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SOME APPLICATIONS OF ARCWISE CONNECTED FUNCTIONS FOR MINIMAX INEQUALITIES AND EQUALITIES

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Some interesting applications of arcwise connected functions, whose properties were discussed by Singh (*J. Optimization Theory Applic.* 41 (1983), 377-87) are given in the area of minimax inequalities and equalities in the present note.

§1. Singh[3] discussed some elementary properties of arcwise connected sets and functions. The purpose of the present note is to study some of the nontrivial applications of arcwise functions in the area of minimax inequalities and equalities. In this context it is to be observed that the arcwise connected property infact is a generalization of the convexity with regard to sets and functions.

§2. *Definition* 2.1—The set $X \subset R^n$ is said to be arcwise connected (AC) if, for every pair of points x^1, x^2 in X, there exists a continuous vector-valued function H_{x^1, x^2}, called an arc, defined on the uuit interval $[0,1] \subset R$ with values in X such that

$$H_{x^1, x^2}(0) = x^1 \text{ and } H_{x^1, x^2}(1) = x^2.$$

For any positive integer k, we let

$$I^k = (a_1, \ldots, a_k) : 0 \leqslant a_i \leqslant 1;$$

i.e., I^k denotes the kth dimensional unit cube. Further let l_i denote the ith unit vector in I^k, i.e.,

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0);$$

i.e., the ith component of e_i is 1 and all other components are zero.

The following proposition is given in Singh[3].

Proposition 2.1—Suppose $X \subset R^n$, then X is AC if and only if, for any positive integer k and $x^1, x^2, \ldots, x^k$ in X, there exists a continuous function $H_{x^1, x^2, \ldots, x^k}$ defined on I^k such that

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$$H_{x^1, x^2, \ldots, x^k}(e_i) = x^i, \text{ for } i = 1, 2, \ldots, k.$$

Definition 2.2—Let $X \subset R^n$ be an arcwise connected set, and let f be a real-valued function defined on X. We say f is arcwise connected convex (CN) if, for every $x^1, x^2 \in X$, there exists an arc H_{x^1, x^2} in X such that

$$f(H_{x^1, x^2}(\theta)) \leqslant (1-\theta) f(x^1) + \theta f(x^2), \text{ for all } \theta,\ 0 \leqslant \theta \leqslant 1.$$

We have similar definition for arcwise connected concave function.

In view of Proposition 2.1 we give the following definition for generalized arcwise connected hull of k points $x^1, x^2, \ldots, x^k$.

Definition 2.3—Let $x^1, x^2, \ldots, x^k \in X \subset R^n$. Then the generalized arcwise connected hull (GACH) of $\{x^1, x^2, \ldots, x^k\}$ is the set $\{H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k) :$

$0 \leqslant a_i \leqslant 1,\ \sum_{i=1}^{k} a_i = 1\} \subset R^n$ for a unique continuous function defined on I^k.

Now let $X \subset R^n$. Consider the set denoted by $GCo\,(X)$ as follows.

$$GCo\,(X) = \{H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k) : 0 \leqslant a_i \leqslant 1,\ \sum_{i=1}^{k} a_i = 1,$$

$$x^1, x^2, \ldots, x^k \text{ are finite number of points in } X\}.$$

Remark : (1) In the above definitions of $GCo\,(X)$ we assume that the function H is unique.

(2) The definition of $GCo\,(X)$ assume the uniqueness of H in the sense that H does not vary from one tuple $\{x^1, \ldots, x^k\}$ to another tuple $\{x'^1, \ldots, x'^k\}$. For example a function of the following type :

$$H_{x^1, \ldots, x^k} = \sum_{i=1}^{k} a_i x^i$$

would work for an ilustration of the functional form of H as in the case of ordinary convex hull.

Definition 2.4—Let $X \subset R^n$. Let f be a real valued function defined on X. Then f is said to be generalized arcwise connected convex on $GCo\,(X)$ is given $x^1, x^2, \ldots, x^k \in X$

$$f[H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k)] \leqslant \sum_{i=1}^{k} a_i f(x^i)$$

with $\sum_{i=1}^{k} a_i = 1$, for all points $H_{x^1, x^2, \ldots, x^k}(a_1, a_2, \ldots, a_k)$ on the GACH of $x^1, x^2, \ldots, x^k$.

We have similar definition for generalized arcwise connected concave function.

§3. As an application of generalized arcwise connected functions we give a minimax inequality derived in Theorem 3.1. We make the following assumptions.

Let $X \subset R^n$ and for each $x \in X$ let a closed set $G(x)$ in R^n be given such that $G(x)$ is compact for at least one $x \in X$. If the GACH of every finite subset $\{x^1, x^2, ..., x^k\}$ of X with respect to the same unique are H is contained in the corresponding union $\bigcup_{i=1}^{n} G(x^i)$ then $\bigcap_{x \in X} G(x) \neq \phi$.

The assumption $\bigcap_{x \in X} G(x) \neq \phi$ with the properties satisfied by $\bigcup_{i=1}^{n} G(x^i)$, for each $x_i \in X$, $i = 1, 2, ..., n$ and other condition mentioned above is a topological property (which we henceforth denote by G-property) which is verified for the case of ordinary convex hull of $\{x_1, x_2, ..., x_n\}$ by Fan's lemmas[1]. The motivation starts at this point when we tag this assumption for the derivation of the minimax inequality given below in Theorem 3.1 for the more general case where X is a compact AC set in R^n with G-property.

Theorem 3.1—Let X be a compact AC set in R^n. Let X also possess G-property. Let f and g be real valued functions on $X \times X$ with the following properties :

(i) For each $x \in X$, $g(x)$ is a lower semi-continuous function on X,

(ii) For each $y \in X$, $f(. , y)$ is a generalized arcwise connected concave funtion on X,

(iii) $g(x, y) \leqslant f(x, y)$ for all $(x, y) \in X \times X$, then the minimax inequality

$$\min_{y \in X} \sup_{x \in X} g(x, y) \leqslant \sup_{x \in X} f(x, x)$$

holds.

Remark : Theorem 3.1 is in fact a generalization of a result (Theorem 1, p.479) of Yen[2] on R^n.

PROOF : Let $t = \sup \{f(x, x) : x \in X\}$. Without loss of generality we may assume that $t < \infty$. For each $x \in X$, let

$$F(x) = \{y \in X : f(x, y) \leqslant t\}$$

$$G(x) = \{y \in X : g(x, y) \leqslant t\}$$

then by (i), (ii) and (iii) we have that, (iv) $G(x)$ is a closed subset of a compact set X and hence $G(x)$ is compact for all $x \in X$.

(v) For any finite set $\{x^1, x^2, ..., x^k\}$ we shall show that GACH of $\{x^1, x^2, ..., x^k\}$ is a subset of $\bigcup_{i=1}^{k} F(x^i)$. Observe that in view of Proposition 2.1 GACH of

$\{x^1, x^2, ..., x^k\}$ is in X. Since $f(. , y)$ is generalized arcwise connected concave for each $y \in X$, we have,

$$\sum_{i=1}^{k} a_i f(x^i, H_{x^1, x^2, ..., x^k}(a_1, a_2, .., a_k))$$

$$\leqslant f(H_{x^1, x^2, ..., x^k}(a_1, a_2, ..., a_k),$$

$$H_{x^2, x^1, ..., x^k}(a_1, a_2, ..., a_k)) \leqslant t. \quad ...(1)$$

From (1) one can see that

$$f(x^i, H_{x^1, x^2, ..., x^k}(a_1, a, ..., a_k)) \leqslant t$$

for at least one index i, which shows that

$$H_{x^1, x^2, ..., x^k}(a_1, a_2, ..., a_k) \in \bigcup_{i=1}^{k} F(x^i).$$

(vi) For each $x \in X$, $F(x) \subset G(x)$, respectively because of (iii). Therefore it follows from (v) and (vi) that

(vii) For any finite subset $\{x^1, x^2, ..., x^k\}$ of X we have the GACH of $\{x^1, x^2, ..., x^k\}$ is a subset of $\bigcup_{i=1}^{k} G(x^i)$. Therefore from the assumption preceeding Theorem 3.1 and the fact that (iv) and (vii) hold, we have that $\cap \{G(x) : x \in X\} \neq \phi$.

Let

$$y_0 \in \{G(x) : x \in X\}.$$

Then

$$g(x, y_0) \leqslant t \text{ for all } x \in X$$

and our minimax inequality holds.

§4. In this section a further application of arcwise connected function is given for minimax equalities on unbounded sets in R^n. The example thus given is inspiredby the work of Hirano and Takahashi[4] for min-max equality for functions $F(. , .)$ which are convex concave type in respective variables. In what follows $\partial_A K$ and $i_A K$ will denote the boundaries points and interior points of a set K imbedded in a set A in R^n respectively.

Theorem 4.1—Let A, B be two non empty closed arcwise connected (AC) subsets of R^n. If F is a function on $A \times B$ such that for each $y \in B$, $F(. , y)$ is an upper semi-continuous generalized arcwise concave function on A and for each $x \in A$, $F(x , .)$ is a lower semi-continuous generalized arcwise convex function on B, then a sufficient condition for the min-max equality

$$\max_{x \in A} \min_{y \in B} F(x, y) = \min_{y \in B} \max_{x \in A} F(x, y)$$

is given as follows :

There exist bounded closed arcwise convex sets $K \subset A$ and $L \subset B$ such that for each $(x, y) \in (\partial_A K \times L) \cup (K \times \partial_B L)$, there exists $(u, v) \in i_A K \times i_B L$ which satisfies $F(u, y) \geqslant F(x, v)$.

PROOF : Let K and L be two bounded closed arcwise connected subsets satisfying the condition stated in the theorem. Then because of upper semi-continuity and lower semi-continuity conditions on $F(. , y)$ and $F(x , .)$ respectively there exists $(x_0, y_0) \in K \times L$ such that $F(x, y_0) \leqslant F(x_0, y_0) \leqslant F(x_0, y)$ for all $(x, y) \in K \times L$. Let $(x_0, y_0) \in i_A K \times i_B L$. Then for each $x \in A$ we can choose $\theta > 0$ so small that $H_{xx_0}(\theta) \in K$. Since $F(. , y)$ is generalized arcwise concave, we have

$F(x_0, y_0) \geqslant F(H_{xx_0}(\theta), y_0) \geqslant \theta F(x, y_0) + (1 - \theta) F(x_0, y_0)$ and hence $F(x, y_0) \leqslant F(x_0, y_0)$. Also we can get similarly $F(x_0, y_0) \leqslant F(x_0, y)$ for all $y \in B$. Now let $(x_0, y_0) \in (\partial_A K \times L) \cup (K \times \partial_B L)$. Then by the hypothesis in the Theorem there exists $(u, v) \in i_A K \times i_B L$ such that $F(u, y_0) \geqslant F(x_0 v)$ since $F(x, y_0) \leqslant F(x_0, y_0) \leqslant F(x_0, y)$ for all $(x, y) \in K \times L$, we have $F(u, y_0) = F(x_0, y_0) = F(x_0, v)$. For each $x \in A$, we choose $\theta > 0$ so small that $H_{xu}(\theta) \in K$. Then

$$\begin{aligned} F(x_0, y_0) &\geqslant F(H_{xu}(\theta), y_0) \\ &\geqslant \theta F(x, y_0) + (1 - \theta) F(u, y_0) \\ &= \theta F(x, y_0) + (1 - \theta) F(x_0, y_0). \end{aligned}$$

Hence we obtain that $F(x, y_0) \leqslant F(x_0, y_0)$. Also we obtain similarly $F(x_0, y_0) \leqslant F(x_0, y)$ for all $y \in B$, which completes the proof of the theorem.

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