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SUBMERSIONS OF *CR*-SUBMANIFOLDS OF A KAEHLER MANIFOLD

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For the submersion $\pi : M \to B$ of a CR-submanifolds of a Kaehler manifold $\overline{M}$ onto an almost hermitian manifold B, Kobayashi proved that B becomes a Kaehler manifold. The object of the present paper is to study the impact of the submersion on the Geometry of CR-submanifold M as well as to obtain conditions under which $\overline{M}$ and B are holomorphically isocurved. We have also obtained the Ricci curvatures as well as scalar curvatures of the manifolds $\overline{M}$ and B.

1. INTRODUCTION

The study of submersion $\pi : M \to B$ of a Riemannian manifold M onto Riemannian manifold B was initiated by O'Neill[7,8]. A submersion naturally gives rise to two distributions on M called the horizontal and vertical distribution of which the vertical distribution is always integrable giving rise to fibres which are closed submanifolds of M. Also on a CR-submanifold M of a Kaehler manifold $\overline{M}$ with almost complex structure J there are two natural distributions D and $D^{\perp}$, D being invariant under J and $D^{\perp}$ being totally real as well as always integrable[1,2,5]. Kobayashi observed this similarity between the total space of the submersion $\pi : M \to B$ of a CR-submanifold M of a Kaehler manifold $\overline{M}$ onto an almost hermitian manifold B such that the distributions D, $D^{\perp}$ of M become respectively the horizontal and vertical distributions required by the submersion and π restricted to D becomes a complex isometry[6]. He has proved that B in such situation becomes a Kaehler manifold and obtained a relation between the holomorphic sectional curvatures of $\overline{M}$ restricted to D and B. With this naturally following questions arise :

(i) What is the impact of the submersion $\pi : M \to B$ on the geometry of CR-submanifold M?

(ii) If $\overline{M}$ is a complex space form, under what conditions B is a complex space form?

The object of this paper is to answer these questions (cf sections 3 and 4) as well as we obtain same relations between the Ricci curvatures and the scalar curvatures of the Kaehler manifold and the base manifold.

2. Preliminaries

Let $\bar{M}$ be a Kaehler manifold of real dimension $2n$ with almost complex structure J and hermitian metric g. Then on $\bar{M}$ we have

$$\bar{\nabla}_X JY = J \bar{\nabla}_X Y, \ X, Y \in \mathscr{X}(\bar{M}). \qquad ...(2.1)$$

$\bar{\nabla}$ being the Riemannian connection on $\bar{M}$ and $\mathscr{X}(\bar{M})$ is the lie-algebra of vector fields on $\bar{M}$. An m-dimensional submanifold M of $\bar{M}$ is said to be a CR-submanifold if on M there exist two distributions D and $D^{\perp}$ satisfying $JD = D$ and $JD^{\perp} \subset \nu$, ν being the normal bundle of M (cf. Benjancu[1]). In what follows we shall always take $JD^{\perp} = \nu$, so that if $\dim D = 2p$, $\dim D^{\perp} = q$ then $m = 2(p + q)$. The Riemannian connection $\bar{\nabla}$ induces Riemannian connections ∇ and $\nabla^{\perp}$ on M and in the normal bundle ν respectively satisfying

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad ...(2.2)$$

$$\bar{\nabla}_X N = -\tilde{A}_N X + \nabla^{\perp}_X N, \quad X, Y \in \mathscr{X}(M), N \in \nu \qquad ...(2.3)$$

where h and $\tilde{A}_N$ are the second fundamental form and the Weingarten map respectively, satisfying $g(h(X, Y), N) = g(A_N X, Y)$. It is known that the distribution D is integrable if and only if[1]

$$h(X, JY) = h(X, YJ), \ \forall \ X, Y \in D. \qquad ...(2.4)$$

If $h \equiv 0$, then M is said to be totally geodesic and if $h(X, Y) = g(X, Y) H$, where $H = 1/m$ (traceh), then M is said to be totally umbilical CR-submanifold of $\bar{M}$. Let $\bar{R}$, R and $R^{\perp}$ be the curvature tensors corresponding to the connections $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ respectively. Then the equations of Gauss, Codazzi and Ricci are

$$\begin{aligned} \bar{R}(X, Y, Z, W) = R(X, Y, Z, W) &- g(h(X, W), h(Y, Z)) \\ &+ g(h(X, Z), h(Y, W)) \end{aligned} \qquad ...(2.5)$$

$$[\bar{R}(X, Y) Z]^{\perp} = (\bar{\nabla}_X h)(Y, Z) - g(\bar{\nabla}_Y h)(X, Z) \qquad ...(2.6)$$

$$\bar{R}(X, Y, N, N') = R^{\perp}(X, Y; N, N') - g([\tilde{A}_N, \tilde{A}_{N'}](X), Y) \qquad ...(2.7)$$

for X, Y, Z, W tangent to M and $N, N' \in \nu$,

where $[\quad]^{\perp}$ denotes the normal component, and

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

For the theory of submersion we follow O'Neill[7]. Let B be an almost Hermitian manifold and we assume that there is a submersion $\pi : M \to B$ of CR-submanifold M onto B such that[6]

(i) $D^{\perp}$ is Kernel of π_*, that is, $\pi_* D^{\perp} = \{0\}$

and

(ii) $\pi_* : D_p \to D^*_{\pi(p)}$ is complex isometry, $p \in M$, where $D^*_{\pi(p)}$ denotes the tangent space of B at $\pi(p)$.

(iii) J interchanges $D^{\perp}$ and ν.

A vector field X on M is said to be basic if

(i) $X \in D$

and

(iii) X is π-related to a vector field on B, i. e., there exists a vector field X_* on B such that $(\pi_* X)_p = X_{*\pi(p)}$ for every $p \in M$.

We have the following lemma for basic vector fields:

Lemma 2.1[7]—Let X and Y be basic vector fields on M. Then

(i) $g(X, Y) = g_*(X_*, Y_*) \, 0 \pi$, g_* being Hermitian metric on B

(ii) The horizontal part $P[X, Y]$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, i. e. $\pi_*[X, Y] = [X_*, Y_*]$.

(iii) $[V, X] \in D$, $V \in D^{\perp}$

(iv) $P(\nabla_X Y)$ is basic vector field corresponding to $\nabla^*_{X_*} Y_*$, where ∇^* is Riemannian connection on B.

Put

$$\bar{\nabla}^*_X Y = P(\nabla_X Y), \; X, Y \in D,$$

then $\bar{\nabla}^*_X Y$ is basic vector field and we have

$$\pi_*(\bar{\nabla}^*_X Y) = \nabla^*_{X_*} Y_*. \qquad \ldots(2.8)$$

Define a tensor field C by

$$\nabla_X Y = \tilde{\nabla}^*_X Y + C(X, Y), \; Q(\nabla_X Y) = C(X, Y). \qquad \ldots(2.9)$$

It has been observed in Kobayashi[6] that C is skew-symmetric and we have

Lemma 2.2[6]—If $X, Y \in D$, then

$$C(X, Y) = \tfrac{1}{2} Q[X, Y].$$

For $X \in D$ and $V \in D^{\perp}$, define A by

$$\nabla_X V = Q(\nabla_X V) + A_X V$$

where $Q(\nabla_X V)$ denotes the vertical part of $\nabla_X V$. Since $[V, X] \in D^{-}$ for $V \in D^{\perp}$, we have

$$P(\nabla_V X) = P(\nabla_X V) = A_X V.$$

The operators A and C are related by

$$g(A_X V, Y) = -g(V, C(X, Y)),\ X, Y \in D,\ V \in D^{\perp}. \quad ...(2.10)$$

The curvature tensors R, $\hat{R}$ and R^* of M, the fibres and B are related by

$$R(X, Y; Z, H) = R^*(X_*, Y_*; Z_*, H_*) - g(C(X, Z), C(Y, H)) + g(C(Y, Z), C(X, H)) + 2g(C(X, Y), C(Z, H)) \quad ...(2.11)$$

$$R(X, V; Y, W) = g((\nabla_X T)_V W, Y) + g((\nabla_V A)_X Y, W) - g(T_V X, T_W Y) + g(A_X V, A_Y W) \quad ...(2.12)$$

$$R(U, V; W, F) = \hat{R}(U, V; W, F) - g(T_V W, T_U F) + g(T_U W, T_V F) \quad ...(2.13)$$

for $X, Y, Z, H \in D$ and $U, V, W, F \in D^{\perp}$.

The operator C in (2.11) is introduced by Kobayashi[6] while in (2.13), the operators T and A are due to O'Neill[7] and are called the fundamental tensors of the submersion π, the operator A in (2.12) coincides with C for horizontal vector fields. The operator T for vertical vector fields will be denoted by L which we shall use in Proposition 3.3.

3. Geometry of CR-Submanifolds

In this section we study the impact of the submersion $\pi : M \to B$ on the geometry of CR-submanifold M. As a first consequence, using (2.2), (2.8) and (2.9) we get

$$C(X, JY) = Jh(X, Y)$$

$$h(X, JY) = JC(X, Y) \quad ...(3.1)$$

from which it easily follows that

$$h(X, JY) + h(JX, Y) = 0,\ X, Y \in D. \quad ...(3.2)$$

Proposition 3.1—Let $\pi : M \rightarrow B$ be a submersion of CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. If D is integrable and $D^{\perp}$is parallel (i. e., $\nabla_X Y \in D$, $X, Y \in D^{\perp}$), then M is the product $M_1 \times M_2$, where M_1 is a complex submanifold and M_2 is a totally real submanifold of $\bar{M}$.

PROOF : If D is integrable, then we have from (2.4) and (3.2) $h(X, JY) = 0$, $X, Y \in D$. As a consequence of this in (3.1) we observe that $C(X, Y) = 0$, and thus $\nabla_X Y \in D$ which proves D is parallel. This completes the proof of the proposition.

Corollary 3.1—Let $\pi : M \rightarrow B$ be a submersion of a CR-submanifold M of a Kaehler manifold $\bar{M}$ with integrable D. Then

$$\bar{H}(X) = H^*(X_*), \ X \in D$$

where $\bar{H}$ and H^* are respectively the holomorphic sectional curvatures of $\bar{M}$ and B. In particular if $\bar{M}$ is of constant holomorphic sectional curvature C, then so is B. The proof follows at once from Proposition 3.1 and eqn. (1.3) Kobayashi[6].

A CR-submanifold is said to be mixed foliate if D is integeable and $h(X, Y) = 0$, $X \in D, Y \in D^{\perp}$. For the submersion of fixed foliate CR-submanifolds we prove the following:

Proposition 3.2—Let $\pi : M$ be a submersion of mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then M is the product $M_1 \times M_2$ where M_1 is complex submanifold and M_2 is totally real submanifold of $\bar{M}$.

PROOF : From Proposition 3.1 it follows that

$$h(X, Y) = 0, \ X, Y \in D.$$

Also M being mixed foliate we have

$$h(X, Y) = 0, \ X \in D, \ Y \in D^{\perp}.$$

Now for $X, Y \in D^{\perp}$, we have

$$(\bar{\nabla}_X J)(Y) = 0$$

i. e.

$$-\tilde{A}_{JY} X + \nabla^{\perp}_X JY = J \nabla_X Y + Jh(X, Y).$$

As $X \in D^{\perp}$, $\tilde{A}_{JY} X \in D^{\perp}$ (Bejancu[1]) and thus equating vertical components in above equation we get $-\tilde{A}_{JY} X = Jh(X, Y)$ which then gives $\nabla^{\perp}_X JY = J \nabla_X Y$, proving that

$\nabla_X Y \in D^{\perp}$ i. e.$D^{\perp}$ is parallel and thus the proof follows from Proposition 3.1.

Proposition 3.3—Let $\pi : M \to B$ be a submersion of a CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then the fibres are totally geodesic submanifolds of M if and only if $h(X, V) = 0$, $X \in D$, $V \in D^{\perp}$.

PROOF : For $U, V \in D^{\perp}$ we define L by

$$\nabla_U V = \hat{\nabla}_U V + L(U, V)$$

where $\hat{\nabla}_U V = Q(\nabla_U V)$ and $L(U, V) = P(\nabla_U V)$. Since $D^{\perp}$ always integrable we get $L(U, V) = L(V, U)$. Now from $(\bar{\nabla}_U J)(V) = 0$ it follows that

$$-\tilde{A}_{JV} U + \nabla^{\perp}_U JV = JL(U, V) + J\hat{\nabla}_U V + Jh(U, V).$$

Equating the horizontal and vertical components we get

$$P(\tilde{A}_{JV} U) = -JL(U, V)$$

and

$$Q(\tilde{A}_{JV} U) = -Jh(U, V).$$

From this it follows that fibres are totally geodesic iff $\tilde{A}_{JV} U \in D^{\perp}$. This proves that fibres are totally geodesic iff $h(U, X) = 0 \ \forall\ X \in D$, $U \in D^{\perp}$.

Proposition 3.4—Let $\pi : M \to B$ be a submersion of a CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then the sectional curvatures of $\bar{M}$ and the fibres are related by

$$\bar{K}(U \wedge V) = \hat{K}(U \wedge V) - g([\tilde{A}_{JU}, \tilde{A}_{JU}]\, U, V)$$

for orthonormal vector fields $U, V \in D^{\perp}$.

PROOF : We defne $\hat{R}$ by

$$\hat{R}(U, V)W = [\hat{\nabla}_U, \hat{\nabla}_V](W) - \hat{\nabla}_{[U,V]} W.$$

Now,

$$\begin{aligned} R(U, V)W &= [\nabla_U, \nabla_V](W) - \nabla_{[U,V]} W \\ &= \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W \end{aligned}$$

(*equation continued on p.* 1191)

$$= \nabla_U (L(V, W) + \hat{\nabla}_V W) - \nabla_V (L(U, W) + \hat{\nabla}_U W)$$

$$- \hat{\nabla}_{[U,V]} W - P(\nabla_{[U,V]} W).$$

Taking inner product with a vertical vector field F in above relation, we get

$$R(U, V; W, E) = \hat{R}(U, V; W, F) - g(L(V, W), L(U, F))$$
$$+ g(L(U, W), L(V, F)).$$

From (2.5) and above relation we have

$$\bar{R}(U, V; W, F) = \hat{R}(U, V; W, F) - g(L(V, W), L(U, F))$$
$$+ g(L(U, W), L(V, F))$$
$$- g(h(V, W), h(U, F)) + g(h(U, W), h(V, F)).$$

The above relation gives

$$\bar{R}(U, V; U, V) = \hat{R}(U, V; U, V) - g(L(U, V), L(U, V)$$
$$+ g(L(U, U), L(V, V))$$
$$- g(h(U, V), h(U, V) + g(h(U, V), h(V, V))$$

which implies that

$$\bar{K}(U \wedge V) = \hat{K}(U \wedge V) - g(L(U, V), L(U, V))$$
$$+ g(L(U, U), L(V, V)) - g(h(U, V), h(U, V))$$
$$+ g(h(U, U), h(V, V))$$

for any orthonormal vectors $U, V \in D^{\perp}$.

Now using $P(\widetilde{A_{JU}} V) = - JL(U, V)$ and $Q(\widetilde{A_{JU}} V) = - Jh(U, V)$ we obtain

$$\bar{K}(U \wedge V) = \hat{K}(U \wedge V) - g(P\widetilde{A_{JU}} V, P\widetilde{A_{JU}} V)$$

$$+ g(P\widetilde{A_{JU}} U, P\widetilde{A_{JV}} V)$$

$$- g(Q\widetilde{A_{JU}} V, Q\widetilde{A_{JU}} V) + g(Q\widetilde{A_{JU}} U, Q\widetilde{A_{JV}} V)$$

$$= \hat{K}(U \wedge V) - g(\widetilde{A_{JU}} V, \widetilde{A_{JU}} V) + g(\widetilde{A_{JU}} V, \widetilde{A_{JV}} V)$$

$$= \hat{K}(U \wedge V) - g(\widetilde{A_{JU}} V, \widetilde{A_{JV}} U) + g(\widetilde{A_{JV}}\widetilde{A_{JU}} U, V)$$

(*equation continued on p.* 1192)

$$= \hat{K}(U \wedge V) - g(\tilde{A}_{JU}\tilde{A}_{JV}U, V) + g(\tilde{A}_{JV}\tilde{A}_{JU}U, V)$$

$$= \hat{K}(U \wedge V) - g([\tilde{A}_{JU}, \tilde{A}_{JV}](U), V)$$

which proves the result.

A CR-submanifold is said to be mixed totally geodesic if $h(X, Y) = 0$ for $X \in D$ and $Y \in D^{\perp}$. For mixed totally geodesic CR-submanifold we have :

Proposition 3.5 Let $\pi : M \to B$ be a submersion of a mixed totally geodesic CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B, then

$$\bar{R}(X, V; Y, W) = -g((\nabla_V C)(X, Y), W) - g(A_X V, A_Y W) + g(h(X, Y), h(V, W)) \quad \ldots(3.3)$$

for $X, Y \in D$ and $V, W \in D^{\perp}$.

PROOF : From definition of R it follows that

$$\begin{aligned} R(V, X)Y &= \nabla_{[X,V]}Y - \nabla_X \nabla_V Y + \nabla_V \nabla_X Y. \\ &= P(\nabla_{[X,V]}Y) + Q(\nabla_{[X,V]}Y) - \nabla_X (P\nabla_V Y + T_V Y) \\ &\qquad + \nabla_V (P\nabla_X Y + C(X,Y)) \\ &= P\nabla_{[X,V]}Y + T_{[X,V]}Y - \nabla_X (P(\nabla_V Y)) \\ &\quad - \nabla_Y (T_V Y) + \nabla_V (P\nabla_X Y) + \nabla_V C(X, Y). \end{aligned}$$

Taking inner product with $W \in D^{\perp}$ and noting that $[X, V] \in D^{\perp}$ we get

$$\begin{aligned} g(R(V, X)Y, W) &= g(T_{[X,V]}Y, W) - g(\nabla_X P(\nabla_V Y), W) \\ &\quad - g(\nabla_X (T_V Y), W) + g(\nabla_V Q(\nabla_X Y), W) \\ &\quad + g(\nabla_V C(X, Y), W) \\ &= g(T_{\nabla_X V} Y, W) - g(T_{\nabla_V X} Y, W) \\ &\quad + g(P\nabla_V Y, \nabla_X W) - g(\nabla_X (T_V Y), W) \\ &\quad - g(P\nabla_X Y, \nabla_V W) + g((\nabla_V C)(X, Y), W) \\ &\quad + g(C(\nabla_V X, Y), W) + g(C(X, \nabla_V Y, W) \end{aligned}$$

here $\nabla_V X \in D$ for $X \in D$ and $V \in D^{\perp}$ follows from $(\bar{\nabla}_V J)(X) = 0$ and $h(X, V) = h(JX, V) = 0$.

From definition of L in Proposition 3.3 and T in O'Neill[7] it easily follows that

$$g(T_V Y, W) = -g(L(V, W)Y).$$

Taking covariant differentiation in above equation with respect to X we obtain

$$
\begin{aligned}
& g(\nabla_X (T_V Y), W) + g(T_V Y, \nabla_X W) \\
& \quad = -g(\nabla_X Y, L(V, W)) - g(\nabla_X L(V, W), Y) \\
& \quad = -g(\nabla_X Y, L(V, W) - g((\nabla_X L)(V, W), Y) \\
& \qquad - g(L(Q \nabla_X V), W), Y) - g(L(V, Q(\nabla_X W), Y)
\end{aligned}
$$

where we define

$$
\begin{aligned}
(\nabla_X L)(V, W) &= \nabla_X L(V, W) - L(Q \nabla_X V, W) \\
&\qquad - L(V, Q \nabla_X W). \\
&= -g(\nabla_X Y, L(V, W)) - g((\nabla_X L)(V, W), Y) \\
&\quad + g(T_{\nabla_X V} Y, W) + g(T_V Y, Q(\nabla_X W))
\end{aligned}
$$

as T is vertical.

From which it follows that

$$
\begin{aligned}
g(\nabla_X (T_V Y), W) = & -g((\nabla_X L)(V, W), Y) - g(\nabla_X Y, L(V, W)) \\
& + g(T_{\nabla V} Y, W). \qquad \ldots(3.5)
\end{aligned}
$$

From (3.4) and (3.5) we obtain

$$
\begin{aligned}
R(V, X; Y, W) = & \; g(T_{\nabla_X V} Y, W) - g(T_{\nabla_V X} Y, W) \\
& + g(P(\nabla_V Y, \nabla_X W) + g(\nabla_X Y, L(V, W)) \\
& - g(T_{\nabla_X V} Y, W) + g((\nabla_X Y, L)(V, W), Y) \\
& - g(P \nabla_X Y, \nabla_V W) \\
& + g((\nabla_V C)(X, Y), W) + g(C(\nabla_V X, Y) W) \\
& + g(C(X, \nabla_V Y), W) \\
= & \; g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W) \\
& + g(P \nabla_V Y, \nabla_X W) - g(T \nabla_V X Y, W) \\
& + g(C(\nabla_V X, Y), W) + g(C(X, \nabla_V Y), W).
\end{aligned}
$$

Using (2.10) we get

$$
\begin{aligned}
R(V, X; Y, W) = & \; g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W) \\
& + g(A_Y V, A_X W) + g(Y, L(Q \nabla_V X, W))
\end{aligned}
$$

(equation continued on p. 1194)

$$
\begin{aligned}
&+ g(A_Y W, \nabla_V X) - g(A_X W, \nabla_V Y)\\
&= g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W)\\
&\quad + g(A_Y V, A_X W) + g(T_W Y, Q \nabla_V X)\\
&\quad + g(A_Y W, P \nabla_V X) - g(A_X W, P\nabla_V Y)\\
&= g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W)\\
&\quad + g(A_Y V, A_X W) - g(T_V X, T_W Y)\\
&\quad + g(A_X V, A_Y W) - g(A_X W, A_Y V)
\end{aligned}
$$

where we have used definitions of A and T and $P\nabla_V X = P\nabla_X V$ as $[V, X] \in D^{\perp}$.

Thus

$$
\begin{aligned}
R(V, X; Y, W) &= g((\nabla_X L)(V, W), Y) + g((\nabla_V C)(X, Y), W)\\
&\quad + g(A_X V, A_Y W) - g(T_V X, T_W Y).
\end{aligned}
$$

Now using equation of Gauss (2.5) in above equation we get

$$
\begin{aligned}
\bar{R}(X, V; Y, W) &= - g((\nabla_X L)(V, W), Y) - g((\nabla_V C)(X, Y), W)\\
&\quad - g(A_X V, A_Y W) + g(T_V X, T_W Y)\\
&\quad - g(h(X, W), h(V, Y)) + g(h(X, Y), h(V, W)).
\end{aligned}
$$

Now if M is mixed totally geodesic from Proposition 3.3 it follows that $L(V, W) = 0$ and

$$
g(T_V X, T_W Y) = - g(X, L(V, Q \nabla_W Y)) = 0.
$$

Hence above equation reduces to (3.3).

Proposition 3.6—Let $\pi : M \to B$ be a submersion of a mixed totally geodesic CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then for the unit vectors $X \in D$ and $V \in D^{\perp}$ we have

$$
\bar{K}(X \wedge V) = - \|\tilde{A}_{JV} X\|^2 + g(h(X, X), h(V, V)). \qquad \ldots(3.6)
$$

PROOF : From Proposition 3.5 for $X = Y, W = V$, and noting $C(X, X) = 0$ we get

$$
\bar{K}(X \wedge V) = g(h(X, X), h(V, V)) - \|A_X V\|^2. \qquad \ldots(3.7)
$$

Using $(\bar{\nabla}_X J)(V) = 0$ we get

$$
- \tilde{A}_{JV} X + \nabla^{\perp}_X JV = JA_X V + JQ \nabla_X V + Jh(X, V).
$$

Since M is mixed-totally geodesic $h(X, V) = 0$ and this implies $\widetilde{A}_{JV} X \in D$ for every $X \in D$. Thus equating horizontal component in above equation we get $J A_X V = -\widetilde{A}_{JV} X$ or

$$A_X V = J\widetilde{A}_{JV} X. \qquad \ldots(3.8)$$

Using (3.8) in (3.7) we get the result.

Proposition 3.7—If $\pi : M \to B$ is a submersion of a mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B, then the curvature tensor $\bar{R}$ of $\bar{M}$ satisfies

$$\bar{R}(X, V; Y, W) = 0, X, Y \in D, V, W \in D^{\perp}.$$

PROOF : If M is foliate, then from Proposition 3.1, it follows that $h(X, Y) = C(X, Y) = 0$. Also

$$g(A_X V, A_Y W) = -g(C(X, A_Y W), V) = 0.$$

Using this in Proposition 3.5 we get the result.

Following Bejancu[2], we say that the normal connection $\nabla^{\perp}$ is D-flat if $R^{\perp}(X, Y; N, N') = 0$, $X, Y \in D$. Now we are in position to prove our main theorem.

Theorem 3.1—If $\pi : M \to B$ is a submersion of a mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B, then the normal connection of M in $\bar{M}$ is D-flat.

PROOF : Using Proposition (3.7) in the Bianchi's identity

$$\bar{R}(X, V; Y, W) + \bar{R}(V, Y; X, W) + \bar{R}(Y, X; V, W) = 0$$

we get

$$\bar{R}(X, Y; V, W) = 0, X, Y \in D, V, W \in D^{\perp}.$$

Now using $\bar{R}(X, Y, V, W) = \bar{R}(X, Y, JV, JW)$ we get

$$\bar{R}(X, Y; JV, JW) = 0.$$

Using the equation of Ricci (2.7) we get

$$R^{\perp}(X, Y; N, N') = g([\widetilde{A}_N, \widetilde{A}_{N'}](X), Y)$$

where $JV = N$ and $JW = N'$ are normals.

Thus

$$R^{\perp}(X, Y; N, N') = g(\widetilde{A}_N \widetilde{A}_{N'} X, Y) - g(\widetilde{A}_{N'} \widetilde{A}_N X, Y)$$

(*equation continued on p.* 1196)

$$= g(\tilde{A}_{JW} X, \tilde{A}_{JV} Y) - g(\tilde{A}_{JV} X, \tilde{A}_{JW} Y)$$

$$= g(A_X W, A_Y V) - g(A_X V, A_Y W)$$

where we have used (3.8). As in proof of (3.7), we get $g(A_X W, A_Y V) = g(A_X V, A_Y W) = 0$. Hence, we get the result.

4. Submersions of Totally Umbilical CR-submanifolds

In last section we have discussed those submersions of CR-submanifolds of a Kaehler manifold in which mostly the CR-submanifolds were turning to be totally geodesic and as such $\bar{M}$ and B were becoming isocurved (cf. Propositions 3.1, 3.2). Next natural question is which non-totally geodesic CR-submanifolds maintain this property of $\bar{M}$ and B being specially the spaces of constant holomorphic curvature. Very natural non-totally geodesic CR-submanifolds are totally umbilical CR-submanifolds of Kaehler manifolds, moreover, they are natural prototypes for the submersion $\pi : M \to B$, because the condition (3.2) is naturally satisfied for $h(X, JY) = g(X, JY) H$ and $h(JX, Y) = g(JX, Y)H$, where H is mean curvature vector which is non-zero for non-totally geodesic submanifolds. In case of totally umbilical CR-submanifolds the equations (2.2), (2.3) and (2.6) take the following forms

$$\bar{\nabla}_X Y = \nabla_X Y + g(X, Y) H \qquad \ldots(4.1)$$

$$\nabla_X N = -g(N, H) X + \nabla^{\perp}_X N. \qquad \ldots(4.2)$$

$$[\bar{R}(X, Y) Z]^{\perp} = g(Y, Z) \nabla^{\perp}_X H - g(X, Z) \nabla^{\perp}_Y H. \qquad \ldots(4.3)$$

In case $\bar{M}$ is complex space form of constant holomorphic sectional curvature c, the curvature tensor $\bar{R}$ is given by

$$\begin{aligned}\bar{R}(X, Y; Z, W) = c/4\, [&g(Y, Z)\, g(X, W) - g(X, Z)\, g(Y, W) \\ &+ g(JY, Z)\, g(JX, W) - g(JX, Z)\, g(JY, W) \\ &+ 2g(X, JY)\, g(JZ, W)]. \qquad \ldots(4.4)\end{aligned}$$

Our main theorem in this section is

Theorem 4.1—Let $\pi : M \to B$ be the submersion of a totally umbilical CR-submanifold M (dim $M \geqslant 5$) of a complex space form $\bar{M}(c)$ onto an almost Hermitian manifold B. Then B is also a complex space form.

Proof : Since in case of submersion $\pi : M \to B$ $JD^{\perp} = \nu$, from Theorem (cf. Blair and Chen)[3] it follows that either $H = 0$ or dim $D^{\perp} = 1$. In case $H = 0$ from eqn. (1.3) of Kobayashi[6] (Theorem 1.3) it follows that B is also a complex space form.

Suppose dim $D^{\perp} = 1$. From eqns. (2.5), (2.11), (4.4) and $h(X, Y) = g(X,Y)H$, we easily get the following expression for the curvature tensor R^* of B.

$$\begin{aligned} R^*(X_*, Y_*; Z_*, W_*) = (c/4 + \|H\|^2)\{ & g(Y, Z)\, g(X, W) - g(X,Z)\, g(Y,W) \\ & + g(JY, Z)\, g(JX, W) - g(JX, Z)\, g(JY, W) \\ & + 2g(X, JY)\, g(JZ, W)\}. \end{aligned}$$

Thus to complete the proof we have to show that $\|H\|^2$ is a constant. Since dim $M \geqslant 5$ we can choose vectors $X, Y \in D$ such that $g(X, Y) = g(X, JY) = 0$. Now from equation (2.6) of Codazzi we have

$$\bar{R}(X, Y; Z, N) = g(Y, Z)\, g(\nabla^{\perp}_X H, N) - g(X, Z)\, g(\nabla^{\perp}_Y H, N).$$

From (4.4) it follows that $\bar{R}(JY, X; JY, N) = 0$. Thus (4.3) gives

$$g(\nabla^{\perp}_X H, N) = 0, \; N \in \nu. \qquad \ldots (4.5)$$

This proves that

$$\nabla^{\perp}_X H = 0 \;\forall\; X \in D.$$

Next let $X \in D^{\perp}$. Then using the following curvature properties of $\bar{M}$

$$\bar{R}(JX, JY; JZ, W) = \bar{R}(X, Y, Z, W)$$

$$\bar{R}(JX, JY; Z, W) = \bar{R}(X, Y; Z, W)$$

and (4.4) we get

$$\bar{R}(X, Y; Y, X) = \bar{R}(X, Y; JY, N') = 0, \; N' = JX.$$

Using linearity of $\bar{R}$ in $\bar{R}(X, Y; Y, X) = 0$ we get

$$\bar{R}(X, Y; JY, X) = 0$$

or

$$\bar{R}(X, Y; Y, N') = 0.$$

Using this in (4.3) we get

$$g(\nabla^{\perp}_X H, N') = 0.$$

As dim $D^{\perp} = \dim \nu = 1$, we get $\nabla^{\perp}_X H = 0$ for $X \in D^{\perp}$. Hence for any vector field X on M we have

$X. \|H\|^2 = X. g(H, H) = 2g(\nabla^{\perp}_X H, H) = 0$, proving that $\|H\|^2$ = constant and hence the theorem.

Theorem 4.2—Let $\pi : M \to B$ be a submersion of a totally umbilical CR-submanifold M of a Kaehler manifold $\bar{M}$ with parallel D. Then M is the product $M_1 \times M_2$ where M_1 is complex submanifold and M_2 is totally real submanifold of $\bar{M}$.

PROOF : Let H be the mean curvature vector of the CR-submanifold M in $\bar{M}$. Since H is normal, JH is vertical.

Using Gauss and Weingarten formulae in $(\bar{\nabla}_X J)(JH) = 0$, obtain

$$\tilde{A}_H X - \nabla^{\perp}_X H = J \nabla_X JH + Jh(X, JH).$$

Now using the definition of totally umbilicalness in above relation we get

$$g(H, H) X - \nabla^{\perp}_X H = J \nabla_X JH + h(X, JH) JH. \qquad ...(4.6)$$

Taking inner product with $X \pm 0 \in D$ in (4.6) we get

$$\|H\|^2 \|X\|^2 = -g(\nabla_X JH, JX)$$
$$= g(JH, \nabla_X JX). \qquad ...(4.7)$$

As D is parallel, $\nabla_X JX \in D$, which implies that $g(\nabla_X JX, JH) = 0$, the above relation (4.7) gives $\|H\|^2 = 0$, i. e M is totally geodesic and hence the result.

Remark : Wherever necessary, the horizontal vector fields are supposed to be basic.

5. RICCI TENSORS AND SCALAR CURVATURE

In this section we obtain relations between the Ricci tensors and scalar curvatures of $\bar{M}$ and the base manifold. We have

Theorem 5.1—Let $\pi : M \to B$ be a submersion of a mixed foliate CR-submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B. Then the Ricci tensors $\bar{S}$ and S^* of $\bar{M}$ respectively B satisfy the relation

$$\bar{S}(X, Y) = S^*(X_*, Y_*) \qquad ...(5.1)$$

for each basic vector fields $X, Y \in D$.

PROOF : From (2.5) and (2.11)

$$\begin{aligned}\bar{R}(Z, X, Y, W) = R^*(Z_*, X_*, Y_*, W_*) &+ g(h(Y, Z), h(X, W)) \\ &- g(h(W, Z), h(X, Y)) - g(C(X, Y), C(Z, W)) \\ &+ g(C(Z, Y), C(X, W)) + 2g(C(Z, X), C(Y, W)).\end{aligned} \qquad ...(5.2)$$

Let $\{E_1, ..., E_{2p}, E_{p+1} = JE_1, ..., E_{2p} = JE_p, F_1, ..., Fq, JF_1, .. JF_q\}$ be a lacol field of orthonormal frames of $\bar{M}$ such that $\{E_1, ..., E_p, E_{p+1} = JE_1, ..., E_{2p} = JE_p\}$ *and* $\{F_1, ..., F_q\}$ are local fields of orthnormal frames of the horizontal distribution D and the vertical distribution $D^{\perp}$ respectively. Then using the definition of of the Ricci tensor in (5.2) we obtain

$$\bar{S}(X, Y) = S^*(X_*, Y_*) + \sum_{i=1}^{2p} g(h(E_i, Y), h(E_i, X)) \quad ...(5.3)$$

$$- g(h(X, Y), \sum_{i=1}^{2p} h(E_i, E_i)) - g(C(X, Y), \sum_{1=1}^{2p} C(E_i, E_i))$$

$$+ \sum_{i=1}^{2p} \{g(C(E_i, Y), C(X, E_i))\} + 2\sum_{i=1}^{2p} \{g\, C(E_i, X),$$

$$C(Y, E_i))\}$$

$$+ \sum_{k=1}^{q} \{\bar{R}(F_k, X; Y, F_k) + \bar{R}(JF_k, X; Y, JF_k)\}.$$

Since M is foliate, the horizontal distribution D is involutive, then M is D-minimal[1]. On the other hand C is skew symmetric. Hence the third and fourth term on the right-hand side of (5.3) vanishes and we get

$$\bar{S}(X, Y) = S^*(X_*, Y_*) + \sum_{i=1}^{2p} g(h(E_i, y), h(E_i, X))$$

$$- 3\sum_{i=1}^{2p} g(C(E_i, X), C(E_i, Y)) \quad ..(5.4)$$

$$+ \sum_{k=1}^{q} \bar{R}(F_k, X; Y, Y_k) + \bar{R}(JF_k, X; Y, JF_k).$$

Now, from $g(h(X, Y), N) = g(A_N X, Y)$ and Lemma 2.1[2] we get

$$\sum_{k=1}^{2p} \{g(h(E_i, X), h\, E_i, Y))\} \quad ...(5.5)$$

$$= \sum_{k=1}^{q} g(\tilde{A}_{JF_k} X, \tilde{A}_{JF_k} Y).$$

Also we have[6] $C(X, JY) = Jh(X, Y)$, which implies

$$\sum_{i=1}^{2p} g(C(E_i, X), C(E_i, Y)) = \sum_{i=1}^{2p} g(-Jh(E_i, JX), -Jh(E_i, JY))$$

(*equation continued on p.* 1200)

$$= \sum_{i=1}^{2p} g(h(E_i, JX, h(E_i, JY))$$

$$= \sum_{k=1}^{q} g(\tilde{A}_{JF_k} JX, \tilde{A}_{JF_k} JY) \quad \ldots(5.6)$$

$$= \sum_{k=1}^{q} g(-J\tilde{A}_{JF_k} X, -J\tilde{A}_{JF_k} Y)$$

$$= \sum_{k=1}^{q} g(\tilde{A}_{JF_k} X, \tilde{A}_{JF_k} Y).$$

Using eqn. (3.8) we obtain

$$\sum_{k=1}^{q} g(\tilde{A}_{JF_k} X, \tilde{A}_{JF_k} Y) = \sum_{k=1}^{q} g(A_X F_k, A_Y F_K). \quad \ldots(5.7)$$

If M is foliate, then from Proposition 3.1, it follow that $h(X, Y) = C(X, Y) = 0$. Also

$$g(A_X F_K, A_Y F_K) = -g(C(X, A_Y F_k), F_k) = 0. \quad \ldots(5.8)$$

Using (5.5), (5.6), (5.7), and (5.8) in (5.4) we get

$$\bar{S}(X, Y) = S^*(X_*, Y_*) + \sum_{k=1}^{q} \bar{R}(F_K, X; Y, F_K) + \bar{R}(JF_k, X, Y, JF_k). \quad \ldots(5.9)$$

From Proposition 3.7.

$$\bar{R}(F_K, X, Y, F_K) = 0 \ \forall\ X, Y \in D. \quad \ldots(5.10)$$

Also, Bianchi identity gives

$$\bar{R}(JF_K, X, Y; JF_K) + \bar{R}(X, Y, JF_K, JF_K) + \bar{R}(Y, JF_K, X, JF_K) = 0.$$

Using Theorem 3.1 and Proposition 3.7 we get

$$\bar{R}(JF_K, X; Y, JF_K) = 0 \quad \ldots(5.11)$$

From (5.9), (5.10) and (5.11) we get the result.

Definition 5.1—The Kaehler manifold $\bar{M}$ is said to be an Einstein space if there exists a constant σ such that the Ricci tensor $\bar{S}$ of $\bar{M}$ satisfies

$$\bar{S}(X, Y) = \sigma g(X, Y) \quad \ldots(5.12)$$

for all tangent vectors X, Y on $\bar{M}$.

As a direct consequence of (5.12) and above theorem, we have

Theorem 5.2—Let M be a mixed foliate CR-submanifold of a Kaehler manifold

$\bar{M}$ and let $\pi : M \to B$ be a submersion of M onto an almost Hermitian manifold B. Then B is an Einstein space if and only if $\bar{M}$ is an Einstein space.

Lastly in this section we estimate the Ricci tensor and scalar curvature of the base manifold B of the submersion $\pi : M \to B$ of M onto an almost Hermitian manifold B, when M is a CR-submanifold of a complex space form $\bar{M}(c)$ of constant holomorphic sectional curvature c.

Let $\{E_m, \ldots, E_m\}$ be a local field of orthonormal frames on M (where m is the dimension of the CR-submanifold M) such that $\{E_1, \ldots, E_p, E_{p+1} = JE_1, \ldots, E_{p2} = JE_p\}$ is a local field of orthonormal frames on D and $\{F_1, \ldots, F_q\}$ is a local field of orthonormal frames on $D^{\perp}$. Then $\{JF_1, \ldots, JF_q\}$ becomes the field of orthonormal frames of the normal bundle ν.

Then from (5.2) and (4.5) we obtain

$$\begin{aligned} R^*(Z_*, X_*; Y_*, W_*) = c/4\, & g(X, Y)\, g(Z, W) - g(Z, Y)\, g(X, W) \\ & + g(JX, Y)\, g(JZ, W) - g(JZ, Y)\, g(JX, W) \\ & + 2g(Z, JX)\, g(JY, W) \\ & + g(h(Z, W), h(X, Y)) - g(h(Z, Y), h(X, W)) \\ & + g(C(X, Y), C(Z, W)) - g(C(Z, Y), \\ & \times C(X, W)) - 2g(C(Z, X), C(Y, W)) \end{aligned} \quad \ldots(5.13)$$

for any basic vector fields X, Y, Z, W on M.

Then from (5.13) we get

$$\begin{aligned} S^*(X_*, Y_*) = c/4\, & [2p.\, g(X, Y) - \sum_{i=1}^{2p} \{g\, E_i, Y)\, g(X, E_i)\} \\ & + \sum_{i=1}^{2p} \{g(JX, Y)\, g(JE_i, E_i)\} - \sum_{i=1}^{2p} \{g(JE_i, Y)\, g(JX, E_i)\} \\ & + 2\sum_{i=1}^{2p} \{g(E_i, JX)\, g(JY, E_i)\}] + g(h(X, Y), \sum_{i=1}^{2p} h(E_i E_i)) \\ & - \sum_{i=1}^{2p} g(h(E_i, Y), h(X, E_i)) + g(C(X, Y), C(E_i, E_i))\} \\ & - \sum_{i=1}^{2p} \{g(C(E_i, Y)\, C(X, E_i)\} - 2\sum_{i=1}^{2p} \{g(C)\, E_i, X), \\ & C(Y, E_i))\}. \end{aligned} \quad \ldots(5.14)$$

Using the skew-symmetry of C, we get

$$S^*(X_*, Y_*) = \frac{pc}{2} g(X, Y) - \frac{c}{4}\sum_{i=1}^{2p} \{g(E_i, Y)\, g(X, E_i)$$

$$- 3g(JE_i, Y)\, g(JX, E_i)\} + g(h(X, Y), \sum_{i=1}^{2p} h(E_i, E_i))$$

$$- \sum_{i=1}^{2p} \{g(h(E_i, Y), h(X, E_i))\}$$

$$+ 3 \sum_{i=1}^{2p} g(C E_i, Y), C(E_i, X))\}. \qquad \ldots(5.15)$$

Now we have the following equation (4.1), (4.2), (4.3) of Bejancu[1]

$$\sum_{i=1}^{m} \{g(JPE_i, Y)\, g(JPX, E_i)\} = -g(PX, PY)$$

$$\sum_{i=1}^{m} \{g(JPE_i. E_i)\} = 0$$

$$\sum_{i=1}^{m} \{g(E_i, JPX)\, g(E_i, JPY)\} = g(PX, PY),$$

for any vector field X, Y on M, so in our case the above equations take the following forms.

$$\sum_{i=1}^{2p} \{g(JE_i, Y)\, g(JX, E_i)\} = -g(X, Y) \qquad \ldots(5.16)$$

$$\sum_{i=1}^{2p} \{g(JE_i, E_i)\} = 0 \qquad \ldots(5.17)$$

$$\sum_{i=1}^{2p} \{g(E_i, JX)\, g(E_i, JY)\} = g(X, Y). \qquad \ldots(5.18)$$

If we use (5.16) – (5.18) in (5.15) we obtain following expression for the Ricci tensor S^* of B

$$S^*(X_*, Y_*) = \frac{pc}{2} g(X, Y) - \frac{c}{4} g(X, Y) + \frac{3c}{4} g(X, Y)$$

$$+ \sum_{i=1}^{2p} \{g(h(X, Y), h(E_i, E_i) - g(h(E_i, Y), h(E_i, X))\}$$

$$+ 3 \sum_{i=1}^{2p} \{g(C(E_i, Y), C(E_i, X))\}$$

(equation contiunued on p. 1203)

$$= \frac{(p+1)c}{2} g(X, Y) + \sum_{i=1}^{2p} \{g(h(X, Y), h(E_i, E_i))$$

$$- g(h(E_i, X), h(E_i, X)\} + 3 \sum_{i=1}^{2p} \{g(C(E_i, Y), C(E_i, X))\}$$

$$S^*(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y) + \sum_{i=1}^{2p} \{g(h X, Y), h(E_i, E_i)$$

$$- g(h(E_i, Y)\}, h(E_i, X)\} + 3 \sum_{i=1}^{2p} \{g(h(E_i, J_U Y), h(E_i, JX))\}$$

$$[\because C(C(E_i, Y) = -Jh(E_i, JY)]. \qquad \ldots(5.19)$$

If we compute the scalar curvature ρ^* of B, we get

$$\rho^* = p(p+1)c + \sum_{i,j=1}^{2p} \{g(h(E_i, E_i), h(E_j, E_j)$$

$$- g(h(E_i, E_j), h(E_i, E_j)\} \qquad \ldots(5.20)$$

$$+ 3 \sum_{i=1}^{2p} \{g(h(E_i, JE_j), h(E_i, JE_j\}.$$

Theorem 5.1—Let $\pi : M \to B$ b be a submersion of a mixed foliate CR-submanifold M of complex space form $\bar{M}(c)$ of constant holomorphic sectional curvature c, onto an almost Hermitian manifold B. Then the Ricci tensor S^* of B satisfies:

$$S^*(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y)$$

for any horizontal vector field $X, Y \in D$. Where 2p is the dimension of D.

As a direct consequence of above theorem we have

Corollary 5.1—Under the hypothesis of above theorem, B is an Einstein space.

PROOF OF THEOREM 5. 1 : Since $\{JF_1, \ldots, JF_q\}$ is a local field of orthonormal frames of the normal bundle ν. We have

$$h(X, Y) = \sum_{k=1}^{q} g(\tilde{A}_{JF_k}, X. Y) JF_k.$$

Using above relation we obtain

$$\sum_{i=1}^{2p} \{g(h(X, Y), h(E_i, E_i))\} = \sum_{k=1}^{q} (\text{tr. } \tilde{A}_K) g(\tilde{A}_{JF_k} X, Y) \qquad \ldots(5.21)$$

$$[\tilde{A}_k = \tilde{A}_{JF_k}]$$

$$\sum_{i=1}^{2p} \{g(h(E_i, X), h(E_i, Y)\} = \sum_{k=1}^{q} g(\tilde{A}_{JF_k} X, \tilde{A}_{JF_k} Y) \quad \ldots(5.22)$$

$$\sum_{i=1}^{2p} \{g(h(E_i, JX), h(E_i, JY)\} = \sum_{k=1}^{q} g(\tilde{A}_J{}^F{}_k\, JX, \tilde{A}_{JF_k} JY)$$

$$= \sum_{k=1}^{q} g(\tilde{A}_{JF_k} X, \tilde{A}_{JF_k} Y) \quad \ldots(5.23)$$

[$\therefore$ M is mixed foliate]

Using (5.19), (5.21), (5.22) and (5.23) we get

$$S^*(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y) + \sum_{k=1}^{q} 2g(\tilde{A}_{JF_k} X, \tilde{A}_{JF_k} Y)$$

$$= \frac{(p+1)}{2} cg(X, Y) + 2 \sum_{k=1}^{q} g(A_X F_k, A_Y F_k).$$

In the proof of Proposition 3.7 we have $g(A_X V, A_Y W) = 0$, hence the above relation transforms into

$$S^*(X_*, Y_*) = \frac{(p+1)c}{2} g(X, Y)$$

which gives the result.

Theorem 5.2—Let M be a foliate CR-submanifold of a complex space form $\bar{M}(c)$ of constant holomorphic sectional curvature c. Let $\pi : M \to B$ be a submersion of M onto an almost Hermitian manifold B. Then M is D-totally geodesic if and only if the scalar curvature of B satisfies

$$\rho^* = p(p+1)c$$

PROOF : Since M is foliate, D is integrable, $h(E_i, JE_j)) = Jh(E_i, E_j))$ and also M is D minimal[1]. Therefore from (5.20) we get

$$= p(p+1)c + 2 \sum_{i=1}^{2p} \|h(E_i, E_j)\|^2$$

which proves the theorem.

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