Indian J. pure appl. Math., **19** (9) : 906-914, September 1988

PROPAGATION CHARACTERISTICS IN DISTENSIBLE TUBES CONTAINING A DUSTY VISCOUS FLUID

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(*Received 23 March* 1987; *after revision* 1 *January* 1988)

Here the propagation characteristics of a dusty viscous fluid, regarded as analogus to blood in arteries, in a distensible tube are studied. The equation of motion of the vessel wall takes into account the pulsatible nature of the wall. Results are analysed for the resistance and the reactance of the fluid and wall impedance. The influence of dust particles on these factors is noted and observation made in reference to the presence of the high haemoglobin contents in blood.

INTRODUCTION

Certain investigations have been made on blood flow in the frequency range of physiological importance[4-7]. Kaimal[2] investigated the propagation characteristics of a visco-elastic fluid in a distensible tube. The present investigation is concerned with the problem of a dusty viscous fluid in distensible vessel. The vessel is modelled as a straight thin walled circular distensible tube and is externally constricted so that the longitudinal movement may be ignored. The study will be helpful in understanding the flow features of blood, especially in the coronary arteries which are subjected to varying external pressure within the muscular walls of the heart, by visualizing blood as a dusty viscous fluid exhibiting characteristics of a visco-elastic fluid as shown by Chadda[1] who compared his result with those of Tandon[6].

MATHEMATICAL FORMULATION

The equations governing the flow of dusty viscous fluid flow in a distensible tube in cylindrical polar coordinates (r, z) with axial symmetry, following Saffman[5] are :

$$\frac{\partial u_1}{\partial z} + \frac{u_2}{r} + \frac{\partial u_2}{\partial r} = 0 \qquad \ldots(1)$$

$$\frac{\partial v_1}{\partial z} + \frac{v_2}{r} + \frac{\partial v_2}{\partial r} = 0 \qquad \ldots(2)$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial z} + u_2 \frac{\partial u_1}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} + \frac{\partial^2 u_1}{\partial z^2}\right) + \frac{k N}{\rho}(v_1 - u_1) \qquad \ldots(3)$$

$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial z} + u_2 \frac{\partial u_2}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} - \frac{u_2}{r} + \frac{\partial^2 u_2}{\partial z^2}\right) + \frac{kN}{\rho}(v_2 - u_2) \quad ...(4)$$

$$m\left\{\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial z} + v_2 \frac{\partial v_1}{\partial r}\right\} = k(u_1 - v_1) \quad ...(5)$$

and

$$m\left\{\frac{\partial v_2}{\partial t} + u_1 \frac{\partial v_2}{\partial z} + v_2 \frac{\partial v_2}{\partial r}\right\} = k(u_2 - v_2) \quad ...(6)$$

where (u_1, v_1) and (u_2, v_2) are respectively, the velocities of the fluid and dust particles, p is fluid pressure ρ the density of the fluid, m the mass of a dust particle, k the Stokes' resistance coefficient, N the number density of dust particles, assumed to be constant, and ν the kinematic coefficient of viscosity.

Ignoring the longitudinal components, the motion of the thin walled tube, under the initial conditions $\xi(0) = \xi_0$, $\dot{\xi}(0) = \xi_0'$ is described by Knowles[3] as

$$\frac{d^2\xi}{d\bar{t}^2} + \left\{\bar{K} - \frac{(\bar{p}_1 - \bar{p}_2)}{\frac{1}{2}\mu\,\bar{\rho}_m}\right\}\xi - \bar{K}\xi^{-3} = 0 \quad ...(7)$$

where

$$\bar{K} = \frac{Kr_1^4}{\nu^2},\ K = \frac{\alpha + \beta}{\rho_m r_1^2},\ \bar{\rho}_m = \frac{\rho_m}{\rho},\ \bar{t} = \frac{t\nu}{r_1^2}.$$

α, β are material constants and ρ_m is density of the material, r_1 is the internal radius of the tube and quantities with bars denote corresponding non-dimensional quantities. For the quantity $(\bar{p}_1 - \bar{p}_2)$, being the difference between pressure on two sides of the tube wall, we assume that

$$\frac{\bar{p}_1 - \bar{p}_2}{\frac{1}{2}u\,\bar{\rho}_m} = A(\bar{z})\,\psi(\bar{\omega}\,\bar{t}) \quad ...(8)$$

where $\psi(\bar{\omega}\,\bar{t})$ is a periodic function, $|\psi(\bar{\omega}\,\bar{t})| < 1$ and $\bar{z} = \frac{z}{r_1}$. The homogeneous part of eqn. (7), using eqn. (8) is

$$\frac{d^2\xi}{d\bar{t}^2} + \{\bar{K} - A\psi(\bar{\omega}\,\bar{t}) \quad ...(9)$$

which is a generalized Hills equation.

Solution of the Problem

The independent solutions of equation (9) are

$$\xi_1(\bar{t}) = C_1 \cos(\bar{K}^{1/2} \int_0^{\bar{t}} \phi(\bar{t})\,d\bar{t}) \quad ...(10)$$

and

$$\xi_2(\bar{t}) = C_2 \sin\left(\bar{K}^{1/2} \int_0^{\bar{t}} \phi(\bar{t})\, d\bar{t}\right) \qquad \ldots(11)$$

where

$$\left.\begin{aligned} &\phi(\bar{t}) = \sigma - (2\sigma\ddot{\sigma} - 3\dot{\sigma}^2)/8\bar{K}\sigma^3 \\ &\sigma^2 = 1 - \bar{\alpha}\psi(\bar{\omega}\bar{t}),\ \bar{\alpha} = A/\bar{K} \\ &C_1 = \xi_0\, \sigma(0)^{1/2} \end{aligned}\right\} \qquad \ldots(12)$$

and

$$C_2 = \bar{K}^{-1/2}\left\{\xi_0'\, \sigma(0)^{1/2} + \tfrac{1}{2}\xi_0\, \dot{\sigma}(0)^{3/2}\right\}/\phi(0).$$

It may be noticed that

$$\left.\begin{aligned} &\xi_1(0) = \xi_0,\ \dot{\xi}_1(0) = \xi_0' \\ &\xi_2(0) = \xi_0,\ \dot{\xi}_2(0) \neq 0. \end{aligned}\right] \qquad \ldots(13)$$

Therefore, the solution of the non linear equation (7) using initial conditions is

$$\xi(t) = \frac{[(\bar{A}^4\phi^2 + \sigma^2) + (\bar{A}^4\phi^2 - \sigma^2)\cos(2\bar{K}^{1/2}\int_0^{\bar{t}}\phi(\bar{t})\, d(\bar{t})]}{\sqrt{2}\,\xi_0\,\sigma(0)^{1/2}\,\phi\sigma^{1/2}}. \qquad \ldots(14)$$

This solution is stable and real only if $\bar{\alpha} < 1$ and $\bar{\alpha}$ signifies the influence of fluid on the motion of wall of the tube.

As a more specific case, let $\psi(\bar{\omega}\ \bar{t}) = \sin\bar{\omega}\,\bar{t}$, $\xi_0 = 1$, $\xi_0' \neq 0$ so that

$$\sigma(0) = 1,\ C_1 = 1. \qquad \ldots(15)$$

If we assume that such a dusty viscous fluid may be regarded as an approximation to blood, then in most physiological situations $\omega^2/\bar{K}$ and higher powers may be neglected. Therefore function $\phi(\bar{t})$ can be approximately taken as $\sigma(\bar{t})$, consequent to which eqn. (14) with eqn. (15) reduces to

$$\xi(\bar{t}) = \frac{1}{\sigma^{1/2}} = (1 - \bar{\alpha}\sin\bar{\omega}\,\bar{t})^{-1/4} \qquad \ldots(16)$$

which suggests that the general solution of the radial component of oscillatory motion of the tube may be given by

$$\xi(t) = \{\perp - \bar{\alpha}\exp i(\bar{\omega}\,\bar{t})\}^{-1/4}. \qquad \ldots(17)$$

Here the motion of the wall is determined by the pressure difference between the two sides of the wall. Equation (17), when modified to include an exponentially decaying factor in the axial direction, takes the form

$$\xi(\bar{z}, \bar{t}) = \{1 - \bar{\alpha} \exp i(\bar{\omega}\bar{t})\}\bar{\xi}^{1/4} - \bar{e}^{i\bar{k}_0 z} \qquad \ldots(18)$$

where $\bar{k}_0$ is the dimensionless wave number. To solve highly nonlinear equations (3)-(6), equations are linearized by introducing long wave-length approximations and are transformed by

introducing non-dimensional quantities :

$$\bar{u} = \frac{u_1 r_1}{\nu}, \quad \bar{u}_2 = \frac{u_2 r_1}{\nu}, \quad \bar{v}_1 = \frac{v_1 r_1}{\nu}, \quad \bar{v}_2 = \frac{v_2 r_1}{\nu}, \quad \bar{t} = \frac{t\nu}{r_1^2}$$

$$\tau = \frac{m\nu}{kr_1^2}, \quad f = \frac{mN}{\rho}, \quad \bar{r}_1 = \frac{r}{r_1}, \quad \bar{z} = \frac{z}{r_1}, \quad \bar{p} = \frac{pr_1^2}{p\nu^2}, \quad \bar{\omega} = \frac{\omega r_1^2}{\nu}.$$

Hence, eqns. (3) — (6) are reduced to

$$\frac{\partial \bar{u}_1}{\partial \bar{t}} + \frac{\partial \bar{p}}{\partial \bar{z}} = \frac{\partial^2 \bar{u}_1}{\partial \bar{r}_2^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}_1}{\partial \bar{r}} + \frac{f}{\tau}(\bar{v}_1 - \bar{u}_1) \qquad \ldots(19)$$

$$\frac{\partial \bar{u}_2}{\partial \bar{t}} + \frac{\partial \bar{p}}{\partial \bar{r}} = \frac{\partial^2 \bar{u}_2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}_2}{\partial \bar{r}} + \frac{f}{\tau}(\bar{v}_2 - \bar{u}_2) \qquad \ldots(20)$$

$$\frac{\partial \bar{v}_1}{\partial \bar{t}} = \frac{1}{\tau}(\bar{u}_1 - \bar{v}_1) \qquad \ldots(21)$$

and

$$\frac{\partial \bar{u}^2}{\partial \bar{t}} = \frac{1}{\tau}(\bar{u}_2 - \bar{v}^2) \qquad \ldots(22)$$

where f is mass concentration of the dust particles and τ is the relaxation time, with boundary canditions :

$$\bar{u} = 0, \quad \bar{u}_2 = \dot{\xi}(\bar{t}) \text{ at } \bar{r} = 1. \qquad \ldots(23)$$

Using the boundary conditions, the velocity components of the fluid and the dust particles are given by

$$\bar{v}_1 = \frac{\bar{u}_1}{1 + \tau_1 \bar{\omega} + \dfrac{5/4\tau i\bar{\omega}\,\bar{\alpha} \exp i(\bar{\omega}\bar{t})}{1 - \bar{\alpha} \exp i(\bar{\omega}\bar{t})}} \qquad \ldots(24)$$

$$\bar{v}_2 = \frac{\bar{u}_2}{1 + \tau_i \bar{\omega} + \dfrac{5/4\ \tau_i\ \bar{\omega}\ \bar{\alpha}\ \exp i\,(\bar{\omega}\bar{t})}{1 - \bar{\alpha}\ \exp i\,(\bar{\omega}\bar{t})}} \qquad \ldots(25)$$

$$\bar{u}_1 = \frac{p_0}{(\delta^2 - n^2)} \left\{ \frac{nJ_0(n)\, J_0(\delta\ \bar{r})}{J_0(\delta)} - n\, J_0(n\bar{r}) \right\} \exp i\,(\bar{\omega}\ \bar{t} - \bar{k}_0\bar{z}) \qquad \ldots(26)$$

$$\bar{u}_2 = \left[\left\{ A_1 + \frac{np_0}{(\delta^2 - n^2)}\, J_1(n) \right\} \frac{J_1(\delta\bar{r})}{J_1(\delta)} - \frac{np_0}{(\delta^2 - n^2)}\, J_1(n\bar{r}) \right] \exp i\,(\bar{\omega}\ \bar{t}) - \bar{k}_0\,\bar{z} \qquad \ldots(27)$$

where

$$\delta^2 = \frac{f}{\tau} \left\{ \frac{1}{1 + \tau i\ \bar{\omega} + \dfrac{5\ \tau i \omega \bar{\alpha}\ \exp i\,(\bar{\omega}\ \bar{t})}{4\,(1 - \bar{\alpha}\ \exp i\,(\bar{\omega}\ \bar{t}))}} - 1 - \frac{\tau i \bar{\omega}}{f} - \frac{5\tau i\ \bar{\omega}\bar{\alpha}\ \exp i\,(\bar{\omega}\ \bar{t})}{4f\,(1 - \bar{\alpha}\ \exp i\,(\bar{\omega}\ \bar{t})} \right\}$$

$$\bar{p} = p_0\, J_0(n\bar{r}) \exp i\ (\bar{\omega}\ \bar{t} - \bar{k}_0\bar{z})$$

and

$$n = i\bar{k}_0. \qquad \ldots(28)$$

Following Womersley[8], the components of velocity of the fluid u_1, u_2 and those of the dust particles v_1, v_2, for $n^2 \ll \delta^2$ are given by

$$\bar{u} = \frac{n\,p_0}{\delta^2}\, J_0(n) \left\{ \frac{J_0(\delta\bar{r})}{J_0(\delta)} - \frac{J_0(n\bar{r})}{J_0(n)} \right\} \exp i\ (\bar{\omega}\ \bar{t} - \bar{k}_0\bar{z}) \qquad \ldots(29)$$

$$\bar{u}_2 = \left[\left\{ A_1 + \frac{np_0}{\delta^2}\, J_1(n) \right\} \frac{J_1(\delta\ \bar{r})}{J_1\delta} - \frac{np_0}{\delta^2}\, J_1(n\ \delta) \right] \exp i\ (\bar{\omega}\ \bar{t} - \bar{k}_0\,\bar{z}) \qquad \ldots(30)$$

$$\bar{v}_1 = \frac{\bar{u}_1}{1 + \tau i\,\bar{\omega} + 5/4\ \tau i\ \ \bar{\omega}\bar{\alpha}\ \exp i\,(\bar{\omega}\ \bar{t})} \qquad \ldots(31)$$

$$\bar{v}_2 = \frac{\bar{u}_2}{1 + \tau i\bar{\omega} + 5/4\ \bar{\omega}\bar{\alpha}\ \exp i\,(\bar{\omega}\ \bar{t})}. \qquad \ldots(32)$$

The rate of flow Q in the lumen is given by

(taking $J_0(n) = 1$, $J_1(n) = n/2$)

$$Q = \frac{\pi n p_0}{\delta^2} \exp i\,(\bar{\omega}\bar{t} - \bar{k}_0\,\bar{z}) \left\{ \frac{2J_1(\delta)}{\delta\, J_0(\delta)} - 1 \right\}. \qquad \ldots(33)$$

The average pressure over the cross section of the tube is given by

$$p(\bar{z}, \bar{t}) = \frac{2p_0}{n} J_1(n) \exp i\ (\bar{\omega}\bar{t} - \bar{k}_0\ \bar{z}). \qquad \ldots(34)$$

The longitudinal impedance Z is given by

$$Z = -\ \frac{\partial^2}{\pi}\ \left\{1 - \frac{2J_1(\delta)}{\delta J_0(\delta)}\right\}^{-1} \quad \text{(approximately)}. \qquad \ldots(35)$$

The limiting value of Z for small value of δ is given by

$$Z_s = -\ \frac{1}{\pi}\ \left\{2\delta^2 - 8\right\}. \qquad \ldots(36)$$

Here in the limiting case, fluid resistance R_s and fluid reactance $(\omega L)_s$, are given by

$$R_s = -\ \frac{2}{\pi}\left[\frac{f}{\tau}\left\{\frac{\tau^2\bar{\omega}^2}{1+\tau^2\bar{\omega}^2} - \frac{5\tau\bar{\omega}\bar{\alpha}}{4(1+\tau^2\bar{\omega}^2)^2}\ (2\tau\bar{\omega}\cos\bar{\omega}\bar{t}\right.\right.$$

$$\left.\left. - \sin\bar{\omega}\bar{t} + \tau^2\bar{\omega}\sin\bar{\omega}\bar{t} + \frac{5\tau\bar{\omega}\bar{\alpha}}{4f}\ \sin\bar{\omega}\bar{t}\right\} - 4\right] \qquad \ldots(37)$$

and

$$(\omega L)_s = \frac{2}{\pi}\left[\frac{f}{\tau}\left\{\frac{\tau\bar{\omega}}{1+\tau^2\bar{\omega}^2} + \frac{5\tau\bar{\omega}\bar{\alpha}}{4(1+\tau^2\bar{\omega}^2)}\ (\cos\bar{\omega}\bar{t} - \tau^2\bar{\omega}^2\cos\bar{\omega}\bar{t}\right.\right.$$

$$\left.\left. + 2\tau\bar{\omega}\ \sin\bar{\omega}\bar{t} + \frac{\tau\bar{\omega}}{f} + \frac{5\tau\bar{\omega}\bar{\alpha}}{4f}\ \cos\bar{\omega}\bar{t}\right\}\right]. \qquad \ldots(38)$$

The polar form of Z_s (for small values of δ) is also given by

$$|Z_s| = \frac{2f}{\pi\tau}\left[\left\{\frac{\tau^2\bar{\omega}^2}{1+\tau^2\bar{\omega}^2} + \frac{4\tau}{t}\right\}^2 + \tau^2\bar{\omega}^2\left\{\frac{1}{1+\tau^2\bar{\omega}^2} + \frac{1}{f}\right\}^2\right.$$

$$+ \frac{5}{2}\ \tau\bar{\omega}\bar{\alpha}\left\{\left(\frac{\tau^2\bar{\omega}^2}{1+\tau^2\bar{\omega}^2} + \frac{4\tau}{f}\right)\right.$$

$$\times\left(\frac{2\tau\bar{\omega}\cos\bar{\omega}\bar{t} - (1-\tau^2\bar{\omega}^2)\sin\bar{\omega}\bar{t}}{(1+\tau^2\bar{\omega}^2)^2}\right.$$

$$\left.\left.\left. - \frac{\sin\bar{\omega}\bar{t}}{f}\right) + \tau\bar{\omega}\left(\frac{1}{(1+\tau^2\bar{\omega}^2)} + \frac{1}{f}\right)\right.\right.$$

(equation continued on p. 912)

$$\times \left(\frac{(1-\tau^2\bar{\omega}^2)\cos\bar{\omega}\bar{t} + 2\tau\bar{\omega}\sin\bar{\omega}\bar{t}}{(1+\tau^2\bar{\omega}^2)^2} + \frac{\cos\bar{\omega}\bar{t}}{f} \right) \Bigg\} \Bigg]^{1/2} \quad \ldots(39)$$

and

$$\psi_s = \frac{\left[\frac{f}{\tau} \left\{ \frac{\tau\bar{\omega}}{1+\tau^2\bar{\omega}^2} + \frac{5\tau\bar{\omega}\bar{\alpha}}{4(1+\tau^2\bar{\omega}^2)^2} (\cos\bar{\omega}\bar{t} - \tau^2\bar{\omega}^2\cos\bar{\omega}\bar{t} + 2\tau\bar{\omega}\sin\bar{\omega}\bar{t}) + \frac{\tau\bar{\omega}}{f} + \frac{5\tau\bar{\omega}\bar{\alpha}}{4f}\cos\bar{\omega}\bar{t} \right\} \right]}{\left[\frac{f}{f} \left\{ \frac{\tau^2\bar{\omega}^2}{1+\tau^2\bar{\omega}^2} + \frac{5\tau\bar{\omega}\bar{\alpha}}{4(1+\tau^2\bar{\omega}^2)^2} (2\tau\bar{\omega}\cos\bar{\omega}\bar{t} - \sin\bar{\omega}\bar{t} + \tau^2\bar{\omega}^2\sin\bar{\omega}\bar{t}) - \frac{5\tau\bar{\omega}\bar{\alpha}}{4f}\sin\bar{\omega}\bar{t} \right\} + 4 \right]}. \quad \ldots(40)$$

Discussion

Noting that parameter τ is a measure of the time for the dust particles to adjust to changes in the fluid velocity, it is called relaxation time and is proportional to the size of the indiviaual particles. Also f signifies the amount of dust particles present. From the results obtained above we can discuss the influence of dust particles on wave propagation characteristics and on the pressure flow relationship in the tube. As the tube wall is subjected to external pressure the results can be applied to study the flow

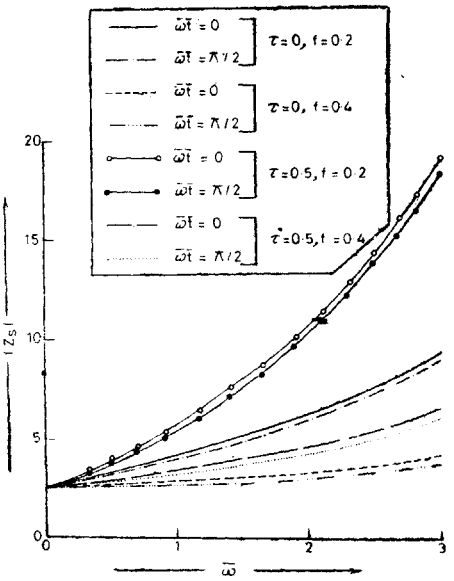

FIG. 1. Magnitude $|Z_s|$ of the wall Impedance Against $\bar{\omega}$.

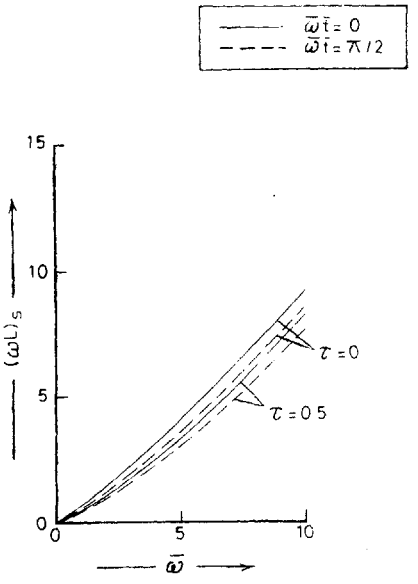

FIG. 2. The impedance angle ψ_s against $\bar{\omega}$.

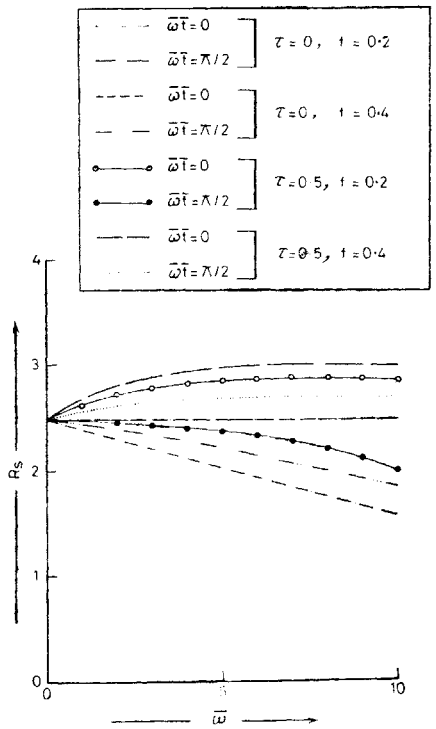

FIG. 3. The fluid Resistance R_s Against $\bar{\omega}$.

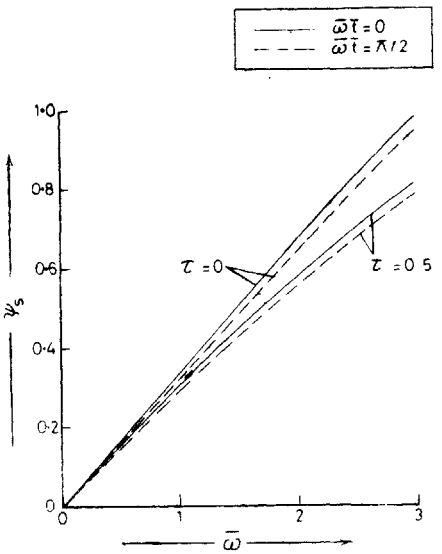

FIG. 4. The fluid Reactance $(\omega L)_s$ against $\bar{\omega}$.

in the coronary arteries where much of their length is imbedded in the muscular walls of the heart and are subjected to varying external pressure. Equation (35) gives the longitudinal impedance due to the flow and equation (36) gives the impedance for small values of δ. The behaviour of transverse wall impedance is plotted in Figs. 1 and 2 against various values of $\bar{\omega}$ when the value of $\bar{\alpha}$ is taken to be 0.1 in all cases. These graphs dipict the effects of wall oscillation and presence of dust particles in fluid resistance and reactance respectively. It is seen that effects of dust particles and wall pulsation are more significant for higher value of $\bar{\omega}$.

Figure 1 shows that the increase in the value of f from 0.2 to 0.4 causes decrease in the impedance. This has the effect of bringing the frequency of oscillation of tube closer to its natural frequency, while Fig. 3 depicts increase in resistance with the increase in the value of f. Since increase in f is due to the increase in the concentration of the dust particles. We see that if there is increase in the cell contents of blood in arteries it will cause increase in the resistance.

ACKNOWLEDGEMENT

Authors are thankful to the learned referee for his valuable comments and suggestions.

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