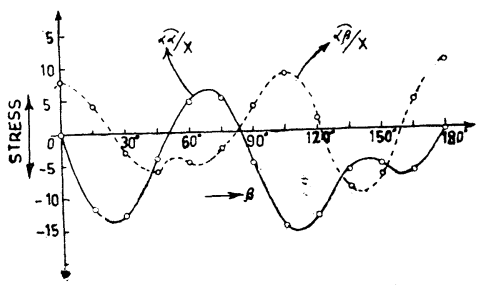

FIG. 1. Stresses on the boundary $\alpha_1 = 0.8$ under the force X along the x-axis.

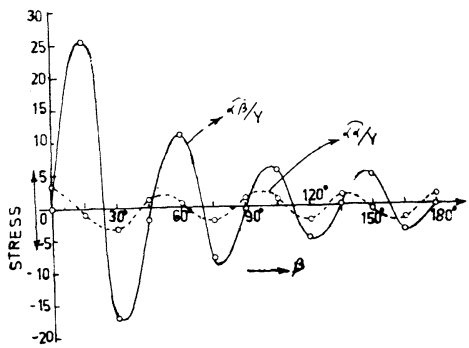

FIG. 2. Stresses on the boundary $\alpha_1 = 0.8$ under the force Y along the y-axis.

TABLE I

Max. value of the stress	For isolated force X parallel to x-axis	For isolated force Y-parallel to y-axis
$\widehat{\alpha\alpha}$	$15.10X$ at $\beta = 109.5°$	$3.13Y$ at $\beta = 0°$
$\widehat{\alpha\beta}$	$10.47X$ at $\beta = 180°$	$25.64Y$ at $\beta = 15°$

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