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ON LINEAR COMBINATIONS OF n ANALYTIC FUNCTIONS IN GENERALIZED PINCHUK AND GENERALIZED MOULIS CLASSES

S. BHARGAVA AND S. NANJUNDA RAO

Department of Mathematics, University of Mysore, Manasagangothri
Mysore 570006

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In this paper, we obtain two Theorems of general nature which unify and generalize some known radii results concerning linear combinations of analytic functions in various well known classes.

1. INTRODUCTION

Let N denote the set of all analytic functions on the unit disc $|z| < 1$ such that $f(0) = 0, f'(0) = 1$. Let $U_k^\lambda(\beta, c)$ be the class of all functions f in N such that

$$\int_0^{2\pi} |\operatorname{Re} e^{i\lambda} J_{f(z)} - \beta \cos \lambda| \, d\theta \leqq k\pi (1 - \beta) \cos \lambda \qquad \ldots(1)$$

where

$$J_{f(z)} = 1 - \frac{1}{c} + \frac{z}{c} \frac{f'(z)}{f(z)}, \qquad \ldots(2)$$

c being a nonzero complex number, $-\pi/2 < \lambda < \pi/2$, $0 \leqq \beta < 1$ and $k \geqq 2$. The class $U_k^\lambda(\beta, c)$ contains as special cases may classes of analytic functions studied in literature, for example, (i) ($c = 1, \lambda = 0, k = 2$) the class $S^*(\beta)$ of starlike functions of order β due to Robertson[13], (ii) ($c = 1, k = 2$) the class $U_2^\lambda(\beta)$ of λ-spirallike functions of order β due to Sizuk[16], (iii) ($c = 1, \beta = 0, \lambda = 0$) the class U_k of bounded radius rotation due to Pinchuk[12] and its generalizations $U_k(\beta)$ due to Padmanabhan and Parvatham[11] and $U_k^\lambda(\beta)$ due to Reddy[5].

Let $V_k^\lambda(\beta, c)$ be the class of functions f in N such that zf' is in $U_k^\lambda(\beta, c)$. This class generalizes many classes of analytic functions such as (i) ($c = 1, \lambda = 0, k = 2$) the class $C(\beta)$ of convex functions of order β due to Robertson[13], (ii) ($c = 1, \beta = 0, k = 2$) the class V_2^λ of Robertson functions namely functions f for which zf' is λ-spirallike[14], (iii) ($c = 1, \lambda = 0, \beta = 0$) the class V_k of functions of bounded boundary rotation due to Paatero[9] and its generalizations $V_k^\lambda(\beta)$due to Moulis[7] and $V_k\,(c)$ due to Nasr[8]. The class $V_k^\lambda\,(\beta, c)$ was introduced by Reddy[6].

In Section 3 of this note we prove the following two Theorems concerning linear combinations of n functions in the classes $U_k^\lambda\,(\beta, c)$ and $V_k^\lambda\,(\beta, c)$.

Theorem 1—Let $f_1, \ldots, f_n$ be n functions in $U_k^\lambda\,(\beta, c)$ and $F=\gamma_1 f_1 + \cdots + \gamma_n f_n$ where $\gamma_1, \ldots, \gamma_n$ are complex numbers such that $0 \leqq \mu = \max\limits_{1\leqq i,\, j\leqq n} \arg(\gamma_i/\gamma_j) < \pi$. Further let $\phi = \arg c$ and

$$g\,(r) = \mu + 2\,|\,c\,|\,(1 - \beta) \cos\lambda\,[k\,|\cos(\lambda - \phi)|\sin^{-1} r + |\sin(\lambda-\phi)|$$

$$\log \frac{(1 + r)^{k/2-1}}{(1 - r)^{k/2+1}}] < \pi,\ (0 \leqq r < 1). \qquad \ldots(3)$$

Then $\operatorname{Re}\left(1 - \frac{1}{c} + \frac{z}{c}\,\frac{F'(z)}{F(z)}\right) > 0$ in $|\,z\,| < R_0$ where R_0 is the least positive root of the equation

$$h\,(r) = 1 + r^2\,[2\,(1 - \beta)\cos^2\lambda - 1] - kr\,(1 - \beta)$$

$$\cos\lambda\,\sec\,[g\,(r)/2] = 0. \qquad \ldots(4)$$

Theorem 2—Let F be as in Theorem 1 with $f_1, \ldots, f_n$ in $V_k^\lambda\,(\beta, c)$. Let $g\,(r)$ be as in (3). Then $\operatorname{Re}\left(1 + \frac{z}{c}\,\frac{F''(z)}{F'(z)}\right) > 0$ in $|\,z\,| < R_0$ where R_0 is as above.

The above two Theorems unify and generalize many results found in literature. In fact the following corollaries are immediate.

Corollary 1—The conditions on $\gamma_1, \ldots, \gamma_n$ being as in Theorem 1 a region $|\,z\,| < R_0$ of starlikeness of $\gamma_1 f_1 + \ldots + \gamma_n f_n$, where $f_1, \ldots, f_n$ are λ-spirallike of order β, is obtained by putting $c = 1$ and $k = 2$ in Theorem 1.

As a further special case of this Corollary (when $\lambda = 0$) we obtain a region of starlikeness of linear combinations of starlike functions of order β.

Corollary 2—Let $n = 2$ and the conditions on γ_1 and γ_2 be as in Theorem 1. Then a region $|\,z\,| < R_0$ of starlikeness of $\gamma_1 f_1 + \gamma_2 f_2$ where f_1, f_2 are functions of bounded radius rotation is obtained by putting $\lambda = 0 = \beta$ and $c = 1$ in Theorem 1. We thus obtain Theorem 4 in Padmanabhan and Parvatham[10].

Corollary 3—The conditions on $\gamma_1, \ldots, \gamma_n$ being as in Theorem 1 a region $|\,z\,| < R_0$ of convexity of $\gamma_1 f_1 + \ldots + \gamma_n f_n$ where $f_1, \ldots, f_n$ are convex functions of order β, is obtained by putting $c = 1$, $k = 2$ and $\lambda = 0$ in Theorem 2. We thus obtain Theorem 2 in Bhargava and Rao[2]. In particular when $n = 2$ we get Theorem 2 of Silverman and Silvia[15].

Next, a region of convexity of $F = \gamma_1 f_1 + \ldots + \gamma_n f_n$ is obtained when each f_j is convex, by putting $\beta = 0$ in the above. In particular when $n = 2$ this yields Theorem 1 of Stump[17] which in turn contains the results of MacGregor[6] and Labelle and Rahman[4].

Corollary 4—The conditions on $\gamma_1, \ldots, \gamma_n$ being as in Theorem 1, a region $| z | < R_0$ of convexity of $\gamma_1 f_1 + \ldots + \gamma_n f_n$ where $zf_1', \ldots, zf_n'$ are λ-spirallike functions of order β is obtained by putting $c = 1$ and $k = 2$ in Theorem 2.

Corollary 5—The conditions on $\gamma_1, \ldots, \gamma_n$ being as in Theorem 1, a region $| z | < R_0$ of convexity of $\gamma_1 f_1 + \ldots + \gamma_n f_n$ where $f_1, \ldots, f_n$ are functions of bounded boundary rotation is obtained by putting $c = 1$ and $\lambda = 0 = \beta$. Again when $n = 2$ and $k = 2$ this reduces to Theorem 1 in Stump[17].

Lemma 1 stated below in Section 2 and proved by the authors[2] is a key for all our discussions. It provides a direct generalization of a Lemma Stump[17]. Stump devised his Lemma for discussing linear combinations $\gamma_1 f_1 + \gamma_2 f_2$ of two convex functions f_1 and f_2. Padmanabhan and Parvatham[10] have used Stump's Lemma while discussing $\gamma_1 f_1 + \gamma_2 f_2$ when f_1 and f_2 are in the class U_k. Campbell[3] has given an excellent treatment of various radii results for linear combinations of n functions in various classes of analytic functions. However, Campbell has not considered the spaces considered by us here. Moreover, unlike us, Campbell felt that Stump's formulation of determining the radii results for $\gamma_1 f_1 + \gamma_2 f_2$ in terms of the joint parameter $\alpha = \arg (\gamma_1/\gamma_2)$ discouraged a generalization of the problem to arbitrary finite combinations $\sum_1^n \gamma_j f_j$ $(\sum_1^n \gamma_j = 1)$ and thus he reformulated the problem in terms of the bound on the parameters $\arg \gamma_j$.

2. Some Preliminary Results

Lemma 1—If a, u_j and $\beta_j \neq 0$ $(j = 1, \ldots, n)$ are complex numbers with $\sum_1^n \beta_j = 1$ and $d \geqq 0$ such that $| u_j - a | \leqq d$ $(j = 1, \ldots, n)$ and $0 \leqq \theta = \max\limits_{1 \leqq i, j \leqq n} \arg (\beta_i/\beta_j) < \pi$, then

$$\operatorname{Re} \sum_1^n u_j \beta_j \geqq \operatorname{Re} a - d \sec \theta/2.$$

Proof : See Bhargava and Rao[2].

Lemma 2—Let $g(r)$ be defined by

$$g(r) = \mu + A_0 \sin^{-1} r + A_1 \log \frac{(1 + r)^{k/2-1}}{(1 - r)^{k/2+1}}$$

where $0 \leqq r < 1$, $A_0 > 0$, $A_1 \geqq 0$, $\mu < \pi$ and $p \geqq 0$. Then $g(r)$ increases strictly in $[0, 1)$ from μ to ∞ and therefore there exists a unique r_0 in $(0, 1)$ such that $g(r_0) = \pi$.

PROOF : It is easy to see that $g(0) = \mu$ and $g(r) \to \infty$ as $r \to 1 - 0$. Now

$$g'(r) = \frac{A_0}{\sqrt{1 - r^2}} + \frac{A_1 (k + 2r)}{1 - r^2} > 0$$

and hence the result is true.

Lemma 3—If f is in $U_k^\lambda (\beta, c)$ and $\phi = \arg c$ then

$$|\arg f/z| \leqq |c| (1 - \beta) \cos\lambda \, [k \, |\cos(\lambda - \phi)| \sin^{-1} r + |\sin(\lambda - \phi)| \log \frac{(1 + r)^{k/2 - 1}}{(1 - r)^{k/2+1}}], \; (0 \leqq r = |z| < 1).$$

PROOF : Since f is in $U_k^\lambda (\beta, c)$, we may choose g in U_k such that[1],

$$(f/z) \frac{e^{i\lambda} \sec\lambda}{c(1 - \beta)} = g/z.$$

Here using[1],

$$|\arg g/z| \leqq k \sin^{-1} r, \; \log |g/z| \leqq \log \frac{(1 + r)^{k/2-1}}{(1 - r)^{k/2+1}}$$

we have the required inequality.

3. PROOFS OF THE MAIN THEOREMS

Proof of Theorem 1—$F(z) = \gamma_1 f_1(z) + \ldots + \gamma_n f_n(z)$ where f_j are in $U_k^\lambda (\beta, c)$. Hence

$$1 - \frac{1}{c} + \frac{z}{c} \frac{F'(z)}{F(z)} = \sum_1^n u_j(z) \, \beta_j(z).$$

where

$$u_j(z) = 1 - \frac{1}{c} + \frac{zf_j'(z)}{cz_j(z)} \text{ and } \beta_j(z) = \frac{\gamma_j f_j(z)}{\sum_1^n \gamma_j f_j(z)}.$$

Since each f_j is in $U_k^\lambda(\beta, c)$, we have from Bhargava and Nanjunda Rao[1] $|u_j - a| \leqq d$ where

$$a = \frac{1 + r^2 (2e^{-i\lambda} (1 - \beta) \cos\lambda - 1)}{1 - r^2}, \; d = \frac{kr(1 - \beta)\cos\lambda}{1 - r^2}.$$

Hence from Lemma 1, Re $\left(1 - \frac{1}{c} + \frac{z}{c}\frac{F'}{F}\right) > 0$ for $|z| < R_0$ where R_0 is the least positive root of the equation Re $a - d \sec g(r)/2 = 0$, that is, R_0 is the least positive root of the equation $h(r) = 0$. That $h(r) = 0$ has a positive root follows since $h(0) = 1$ and $h(r) \rightarrow -\infty$, since $g(r) \rightarrow \pi$ (as $r \rightarrow r_0$ where r_0 is as in Lemma 2).

Proof of Theorem 2—Since f_j is in $V_k^\lambda(\beta, c)$ we see that zf'_j is in $U_k^\lambda(\beta, c)$ (Bhargava and Rao[7]). The Theorem now follows immediately from Theorem 1 on changing f_j to zf'_j there.

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