INDEFINITE QUADRATIC FORMS IN MANY VARIABLES

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In this paper we investigate the positive and the nonnegative inhomogeneous spectra for indefinite binary quadratic forms of rank at least 21. (Recent work of Margulis should allow the restriction \( n \geq 21 \) to be reduced.) For the positive spectra we prove that the sharpest constant depends only on the congruence class (mod 8) of the signature of the form; all extreme forms and grids are also obtained. In addition we bound the best possible constant for all nonnegative inhomogeneous spectra.

1. INTRODUCTION

Let \( f(X) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j \) be a nonsingular quadratic form with real coefficients.

Using the notation in Watson\textsuperscript{31}, we let

\[
A = \begin{pmatrix}
2a_{11} & a_{12} & \cdots & a_{1n} \\
2a_{21} & 2a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1n} & a_{2n} & \cdots & 2a_{nn}
\end{pmatrix}
\]

be called the matrix associated with \( f \) and set \( D = \det f \) to equal the determinant of \( A \). Also, the discriminant of \( f \) is defined to be

\[
d(f) = (-1)^{[n/2]} \left\{ \begin{array}{ll}
\det f & \text{if } n \text{ is even} \\
\left(\frac{\det f}{2}\right) & \text{if } n \text{ is odd}
\end{array} \right.
\]

(1)

In Watson\textsuperscript{31} (p. 3), it is shown that the discriminant of any integral quadratic form is an integer; the nonsingularity of \( f \) implies that each of \( \det f \) and \( d(f) \) is nonzero. Further, in Watson\textsuperscript{31} (p. 3), it is shown that, if \( f_1 + f_2 \) is a disjoint sum, then

\[
d(f_1 + f_2) = \left\{ \begin{array}{ll}
d(f_1) d(f_2) & \text{if } n_1 n_2 \text{ is even} \\
-4d(f_1) d(f_2) & \text{if } n_1 n_2 \text{ is odd}
\end{array} \right.
\]

(2)

Also, \( d(f) \) is an integer and by Watson\textsuperscript{31} (p. 21),

\[
d(f) \equiv 0, 1 \pmod{4}, \text{ if } n \text{ is even.}
\]

(3)
For any $V \in \mathbb{R}^n$, we define
\[ P^+ (f, V) = \inf \{ f(X + V) > 0 : X \text{ is an integral point}; \]
\[ P (f, V) = \inf \{ f(X + V) > 0 : X \text{ is an integral point} \}
\]

where for the special case $V = 0$ we restrict to $X \neq 0$. We define
\[ P_l (f) = \sup \{ P^+ (f, V) : V \in \mathbb{R}^n \}; \]
\[ P_l (f) = \sup \{ P (f, V) : V \in \mathbb{R}^n \}; \]

which may be called, respectively, the positive and nonnegative nonhomogeneous minima of the form $f$. For fixed $n > 1$, $|s| \leq n$, we consider the sets
\[
\{ P_l (f)^{+} | D | : f \text{ indefinite of rank } n \text{ and signature } s \};
\]
\[
\{ P_l (f)^{n} | D | : f \text{ indefinite of rank } n \text{ and signature } s \}
\]

which we shall call, respectively, the positive and nonnegative inhomogeneous spectra.

Bambah et al.\textsuperscript{2,3} have considered the positive spectrum. They show that, for $n > 2$ and $s = 0, 1, 2, 3; n \geq 7$ and $s = -1$,
\[ P_l (f)^{+} | D | \leq (|s| + 1)^{-1}. \]

They also prove that this bound is best possible for each of these values of $n, s$. Hence, this extends the work of many authors, among them: Davenport and Heilbronn\textsuperscript{11}, Blaney\textsuperscript{6}, Barnes\textsuperscript{4}, Dumir\textsuperscript{14,15}, Hans-Gill and Raka\textsuperscript{18}.

In addition, Dumir and Hans-Gill\textsuperscript{16} showed that 1 is the best constant for $n = 4, s = -2$; in Bambah et al.\textsuperscript{3}, (7/8)\textsuperscript{2} is proved to be best possible for $n = 5, s = -1$.

In this paper we investigate these spectra for $n \geq 21$. We prove that, for any $n, s$, the best possible constants depend only on $s \pmod{8}$; for the various nonnegative spectra an upper bound of $1/5$ is obtained. The statements of these results are given in Theorems 1 to 3 below.

For convenience, we set
\[ E_1 = x_1^2; \quad E_2 = x_1^2 + x_1 x_2 + x_2^2; \]
\[ E_3 = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3; \]
\[ E_5 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + (x_1 + x_2 + x_3) x_4 + (x_1 - x_2) x_5; \]
\[ E_4 = E_3 (x_1, x_2, x_3, x_4, 0); \]
\[ E_6 = \frac{1}{4} \sum_{i=1}^{4} (x_i^2 + (2x_{i+4} + x_i - \frac{4}{5} \sum_{i=1}^{4} x_i)^2); \]
\[ E_6 = E_3 (x_1, x_2, x_3, x_4, x_5); \]
\[ \hat{E}_6 = E_8 (0, 0, x_1, \ldots, x_6). \]
These forms are our building blocks; each $E_t$ is a positive definite integral quadratic form with minimum nonzero value equal to 1. Also,

| Table I |
|---|---|---|---|---|---|---|---|
| $f$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ | $E_8$ |
| det $f$ | 2 | 3 | 4 | 4 | 4 | 3 | 4 | 1 |
| $d(f)$ | 1 | -3 | -2 | 4 | 2 | -3 | -4 | 1 |

Define each of $F_0$ and $H_0$ to be the form which is identically zero, and for any nonzero integer $m$, define

$$F_m = \sum_{0 \leq i \leq m} E_8 (x_{0i-1}, \ldots, x_{0i+8});$$

for any positive integer $t$, let

$$H_t = \sum_{1 \leq i \leq t} x_{ti-1} x_{2t}.$$

Then

$$s(F_m) = 8m, \det F_m = 1; s(H_t) = 0, \det H_t = (-1)^t.$$

Definition—$f$ is said to be (integrally) equivalent to $g$ [denoted by $f \sim g$] if there exists an integer matrix $N$, with $\det N = \pm 1$, such that $f(NX) = g(X)$.

Theorem 1—Let $f$ be an indefinite quadratic form of rank $n \geq 21$ and signature $s = 8q + k, \quad -3 \leq k \leq 4$. Then for all $V \in \mathbb{R}^n$

$$P^r (f, V)^\sim \leq 1/\min \{ |k| + 1, 4 \}.$$

Moreover, the equality sign is required only when either

(i) $f$ is equivalent to a positive multiple of $\text{sgn}(q) \cdot F_m + H_t \sim g$, for $2t = n - 8m - n (g)$ and $m$ and $g$ are given in Table II; under this equivalence $V$ becomes $O \bmod 1$

| Table II |
|---|---|---|---|---|---|
| $k$ | 0 | $\pm 1$ | $\pm 2$ | $\pm 3$ | $\pm 3$ |
| $s$ | all | $|s| \leq n-2$ | $|s| = n-2$ | $|s| \leq n-4$ | $|s| > n-4$ |
| $m$ | $|q|$ | $|q|$ | $|q|$ | $|q|$ | $|q|$ |
| $g$ | 0 | $\pm E_1$ | $\pm E_2$ | $\mp E_6$ | $\pm E_3$ |

or

(ii) $f$ is equivalent to a positive multiple of $-x_1^2 - \ldots - x_1^2 + x_{t_1+1}^2 + \ldots + x_{t_1+t_2}^2 + h$, for suitable choice of $t_1$ and $t_2$, and $V$ becomes $(1/2, V_2) \bmod 1$, where $h$ and $V_2$ are given below (Table III):
This extends the following theorem proved by Watson\textsuperscript{23}. [Note: The fourth exceptional form was not recognized until Watson\textsuperscript{23}].

Let \( f \) be an integral, primitive, nonsingular quadratic form of rank \( n \) and signature \( s(f) = 8m + k \), for \(-3 \leq k \leq 4\). If \( f \) is definite, we assume it is positive definite and of rank \( n \leq 8 \). Then, if \( f \) is equivalent to none of the forms \( 4x_1 x_2 - x_3^2 \), \( 16x_1 x_2 - x_3^2 \), \( 3x_1 x_2 - 3x_3 x_4 - 3x_4^2 \), \( x_2 (6x_1 - x_2 + 3x_3) - 3x_4^2 \),

\[
\frac{P^+ (f, 0) n}{|D|} \leq \frac{1}{\min \{ |k| + 1, 4 \}}.
\]

The sign of equality is necessary only if either

\[ |\det f| = \min \{ |k| + 1, 4 \} \text{ or } f \sim 8x_1 x_2 - x_3^2. \]

In Watson\textsuperscript{23}, the author finds other elements of the spectrum \( \{P^+ (f, 0)^n/ |D| : n(f) = n \} \), for \( n = 3 \). In that article, he also gives an inductive argument for estimating \( P^* (f, 0)^n/ |D| \) for zero forms of rank \( n \geq 4 \) from information on forms of rank \( n - 2 \).

\textbf{Theorem 2}—Let \( f \) be an indefinite quadratic form of rank \( n \geq 21 \) and signature \( s = 8q + k \), \(-3 \leq k \leq 4\). If \( f \) is none of the exceptional forms listed in Theorem 1, then \( P^+ (f, V)^n/ |D| \leq 1/4 \). Moreover, for \( |k| \leq 2 \) equality is required only if either

(i) \( f \) is equivalent to a positive multiple of \( \text{sgn}(q) \cdot F_m + H_t + g \), for \( 2t = n - 8m - n(g) \) and \( m \) and \( g \) are given in Table IV.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 0 )</th>
<th>( \pm 1 )</th>
<th>( \pm 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
<td>( 0 )</td>
<td>( \pm 2E_1 )</td>
<td>( \pm 2E_3 )</td>
</tr>
<tr>
<td>( V ) (mod 1)</td>
<td>( 0 )</td>
<td>( \frac{1}{2} )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

and under this equivalence \( V \) becomes \( O \) (mod 1) except for \( g = E_1 - E_1 \) we also have \( V \equiv (0, 1, 1)/2 \) (mod 1);


<table>
<thead>
<tr>
<th>( k )</th>
<th>( 0 )</th>
<th>( \pm 1 )</th>
<th>( \pm 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>(</td>
<td>q</td>
<td>)</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( E_1 - E_1 )</td>
<td>( H_t )</td>
<td>( \pm 2E_1 )</td>
</tr>
</tbody>
</table>

or

(ii) \( f \) is equivalent to a positive multiple of

\[-x_1^2 - \cdots - x_i^2 + x_{i_1 + 1}^2 + \cdots + x_{i_2 + 2}^2 + h\]
for suitable choice of $t_1$, $t_3$, and under this equivalence $V$ becomes $(1/2, \tilde{V}) (\mod 1)$ for $h$ and $\tilde{V}$ given in Table V.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$\tilde{V}$ (mod 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\pm 2 (E_1 + E_1)$</td>
<td>$1/2 (u_1, u_2)$; $u_2 \not\equiv u_2 (\mod 2)$</td>
</tr>
<tr>
<td></td>
<td>$\pm 2 (E_1 - E_1)$</td>
<td>$1/2 (u_1, u_2)$; $u_1 = u_2 (\mod 2)$</td>
</tr>
<tr>
<td></td>
<td>$\pm 4 E_1$</td>
<td>$\pm 1/4$</td>
</tr>
<tr>
<td></td>
<td>$\pm 2 E_3$</td>
<td>$\pm 1/4 (1, 1, 1)$</td>
</tr>
<tr>
<td></td>
<td>$4 H_1$</td>
<td>$1/2 (u_1, u_2)$; $u_1 u_2 = 0 (\mod 2)$</td>
</tr>
<tr>
<td></td>
<td>$2 E_4$</td>
<td>$1/2 (u + v, u, v, 0)$; $(u, v) \not\equiv (0, 0)$</td>
</tr>
<tr>
<td></td>
<td>$\pm 2 E_5$</td>
<td>$\pm 1/4 (1, 1, 2, 0, 2)$</td>
</tr>
<tr>
<td>$\pm 1$</td>
<td>$\pm 4 E_1$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$\pm 2 E_3$</td>
<td>$1/2 (1, 1, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\pm 2 E_5$</td>
<td>$1/2 (1, 1, 0, 0, 0)$</td>
</tr>
<tr>
<td>$\pm 2$</td>
<td>$\mp 2 (E_1 + E_1)$</td>
<td>$(0, 0)$</td>
</tr>
<tr>
<td></td>
<td>$\mp 2 (E_1 + E_1)$</td>
<td>$1/2 (1, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\pm 2 E_5$</td>
<td>$1/2 (0, 1)$</td>
</tr>
<tr>
<td></td>
<td>$\mp 2 E_4$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$\pm 2 E_4$</td>
<td>$1/2 (0, 0, 1, 1, 0, 0)$</td>
</tr>
</tbody>
</table>

Moreover, if $P_1^+ (f)^n/ |D| < 1/4$, then $P_1^+ (f)^n/ |D| \leq 1/5$.

For the nonnegative spectra we have

**Theorem 3**—If $f$ is an indefinite quadratic form of rank $n \geq 21$, then $P_1 (f)^n/ |D| \leq 1/5$.

Because of the recent work of Margulis, the restriction $n \geq 21$ may be a technical one, which is used only to obtain the bounds in the proof of Lemma 6. In particular, the work of Bambah et al., shows that the constant given in Theorem 1 also holds for $n \geq 2$, $s = 0, 1, 2$; for $n \geq 7$, $s = -1$.

2. A Conjecture of Oppenheim (Also called Davenport's Conjecture)

In Watson\textsuperscript{22} the author defines the function $G (f, V)$, for fixed $V \in \mathbb{R}^n$, to be the infimum of $\gamma$ for which the inequality

$$c < f(X + V) \leq c + \gamma$$

is solvable in integral $X \in \mathbb{Z}^n$, for all $c$. Hence, for all $V \in \mathbb{R}^n$,

$$P(f, V) \leq P^+ (f, V) \leq G(f, V).$$

Setting

$$G(f) = \sup \{G(f, V) : V \in \mathbb{Z}^n\}.$$
Theorem 2 in Watson\textsuperscript{22} shows that, for \( n \geq 21 \), \( G(f)^n | D | \leq 1 \). Jackson\textsuperscript{20} proves the same inequality for zero forms of arbitrary rank. More recently, Bambah et al.\textsuperscript{3}, proved that, for \( \alpha, \beta \geq 0, x + \beta = 2 | D | ^{1/n} \), the strict inequality \(-\alpha < f(X + V) < \beta\) holds for any nonzero form \( f \) with \( | s(f) | \leq 2 \).

Restricting our attention to indefinite forms, we define

\[ M(f) = \inf \{ | f(X) | \neq 0 : X \text{ is an integral point} \}. \]

In Oppenheim\textsuperscript{22} it was conjectured that, if \( M(f) \neq 0 \) and \( n \geq 5 \), then \( f \) is a rational form. In Theorem 1 of Margulies\textsuperscript{51} it is proved that if \( f \) is an indefinite quadratic form of rank \( n \geq 3 \) which is not a multiple of an integral form, \( M(f) = 0 \). We shall see below that this result allows us to consider only rational forms.

\textbf{Lemma 1}—Let \( f \) be an indefinite quadratic form of rank \( n \geq 3 \). If \( f \) is not a multiple of a rational form, then \( G(f) = 0 \). Moreover, if \( f \) is rational form of rank \( n \geq 5 \) and \( V \) is not a rational point, then \( G(f, V) = 0 \).

\textbf{Proof}: We first suppose that \( f \) is not a rational form. Then, by Oppenheim's conjecture, \( M(f) = 0 \). In Oppenheim\textsuperscript{23} it is shown that, for \( n \geq 3 \), \( P^+(f, 0) = 0 \) implies \( P^+(-f, 0) = 0 \); hence, from \( M(f) = 0 \) we obtain \( P^+(f, 0) = 0 \). In Theorem 1 of Watson\textsuperscript{20} it is proved that \( G(f) = 0 \) follows from \( P^+(f, 0) = 0 \). This proves the first assertion.

If \( f \) is an indefinite integral quadratic form with \( n \geq 5 \), then by Meyer's Theorem [Cassels\textsuperscript{7}, p. 75, Corollary 1], \( f \) represents zero nontrivially. Hence, by Theorem 2 in Watson\textsuperscript{50}, \( G(f, V) = 0 \) for any nonrational point \( V \).

\section{Results on Equivalence}

In this section \( f \) and \( g \) denote nonsingular primitive integral (not necessarily indefinite) quadratic forms of respective ranks \( n(f), n(g) \), and respective signatures \( s(f), s(g) \). Because in any of our spectra the values for two equivalent forms are equal, it suffices to consider a complete set of representatives for the equivalence classes. It has been proved, for instance Theorem 1.1 in Cassels\textsuperscript{7}, that the number of equivalence classes of forms of fixed rank and discriminant is finite. Because results of Watson\textsuperscript{22} will allow us, for any fixed \( n \), to consider only a finite set of discriminants, our first step will be to determine a set of representatives for forms with the discriminants which occur in our problem. The following notions are useful in this determination.

\textbf{Definition}—For prime \( p \), \( f \) is said to be \( p \)-adically equivalent to \( g \) (denoted by \( f \sim p g \)) if, for each positive integer \( t \), there exists an integral matrix \( N \) such that \( p \) does not divide \( \det N \) and \( f(N X) \equiv g(X) \pmod{p^t} \).

\textbf{Definition}—Let \( d(f) = d(g) = d \). Then \( f \) is said to be congruentially equivalent to \( g \) (denoted by \( f \equiv g \)) if \( f \sim p g \), for all primes \( p \) dividing \( d \).
Note: In Watson\textsuperscript{31}, Theorem 43 it is shown that if \( f \equiv g \) then \( s(f) \equiv s(g) \) (mod 8).

Lemma 2—Let \( f \) and \( g \) be indefinite forms of the same rank \( n \geq 3 \), and the same discriminant \( d \). If, for any integer \( m \geq 5 \), \( d \) is not divisible by \( m^{(n-1)/2} \), then \( f \sim g \) if and only if \( f \equiv g \) and \( s(f) = s(g) \).

Proof: Watson\textsuperscript{31}, Corollary 2, p. 111.

We shall later prove that all of our discriminants are small, relative to the hypothesis of Lemma 2.

Lemma 3—Let \( p \) be any prime dividing the integer \( d \). Then there are only finitely many \( p \)-adic equivalence classes of fixed rank and discriminant \( d \). Each class is represented by a disjoint sum
\[
f_0 + pf_1 + \ldots + p^k f_k
\]
where, for each \( 0 \leq i \leq k \), either \( f_i = 0 \) or there exists \( t_i > 0 \) such that \( f_i = H_i + g \), where either \( g = 0 \) or, if \( p = 2 \),
\[
g = E_2, \pm cx^2, x^2 \pm cy^2, -x^2 - cy^2 \text{ for } c = 1, -3;
\]
if \( p \) is odd,
\[
g = x^2, x^2 - by^2, bx^2
\]
where \( b \) is quadratic nonresidue (mod \( p \)).

Proof: Watson\textsuperscript{31}, Theorems 32 and 35, p. 54 ff.

We note that since the sum in (4) is disjoint, the determinant of the form in (4) is \( \det f_0 \cdot \det (pf_1) \cdot \ldots \cdot \det (p^k f_k) \).

Lemma 4—Let \( n(f) = n(g) \) and \( p \) be any prime. Then \( \overset{\sim}{f} \overset{\sim}{g} \) if and only if \( \det f \cdot \det g \) is a \( p \)-adic square, \( p^i \| d(f), d(g) \), and there exists an integral matrix \( N \) such that \( p \) does not divide \( \det N \) and \( f(NX) \equiv g(X) \) (mod \( p^i \)).

Proof: We note that the quotient \( \det f \cdot \det g / ((-1)^{[n/2]} \cdot d(f) \cdot d(g)) \) equals 1 or 4. This lemma thus follows from the analogues for determinants given in Watson\textsuperscript{31} the sufficiency is given in Theorem 33 (ii); the necessity can be found on p. 50.

The last result will be used to eliminate replications in the list obtained from Lemma 3. To obtain the congruential classes for a fixed rank and discriminant \( d \), the information from the \( p \)-adic classes for all \( p \) dividing \( d \) must be combined.

4. A Reduction of our Problem

We recall that in Lemma 1 we showed if the rank of \( f \) is at least 5, and either \( f \) is not a multiple of a rational form or \( V \) is not a rational point, then \( G(f, V) = 0 \).
We now return to the function $G(f, V)$ and restrict our attention to primitive integral forms $f$ and rational points $V$.

Writing $V = q^{-1}(u_1, \ldots, u_n)$, where $gcd(u_1, \ldots, u_n) = 1$, expansion of $f(X + V)$ yields $f(X + V) = f(X) + 1/c_1 L(X) + r$, where $r$ is a rational number and $c_1$ is the integer such that $L$ is a linear integral form in which any common divisor of its coefficients is relatively prime to $c_1$.

**Lemma 5**—$q$ divides $c_1 D$.

**Proof:** For $f(X) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j$, expansion of $f(X + V)$ yields the linear form

$$L = \frac{c_1}{q} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} u_i x_j$$

where

$$b_{ij} = \begin{cases} 2a_{ij} & \text{for } i = j \\ a_{ii} & \text{for } i \neq j. \end{cases}$$

Setting $N_j = \frac{c_1}{q} \sum_{i=1}^{n} b_{ij} u_i$, the fact that $L(X)$ is an integral form implies that $N_j$ is an integer, for all $1 \leq j \leq n$. Thus, for $N = (N_1, \ldots, N_n)$, $U = (u_1, \ldots, u_n)$, and $B = (b_{ij})$, we have $q N = c_1 U B$.

Since $\det B = \det f \neq 0$, then by Cramer's Rule

$$u_i = \frac{q B_i}{c_1 \det f}$$

where $B_i$ is the correct minor of $B$ augmented by $N$. We recall that $U$ is an integral point with $gcd(u_1, \ldots, u_n, q) = 1$ and so $q$ divides $c_1 \det f$.

The following lemma follows from Theorem 3 of Watson. Watson's proof uses the restriction $n \geq 21$; in light of Margulis' work a modification of Watson's proof should allow a weakening of $n \geq 21$. As noted in Watson, p. 567 the result may be true for all $n \geq 9$, but there is a counterexample for all $n \leq 8$; namely,

$$f = x_1^2 \pm 2(x_2 + 1/2)^2 \pm \cdots \pm 2(x_n + 1/2)^2$$

has $|\det A| = 2^{n-1}, f(X) \equiv (s \pm 1)/2 \pmod{4}$, for all integral $X$.

**Lemma 6**—Let $n \geq 21$. If $G(f, V)^n/|D| > 1/5$, then $c_1 = 1$ and one of the following holds:

(i) $G(f, V) = 1$ and $|D| < 5$;

(ii) $G(f, V) = 2$, $|D| < 5 \cdot 2^n$, and $f \equiv L^2 \pmod{2}$.  

Recalling that \( G (f, V) \geq P^+ (f, V), \) \( > P (f, V), \) our analysis of each spectrum above \( 1/5 \) will be divided into the cases given in Lemma 6.

5. **Proof of the Theorems for the Case** \( G (f, V) = 1. \)

We consider primitive integral forms \( f \) with \( n \geq 21 \) and rational points \( V = \frac{1}{q} \) for which \( G (f, V) = 1. \)

**Lemma 7**—If \( n \geq 5 \) and \( P^+ (f, V)^n/ \mid D \mid > 1/5, \) then \( P^+ (f, V) = 1. \)

**Proof:** Since \( G (f, V) \geq P^+ (f, V), \) then \( G (f, V)^n/ \mid D \mid > 1/5 \) and the hypotheses of Lemma 6 hold. Thus, \( c_1 = 1 \) and from Lemma 5 we obtain that \( q \) divides \( D. \) Hence, the denominator of the rational number \( P^+ (f, V) \) divides \( D. \) Hence, either \( P^+ (f, V) = 1 \) or \( P^+ (f, V) \leq 1 - 1/D^2. \)

If \( P^+ (f, V) \neq 1, \) then \( P^+ (f, V) \leq 1 - 1/D^2. \) Setting \( T (|D|, n) = (1 - D^{-2})^n/ |D|, \) we obtain \( P^+ (f, V)^n/ |D| \leq T (|D|, n). \)

Since \( n \geq 21 \) and \( 1 \leq |D| \leq 4, \) \( T (|D|, n) \) is a decreasing function of \( n \) and an increasing function of \( |D|. \) Therefore, \( T (|D|, n) \leq T (4, 21) < 1/5, \) contrary to hypothesis.

**Lemma 8**—Let \( t > 1 \) be an integer, \( p \) be an odd prime, and \( b \) be any quadratic nonresidue \( (\text{mod } p). \) If \( d_1 \) and \( d_2 \) are integers which are not divisible by \( p, \) then

\[
d_1 x_1^2 + p' d_2 x_2^2 \equiv c \left( k x_1^2 + p' x_2^2 \right)
\]

where \( cd_2 \equiv 1 \) \( (\text{mod } p') \) and \( k = 1 \) or \( b \) is chosen so that \( kd_1 d_2 \) is a \( p \)-adic square.

**Proof:** For convenience, we define the forms

\[
g_1 (X) = d_1 x_1^2 + p' d_2 x_2^2 \quad \text{and} \quad g_2 (X) = c \left( k x_1^2 + p' x_2^2 \right)
\]

for any \( c, k \) as given in the conclusion. Since \( kd_1 d_2 \) a \( p \)-adic square, there exists \( m \) such that \( kd_1 d_2 \equiv m^2 \) \( (\text{mod } p'). \) Also,

\[
det \ g_1 \cdot det \ g_2 = 16 \ c^2 \ kd_1 d_2 p^{2i} \text{ is a } p \text{-adic square and } p' \| d (g_1), d (g_2). \quad \text{For} \quad N = \begin{pmatrix} mk^{-1} & 0 \\ 0 & d_2 \end{pmatrix}, \text{where det } N \text{ is not divisible by } p,
\]

\[
g_2 (N X) = c \left( k (mk^{-1} x_1)^2 + p' \left( d_2 x_2 \right)^2 \right)
\]

\[
= cm^2 k^{-1} x_1^2 + p' c d_2 \ x_2^2,
\]

where

\[
c \ m^2 k^{-1} \equiv cd_1 d_2 \equiv d_1 \ (\text{mod } p')
\]

and

\[
bd_2 \equiv d_2 \ (\text{mod } p').
\]
Hence,
\[ g_2 \left( N X \right) \equiv g_1 \left( X \right) \pmod{p'} \] and, by Lemma 4, \( g_1 \equiv p g_2 \).

**Lemma 9**—Let \( f \) be a primitive integral quadratic form with \( n \left( f \right) = n \geq 4 \). If \( \mid D \mid \leq 4 \), then, for some \( g \) as given in Table VI, \( f \equiv H_m + g \), where the sum is disjoint and \( 2m = n - n \left( g \right) \).

<table>
<thead>
<tr>
<th>( D )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>0</td>
<td>( \pm E_1 )</td>
<td>( \pm E_2 )</td>
<td>( E_1 \pm E_1; E_1 - E_1; 2H_1; E_4; \pm 2E_1; \pm E_1 )</td>
</tr>
</tbody>
</table>

Moreover, these forms are pairwise incongruent.

**Proof**: If \( \mid D \mid = 1 \), then the integralness of \( d \) implies that \( n \) is even. Since \( n \) is even, then \( d \equiv 0, 1 \pmod{4} \) by (3) and so \( d = 1 \). Moreover, \( d \left( H_m \right) = 1 \) implies that \( f \equiv H_m \) by definition of congruential equivalence.

If \( \mid D \mid = 2 \) and \( n \) were even, then \( \mid D \mid = \mid d \mid \equiv 2 \pmod{4} \), a contradiction to \( d \equiv 0, 1 \pmod{4} \). Hence, \( n \) is odd and \( \mid d \mid = 1 \). Also, \( d \left( H_m \pm E_1 \right) = \pm 1 \). Therefore, choosing the sign so that \( d = d \left( H_m \pm E_1 \right) \), we have that \( f \equiv H_m \pm E_1 \).

If \( \mid D \mid = 3 \), the integralness of \( d \) again implies that \( n \) is even and so \( d = -3 \equiv d \left( \pm E_1 \right) \equiv d \left( H_m \pm E_1 \right) \). Hence, it is sufficient to show that \( f \equiv H_m \pm E_1 \). By Lemma 3, \( f \equiv f_0 + 3 f_1 \). Also, Lemma 4 implies \( 3 \parallel d \left( f_0 + 3 f_1 \right) \) and so \( n \left( f_1 \right) = 1 \). Since \( n \) is even, \( n \left( f_0 \right) \) must be odd and
\[ f \equiv H_m + d_1 n_1^2 + 3d_2 n_2^2 \]
for some \( d_1, d_2 \equiv \theta \pmod{3} \). Applying Lemma 8 we obtain
\[ f \equiv H_m + c \left( k n_1^2 + 3n_2^2 \right) \]
for \( cd_2 \equiv 1 \pmod{3} \) and \( k = \pm 1 \) chosen so that \( kd_1d_2 \) is a \( 3 \)-adic square. For the choice \( k = -1 \), we have
\[ d \left( f \right) \cdot d \left( H_m + c \left( -x_1^2 + 3x_2^2 \right) \right) = -36c^2 \]
which is not a \( 3 \)-adic square. Therefore,
\[ f \equiv H_m \pm x_1^2 + 3x_2^2 \].

For \( N = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), we obtain \( E_2 \left( NX \right) = x_1^2 + 3x_2^2 \). Since \( \det N = -2 \), by Lemma 4 \( f \equiv H_m \pm E_2 \), proving \( f \equiv H_n \pm E_2 \).
For the case \(|D| = 4, f \sim f_0 + 2f_1\), Lemma 4 implies that \(4 \parallel \det (f_0 + 2f_1)\).

Hence, \(n (f_1) \leq 2\) and each of \(f_0, f_1\) is of the form \(H_m + g\), where

\[g = 0, E_2, \pm cE_1, E_1 \pm cE_1, -E_1 - cE_1, \text{ or } -E_1 - cE_1, \text{ for } c = 1, -3.\]

Moreover, \(4\|2^n (f_1) \det f_0 \cdot \det f_1\) restricts \(f_1\) to \(0, H_1, E_2, \pm cE_1\). We consider each of these cases in Table VII, where we let \(R = 0, 1, 2\) be such that \(2^R\| \det f_0\).

**Table VII**

<table>
<thead>
<tr>
<th>(f_1)</th>
<th>(R)</th>
<th>(f_0)</th>
<th>(\det f \cdot \det (f_0 + 2f_1))</th>
<th>(c)</th>
<th>(f_0 + 2f_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>(H_m + E_1 \pm cE_1)</td>
<td>(\pm 16c)</td>
<td>1</td>
<td>(H_m + E_1 \pm E_1)</td>
</tr>
<tr>
<td>(H_1)</td>
<td>0</td>
<td>(H_m)</td>
<td>(\pm 16)</td>
<td>-</td>
<td>(H_m + 2H_1)</td>
</tr>
<tr>
<td>(E_2)</td>
<td>0</td>
<td>(H_m)</td>
<td>(\pm 48)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(\pm cE_1)</td>
<td>0</td>
<td>(H_m)</td>
<td>(\pm 48)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(H_m + E_2)</td>
<td>(\pm 144)</td>
<td>-</td>
<td>(H_m + E_2 \pm 2E_2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(H_m + E_2)</td>
<td>(\pm 16c)</td>
<td>1</td>
<td>(H_m \pm 2E_1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(H_m + E_2)</td>
<td>(\pm 48c)</td>
<td>-3</td>
<td>(H_m + E_2 \pm 6E_1)</td>
</tr>
</tbody>
</table>

We next apply Lemma 4 to show that \(E_2 + 2E_2 \cong E_4\) and \(E_2 \pm 6E_1 \cong E_8\). We recall that \(2^R\|d(f)\), set \(g = f_0 + 2f_1\), and note that

**Table VIII**

<table>
<thead>
<tr>
<th>(g)</th>
<th>(d (g))</th>
<th>(t)</th>
<th>(N)</th>
<th>(\det N)</th>
<th>(g (NX) \pmod{2^1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_2 + 2E_2)</td>
<td>36</td>
<td>2</td>
<td>(\begin{pmatrix} 2 &amp; 2 &amp; 0 &amp; 1 \ 1 &amp; 1 &amp; -1 &amp; -1 \ 0 &amp; 1 &amp; 0 &amp; 0 \ 1 &amp; 1 &amp; 0 &amp; 0 \end{pmatrix})</td>
<td>(-1)</td>
<td>(E_4 (X))</td>
</tr>
<tr>
<td>(E_2 \pm 6E_1)</td>
<td>18</td>
<td>1</td>
<td>(\begin{pmatrix} 1 &amp; 0 &amp; 1 \ 1 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix})</td>
<td>1</td>
<td>(E_8 (X))</td>
</tr>
</tbody>
</table>

Hence, \(E_2 + 2E_2 \cong E_4\) and \(E_2 \pm 6E_1 \cong E_8\) and Table VI follows from Table VII.

Finally, we note that \(s (H_m + g) = s (g)\). Since the signatures of congruent forms are in the same congruence class \(\pmod{8}\), to complete the proof of Lemma 9 it suffices to check that there exists no matrix \(N\) of odd determinant such that \(2H_1 (NX) \equiv (E_1 - E_1) (X) \pmod{4}\).

**Lemma 10**—Let \(f\) be a primitive integral quadratic form with \(n (f) \equiv n \geq 4\) and \(s (f) \equiv k \pmod{8}\), where \(-3 \leq k \leq 4\). If \(|D| \ll 4\), then \(f \equiv H_m + g\), where \(2m = n - n (g), s (g) = k\), and \(g\) is given in Table IX.
Moreover, the forms are incongruent.

**Proof:** From Lemma 9 we obtain \( f \cong H_m + g \), for \( 2m = n - n(g) \) and \( g \) listed in Table VI. Since \( s(H_m) = 0 \) and \( f \cong H_m + g \), then \( s(f) \equiv s(H_m + g) \equiv s(g) \) (mod 8). For all \( g \) listed in Table VI we have \(-3 \leq k \leq 4\). Hence, \( s(g) = k \) and Lemma 10 is proved.

**Proposition 1**—Let \( f \) be a primitive integral quadratic form with \( n(f) = n \geq 5 \), \( |D| \leq 4 \), and \( s(f) = 8q + k \), where \(-3 \leq k \leq 4\). As in Lemma 10, \( f \cong H_m + g \), where \( 2m = n - n(g) \) is given in Table IX. Let \( 2t = n - n(g) - 8|q| \).

(i) If \( t \geq 0 \), then \( f \sim sgn(q) \cdot F_{14l} + H_t + g \), where \( g \) is given in Table IX.

(ii) If \( t < 0 \), then \( f \sim sgn(q) \cdot F_{14l-1} + h \), where \( h \) is given in Table X.

### Table IX

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 0 )</th>
<th>( \pm 1 )</th>
<th>( \pm 2 )</th>
<th>( \pm 3 )</th>
<th>( 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g )</td>
<td>( 0; E_1 - E_1; 2H_1 )</td>
<td>( \pm E_1; \pm 2E_1 )</td>
<td>( \pm E_1; \pm (E_1 + E_1) )</td>
<td>( \pm E_3 )</td>
<td>( E_4 )</td>
</tr>
</tbody>
</table>

Moreover, these forms are pairwise inequivalent.

**Proof:** Since equivalent forms are congruent and have the same signature, the given forms are pairwise inequivalent.

Applying Lemma 10 we have that \( f \cong H_{4l+1} + g \), where \( g \) is given in Table IX. We note that since \( d(\pm E_8) = d(\pm H_1) = 1 \) and \( n(\pm E_8) = n(\pm H_1) = 8 \), by definition we have \( \pm E_8 \cong \pm H_1 \) and so for all \( r \geq 0 \), \( \pm F_r \cong \pm H_{4r} \).

Considering the case when \( t \geq 0 \), we have \( f \cong sgn(q) \cdot F_{14l} + H_t + g \) with \( s(sgn(q) \cdot F_{14l} + H_t + g) = 8q + s(g) = 8q + k = s(f) \). From Lemma 2 we thus have \( f \sim sgn(q) \cdot F_{14l} + H_t + g \).

We next note that \( |s(f)| = 8|q| \pm |s(g)| \). If \( t < 0 \), then \( q \neq 0 \) and

\[
-2 \geq 2t = n - n(g) - 8|q| = n - n(g) - (|s(f)| + |s(g)|)
\]

\[
= n - s(f) - (n(g) + |s(g)|) \geq 2 - (n(g) + |s(g)|)
\]

Since \( f \) is indefinite. In particular, \( n(g) \pm |s(g)| \geq 4 \). Considering the choices of \( g \), we thus have that \( |s(g)| \geq 2 \) and \( |s(f)| = 8|q| - |s(g)| \). An inspection of the forms in Table IX, combined with \( 2t = n(f) - n(g) - (|s(f)| + |s(g)|) \), yields...
TABLE XI

| \( k \) | \( g \) | \( n(f) - |s(f)| \leq \) | \( d(f) \) | \( t \) |
|-------|-------|----------------|-----------|-------|
| \( \pm 2 \) | \( \pm E_2; \pm (E_1 + E_1) \) | 2 | -3; -4 | -1 |
| \( \pm 3 \) | \( \pm E_3 \) | 4 | -2 | -1; -2 |
| \( 4 \) | \( E_4 \) | 6 | 4 | -1; -2; -3 |

Hence, when \( t < 0, f \cong H_{41l+1} + g \cong \text{sgn} (q) \cdot F_{lq-1} + H_{l+4} + g, \) for \( g \) given in Table XI.

To prove that \( f \sim \text{sgn} (q), F_{lq-1} + h, \) for the forms \( h \) given in Table X it thus suffices to show that

\[
H_{l+4} + g \cong h \text{ and } s (h) = k - 8 \cdot \text{sgn} (k) \quad \ldots (5)
\]

the latter since we want \( s (f) = 8 \cdot \text{sgn} (q) \cdot (|q| - 1) + s (h); s (h) = (8q + k) - 8q + 8 \cdot \text{sgn} (q) = k + 8 \cdot \text{sgn} (q). \) We recall that \( |8q + k| = |s (f)| = 8 \cdot |q| - |k| \) and so \( q < 0 \) if and only if \( k > 0. \) Hence, \( k + 8 \cdot \text{sgn} (q) = k - 8 \cdot \text{sgn} (k), \) which gives the second condition of (5).

We now consider the positive definite forms \( E_6, \hat{E}_6, E_5, E_4 \) with respective discriminants \( -3, -4, 2, 4. \) By Lemma 10, each of these is congruent to \( H_m + g, \) for appropriate \( m > 0 \) and some \( g \) in Table IX. An analysis of discriminants, ranks, and signature (mod 8) yields

\[
E_6 \cong H_2 - E_2; \quad \hat{E}_6 \cong H_2 - (E_1 + E_1); \quad E_5 \cong H_1 - E_2; \quad E_4 \equiv \pm E_4.
\]

The conclusions of the proposition thus follow from our above analysis and the information in Table XI.

**Proposition 2**—Let \( f \) be a primitive integral quadratic form of rank \( n \geq 21 \) and let \( V \in \mathbb{R}^n \) for which \( G (f, V)^{\sim} / |D| > 1/5. \) If \( G (f, V) = 1, \) then \( f \sim \pm F_r + H_v + g, \) for the appropriate choice of sign, and \( r, v, g \) given in Proposition 1. Under this equivalence \( V \) becomes \( (0, \tilde{V}) \) for \( \tilde{V} \in \mathbb{R}^{n(\tilde{r})}. \) Moreover, \( \{ f (X + V) : X \text{ is an integral point} \} \sim (\tilde{V}) + Z. \)

**Proof:** Since \( G (f, V)^{\sim} / |D| > 1/5 \) and \( G (f, V) = 1, \) by Lemma 6 we have \( |D| \leq 4 \) and Proposition 1 implies that \( f \sim \pm F_r + H_v + g, \) for the appropriate choice of sign, and \( r, v, g. \) We write \( V = q^{-1} (u_1, \ldots, u_n), \) where \( \text{gcd} (u_1, \ldots, u_n) = 1, \) and as in Lemma 5 \( f (X + V) = f (X) + L (X) |c_i + r, \) where \( L \) is an integral linear form and, by Lemma 6, \( c_i = 1. \) Since \( \pm F_r + H_v + g \) is a disjoint sum and the statement of this proposition only gives information about \( \pm F_r + H_v, \) it is likewise sufficient to consider only \( E_5 \) and \( H_1. \) But
\[ H_1(x + u/q, y + u'/q) = H_1(x, y) + q^{-1}(u'x + uy) + uu'q^2 \]

and the linear portion is an integral form; for all \(8r \leq i \leq 2m\), \(v_i \equiv O \pmod{1}\). Expanding \(E_h(X + V)\) we obtain that the linear portion equals

\[
\frac{1}{q} \left[ \sum_{i=1}^{4} (2u_{i} + \sum_{j \neq i} (u_j - u_{j+i})) x_i + \sum_{i=1}^{4} (2u_{i+4} - \sum_{j \neq i} u_j) x_{i+4}. \right]
\]

Since this is an integral form,

\[ q \text{ divides } 2u_i + \sum_{j \neq i} (u_j - u_{j+i}) \]  \(\ldots\) \((6)\)

and

\[ q \text{ divides } 2u_{i+4} - \sum_{j \neq i} u_j \]  \(\ldots\) \((7)\)

for all \(i\). Summing over \(i\) in \((7)\) we obtain

\[ q \text{ divides } 2U_1 - 3U_2 \]  \(\ldots\) \((8)\)

where

\[ U_1 = \sum_{i=1}^{4} u_{i+4} \text{ and } U_2 = \sum_{i=1}^{4} u_i. \]

We recall from Lemma 5 that \(q\) divides \(c_1 D = D\) and \(|D| \leq 4\). Therefore, 2 and 3 are the only prime divisors of \(q\).

If 3 divides \(q\), then \((8)\) implies that 3 divides \(U_1\). Adding \((6)\) and \((7)\)

\[ q \text{ divides } 2u_i - U_1 + 3u_{i+4} \]

and so 3 divides each \(u_i\). Hence, from \((6)\) we obtain that 3 divides each \(U_1 - u_{i+4}\). Therefore, if 3 divides \(q\), then

\[ u_i \equiv O \pmod{3} \text{ for all } i \leq \lfloor 8r \rfloor. \]

If \(q\) is divisible by a power of 2, then \((8)\) implies that \(U_2\) is even. Subtracting \((7)\) from \((6)\) we thus have that \(U_1 + u_{i+4}\) is even;

\[ \sum_{i=1}^{4} (U_1 + u_{i+4}) = 5U_1 \text{ is even}; \]

and so \(u_{i+4}\) is even, for all \(1 \leq i \leq 4\). The evenness of \(u_i\), for \(i \leq 4\), follows from \((6)\) and the evenness of each \(u_{i+4}\) and \(U_2\). Hence, there exist integers \(w_i\) such that \(u_i = 2w_i\), for all \(i \leq 8\). If 4 divides \(q\), then \((6), (7), (8)\) can be rewritten in terms of the \(w_i\); the above argument applies and yields \(w_i \equiv O \pmod{2}\), for all \(i \leq 8\).

Therefore, \(u_i \equiv O \pmod{q}\) for all \(i \leq 2m\) and we write \(V = (0, \widetilde{V})\), where \(\widetilde{V} \in R^{n(x)}\).
To complete the proof, we show that

\[ \{ f(X + V) : X \text{ is integral} \} = g(V) + \mathbb{Z}. \quad \ldots(9) \]

Writing \( X = (Y, Z) \), where \( Z \in \mathbb{Z}^{n(\omega)} \), we have \( X + V = (Y, Z + \tilde{V}) \). We recall, from Proposition 1, that \( n(g) \leq 6 \) and so \( |8r| + 2v > 15 \), implying that either \( |r| > 0 \) or \( v > 0 \).

If \( t = 0 \) then \( \{ (\pm F_r + H_r)(Y) : \text{integral } Y \} = \mathbb{Z} \) and, for fixed \( Z \), \( \{ f(Y, Z + \tilde{V}) : \text{Y is integral} \} = g(Z + \tilde{V}) + \mathbb{Z} \). \( G(f, V) = 1 \) thus implies (9).

On the other hand, if \( v = 0 \) and \( s(\pm F_r) > 0 \), then \( \{ \pm F_r(X) : \text{integral } X \} = \mathbb{N}^* \), the set of nonnegative integers.

Therefore, for fixed \( Z \in \mathbb{Z}^{n(\omega)}, \{ f(Y, Z + \tilde{V}) : \text{integral } Y \} = g(Z + \tilde{V}) + \mathbb{N}^* \).

From \( G(F, V) = 1 \) we thus obtain

\[ \tilde{g}(Z + \tilde{V}) - g(V) \in \mathbb{Z}, \text{ for all } Z \in \mathbb{Z}^{n(\omega)}. \]

Moreover, since \( f \) is indefinite and \( \{ F_r(Y) : \text{integral } Y \} = \mathbb{N}^* \), there exist integral vectors \( Z_i \) such that \( g(Z_i + V) \underset{i \to \infty}{\longrightarrow} -\infty \). Writing \( g(Z_i + V) - g(V) = n_i \), we have, for fixed \( i \), \( \{ f(Y, Z_i + V) : \text{integral } Y \} = g(V) + n_i + \mathbb{N}^* \), from which (9) follows. An analogous argument applies for \( s(\pm F_r) < 0 \).

**Proof of the Theorems of the case** \( G(f, V) = 1 \)

We assume that \( P^r(f, V)^{n/D} > 1/5 \) and \( G(f, V) = 1 \). By Proposition 1, \( f \sim F_r + H_r + g \) for some \( r, v \) and \( g \) as given in Proposition 1. Moreover, from Proposition 2, under this equivalence we have \( V = (0 \tilde{V}) \). We claim that all non-integral \( V \) which satisfy \( G(f, V) = 1 \) are given in Table XII.

Column 4 (Table XII) follows from Lemmas 5 and 6. We shall complete the argument for \( g = \pm E_3 \). Recalling (9), we obtain that, for any integral \( N = (n_1, n_2, n_3), E_3(V + N) - E_3(V) \) is an integer. Setting \( \tilde{V} = 1/4(u_1, u_2, u_3) \) we thus obtain that

\[ u_1(2n_1 + n_2 + n_3) + u_2(n_1 + 2n_2 + n_3) + u_3(n_1 + n_2 + 2n_3) \equiv 0 \pmod{4} \]

for all choices of integral \( N \). Using \( N = (1, 0, 0), (0, 1, 0), (0, 0, 1) \) we thus obtain that \( u_1 = u_2 = u_3 \equiv 1 \pmod{4} \), completing the argument for \( g = \pm E_3 \).

Continuing with the proof of the theorems for the case when \( G(f, V) = 1 \), we assume that \( P^r(f, V)^{n/D} > 1/5 \), where
## Table XII

| $k$ | $\pm g$ | $| D |$ | $q \neq 1$ | $\tilde{V}$ (mod 1) |
|-----|---------|-------|-----------|--------------|
| 0   | 0       | 1     | -         | -            |
| $E_1 - E_1; \ 2H_1$ | 4     | 2, 4  | $\frac{1}{2}$ ($u_1, u_2$) |
| $\pm 1$ | $E_1$  | 2     | 2         | $\frac{u}{2}$ |
|       | $2E_1$ | 4     | 2, 4      | $\frac{u}{4}$ |
| $\pm 2$ | $E_2$  | 3     | 3         | $\frac{u}{3}$ (1, 1) |
|       | $E_1 + E_1$ | 4   | 2, 4      | $\frac{1}{2}$ ($u_1, u_2$) |
|       | $E_3$  | 4     | 2, 4      | $\frac{u}{4}$ (1, 1, 1) |
|       | $E_5$  | 4     | 2, 4      | $\frac{u}{4}$ (1, 1, 2, 0, 2) |
| 4   | $E_6$  | 4     | 2, 4      | $\frac{1}{2}$ ($u+v, u, v, 0$) |

## Table XIII

| $k$ | $\pm g$ | $\pm g \tilde{V}$ | $\tilde{V}$ (mod 1) | $P^+ (f, \tilde{V})$ \(|D|\) |
|-----|---------|-------------------|---------------------|----------------------|
| 0   | 0       | 0                 | 0                   | 1                    |
| $E_1 - E_1$ | $\frac{1}{4}$ ($u_1 - u_2$) | $\frac{u}{2}$ (1, 1) | $1/4$ |
| $H_1$ | $\frac{1}{4}$ ($u_1 u_2$) | 0 | 1/4 |
| $\pm 1$ | $E_1$  | $u^2/4$ | 0 | 1/2 |
|       | $2E_1$ | $u^2/8$ | 0 | 1/4 |
| $\pm 2$ | $E_2$  | $u^2/3$ | 0 | 1/3 |
|       | $E_1 + E_1$ | $\frac{1}{4}$ ($u_1^2 + u_2^2$) | 0 | 1/4 |
|       | $E_6$  | $2u^2/3$ | 0 | 1/3 |
|       | $E_8$  | $\frac{3}{4}$ ($u^2 + v^2$) | 0 | 1/4 |
| $\pm 3$ | $E_3$  | $3u^2/8$ | 0 | 1/4 |
|       | $E_5$  | $5u^2/8$ | 0 | 1/4 |
| 4   | $E_4$  | $\frac{1}{4}$ ($u^2 + v^2 + uv$) | 0 | 1/4 |
INDEFINITE QUADRATIC FORMS IN MANY VARIABLES

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\[ P^+ (f, V) = \inf \{ fX + V \geq 0 : \text{integral } X \} \]

By Lemma 7, \( P^+ (f, V)^n / |D| > 1/5 \) implies that

\[ P^+ (f, V) = 1; \tilde{g} (V) \in \mathbb{Z}, \text{ by (9).} \]

Using this and Table XII, we obtain column 5 Table XIII below and so Theorems 1 and 2 for the case when \( G (f, V) = 1 \).

We note that \( P (f, V)^n / |D| > 1/5 \) implies that \( P^+ (f, V)^n / |D| > 1/5 \). Therefore, if \( G (f, V) = 1 \) and \( P (f, V)^n / |D| \geq 1/5 \), then the above argument applies with \( \tilde{V} \) and \( \tilde{g} \) as listed in Table XI. Since

\[ P (f, 0) = \inf \{ f(X) \geq 0 : X \text{ is nonzero integral point} \} = 0 \]

for any of these \( f \), then \( P_l (f)^n / |D| \leq 1/5 \) for all \( f \) and Theorem 3 is proved for the case when \( G (f, V) = 1 \).

6. PROOF OF THE THEOREMS FOR THE CASE \( G (f, V) = 2 \)

By Lemma 6, we have that \( f \equiv l^2, \bmod 2 \) and so, for \( i 
eq j \), \( a_{ij} \) is even. By definition, the diagonal entries of the matrix \( A \) are even.

Lemma 11—If \( f \sim g = f_0 + 2f_1, \) then \( f_0 \equiv cx_i^2 \) or \( \pm x_i^2 + cx_i^2 \).

PROOF Since \( f \sim g \), by definition there exists an integral matrix \( M \) with odd determinant such that \( g (MX) \equiv f(X) \), \( \bmod 2 \). A comparison of coefficients yields that, for \( Y = m_{ij} x_i + \ldots + m_{n} x_n, f_0 (Y) \equiv f_0 (X), \bmod 2 \). If \( f_0 \equiv E_2 \) or there exists \( t > 0 \) such that \( f_0 \equiv H_t + g_0 \), then for \( i 
eq j \)

\[ m_{ij} m_{ij} + m_{ij} m_{ij} \equiv a_{ij} (\bmod 2). \]

Since \( a_{ij} \) is even, then also

\[ m_{ij} m_{ij} - m_{ij} m_{ij} \equiv O (\bmod 2). \]

Letting \( M_{ij} \) be the \( n-2 \) order minor of \( M \) with the first two rows and the \( i, j \) columns removed, then (cf. Hadley\textsuperscript{17}, page 98, eqn. (3-100)).

\[ \det M = \sum_{i 
eq j} \epsilon_{ij} (m_{ij} m_{ij} - m_{ij} m_{ij}) \det M_{ij} \]

for the correct choice of \( \epsilon_{ij} = \pm 1 \), contrary to the oddness of \( \det M \).

Lemma 12—Let \( n \geq 5 \) and let \( a_{ij} \) be even, for all \( i \neq j \). If \( |D| \leq 2^n + 2 \), then \( f \sim \pm x_i^2 + g \), where the sign can be chosen so that \( g \) is an indefinite form of rank \( n - 1 \).

PROOF: We first show that the result follows if \( f \) represents both \( \pm 1 \). Let \( B \) be an integral point for which \( f(B) = \pm 1 \). Then \( B \) is primitive and \( B \) can be extended to a basis, thus obtaining a unimodular matrix \( B \) in which \( B \) is the first column, Thus,
\[ f(BX) = \pm x_i^2 + x_1 M(x_2, \ldots, x_n) + h(x_2, \ldots, x_n), \]

where \( h \) is a quadratic form and \( M \) is the linear form given by

\[ M(X) = \sum_{i=1}^{n} 2a_{ii} \sum_{k=2}^{n} b_{ik} x_k + \sum_{i<j}^{n} a_{ij} \sum_{k=2}^{n} (b_{i1} b_{jk} + b_{ik} b_{j1}) x_k \]

Since \( a_{ij} \) is even for all \( i \neq j \), then \( M \) has even coefficients, say

\[ M = 2(c_2 x_2^2 + \ldots + c_n x_n). \]

Hence,

\[ f(BX) = \pm (x_1^{\pm} (c_2 x_2 + \ldots + c_n x_n))^2 + (h \mp M^2/4)(x_2, \ldots, x_n); \]
\[ f \sim \pm x_i^2 + g, \text{ for the quadratic form } g = h \mp M^2/4. \] The sign of \( \pm 1 \) is chosen so that \( g \) is indefinite.

In order to complete the proof of the lemma it suffices to show that \( f \) represents both \( \pm 1 \). For this we shall use Theorem 51 (ii) in Watson: Let \( a \) be a fixed integer. If there exists a primitive solution \( X \) to \( f(X) \equiv a \pmod{d} \), then there exists a form \( h \) which is congruentially equivalent to \( f \), has the same signature, and properly represents \( a \).

We note that \( |D| \ll 2^{n+2} \) and Lemma 2 imply that such \( h \) is equivalent to \( f \) and hence that \( f \) itself properly represents \( a \). Moreover, from \( f \equiv L^2 \pmod{2} \) we obtain that \( 2^n \) divides \( D \) and so: \( D = r \cdot 2^n \), for some \( r \leq 4 \).

For the case when \( r = 3 \), by Lemma 3 we write \( f \sim g_0 + 3g_1 \). Lemma 4 implies that \( n(g_i) = 1 \) and so \( n(g_n) \geq 4 \). Inspecting the choices in Lemma 3, we observe that \( g_0 \equiv H_t + g \), for some \( t \geq 1 \). Therefore, \( g_0 \) represents \( \pm 1 \); that is, each of \( f(X) \equiv \pm 1 \pmod{3} \) is solvable. Suppose we have shown that, for each of \( e = \pm 1 \), there exists primitive \( P \) such that \( f(P) = e + 2^n m \). For primitive \( Q \) such that

\[ f(Q) \equiv e \pmod{3}, \]

we set \( \hat{P} = P + Q \). We note that, for the associated bilinear form defined by

\[ f(X, Y) = \sum_{i<j} a_{ij} x_i y_j, \]

\[ f(P, \hat{P}) = f(P) + f(P, Q); \]

\[ f(P) \equiv f(P) + 2f(P, Q) + \epsilon \pmod{3}. \] Therefore, \( T \equiv 2^{n-2} \pmod{3} \).
\[ f(P + T2^{n-1} P) = f(P) + 2^n \left[ Tf(P, P) + 2^n T^2 f(P) \right] \]
\[ \equiv f(P) + 2^n \left[ T(f(P) + f(P, Q)) + 2^{n-2} (f(P) + 2f(P, Q) + \epsilon) \right] (\text{mod } 3 \cdot 2^n) \]
\[ \equiv f(P) + 2^n \left[ 2^{n-1} f(P) + 2^{n-2} \epsilon \right] \]
\[ \equiv \epsilon + 2^n m + 2^n \left[ 2^{n-1} (\epsilon + 2^m) + 2^{n-2} \epsilon \right] \]
\[ \equiv \epsilon + (2^n + 2^{n-2}) m \]
\[ \equiv \epsilon (\text{mod } 3 \cdot 2^n). \]

Hence, for any \( |D| \ll 2^{n+2} \) it suffices to show that each of

\[ f(X) \equiv \pm 1 (\text{mod } 2^n) \]

is solvable for \( 2^n \parallel d \).

By Lemma 3, we write \( f \sim f_0 + 2g_1 = h \) and consider the choices for \( f_0 \). If, for some \( v \geq 1, f \equiv H_v + f_0 \), then \( f_0 \) properly represents both \( \pm 1 \). We may thus assume \( v = 0 \). Also, since \( f \) is primitive, \( f_0 \) represents some odd integer. Therefore, if \( g_1 = H_w + g \), for some \( w \geq 1 \), we would obtain that \( h \) represents all odd integers. We may thus assume that \( v = w = 0 \). We write \( g_1 \equiv f_1 + 2g_2 \) and observe that \( n(f_0), n(f_1) \leq 2 \). From Lemma 11, \( f_0 \sim E \) and so \( 2^k(f_0) \) divides \( \det f_0 \), and \( 2^{n+2}g_2 \) divides \( D \). Since \( |D| \ll 2^{n+2} \), we obtain that \( g_2 \equiv H_1, E_2, cx_2^2 \), and \( |D| = 2^{n+2}f_1 \equiv E_2 \). Also, since \( h \sim f_1 \), then \( \det h \det f \) is a 2-adic square; that is, its odd part is congruent to 1 (mod 8). Letting \( d \) be the odd part of \( \det g \) and recalling that \( f_0 = bx_1^2 \) or \( \pm x_1^2 + bx_2^2 \) we thus obtain that

\[ bd_{i} \equiv \pm 3 (\text{mod } 8). \]

We also note that, for \( g = -x_1^2 + 2E(x_2, x_3) \).

\[ n(1, 0) = -1 \text{ and } g(-1, 1, 0) = 1. \]

Hence, we may assume that \( b \neq -1 \). In Table XIV below we obtain \( P_1, P_2 \) such that \( h(P_i) \equiv 1 (\text{mod } d) \) and \( h(P_2) \equiv -1 (\text{mod } d) \) for the remaining choices of \( h \). The congruences in (10) and \( n > 5 \) are used to obtain the possibilities for \( f_0 \) given in column 3.

**Proposition 3**—Let \( n > 4 \). If \( f \) is an indefinite primitive integral quadratic form with for each \( i \neq j \), \( a_{ij} \) is even, and \( |D| \ll 2^{n+2} \), then there exists \( r \), such that \( f \sim \pm x_1^2 \pm \cdots \pm x_n^2 \pm h \), for suitable choice of signs, where sum is disjoint and \( h \) is given in Table XV.
Table XIV

| $g_2$ | $d_i$ | $f_{0}$ | $n$ | $|d| |$ | $P_1$ | $P_2$ |
|-------|-------|--------|-----|--------|-----|-----|
| $E_2$ | 3     | $x_1^2$ | 5   | $2^6$  | $(1, 0)$ | $(1, 2, 1, 2, 2)$ |
|       |       | $x_1^2 + x_2^2$ | 6   | $2^8$  | $(1, 0)$ | $(1, 0, 5, 3, 5, 2)$ |
| $H_1$ | 1     | $3x_1^2$ | 5   | $2^6$  | $(1, 1, 0, -1, 1)$ | $(1, 0, 0, -1, 1)$ |
|       |       | $-3x_1^2$ | 5   | $2^6$  | $(1, 0, 0, 1, 1)$ | $(1, 1, 0)$ |
|       |       | $x_1^2 + 3x_2^2$ | 6   | $2^8$  | $(1, 0)$ | $(0, 1, 0, 0, -1, 1)$ |
|       |       | $x_1^2 - 3x_2^2$ | 6   | $2^8$  | $(1, 0)$ | $(0, 1, 1, 0)$ |
| $\pm x_0^2$ | 1     | $x_1^2 + 3x_2^2$ | 5   | $2^6$  | $(1, 0)$ | $(2, 1, 2, 4, 0)$ |
|       |       | $x_1^2 - 3x_2^2$ | 5   | $2^6$  | $(1, 0)$ | $(0, 1, 1, 0)$ |
| $3x_0^2$ | 3     | $x_1^2 + x_2^2$ | 5   | $2^6$  | $(1, 0)$ | $(1, 0, 7, 1, 1)$ |
|       |       | $-3x_0^2$ | 3   | $2^6$  | $(1, 0)$ | $(1, 2, 1)$ |

Table XV

| $2^{-n} |D|$ | $h$ |
|--------|-----|
| 1      | 0   |
| 2      | $2E_1$ |
| 3      | $2E_1; 3E_0$ |
| 4      | $2 \ (E_1 \pm E_1); 4E_1; 2E_1; 4H_1; 2E_4; 2E_6; 2E_8$ |

Proof: Since, for each $i \neq j$, $a_{ij}$ is even, then $2^n$ divides $D$ and $|D| \leq 2^{n+2}$ implies that $|D| = r \cdot 2^n$, for some $1 \leq r \leq 4$.

If $n \geq 5$, we can use Lemma 12 repeatedly until we obtain

$$- \pm x_1^2 \pm \ldots \pm x_i^2 + f_i,$$

for some indefinite integral quadratic form $f_i$. The process stops when either $n(f_i) = 4$ or $f_i$ is not primitive. We set $n_i = n(f_i)$ and $D_i = \det f_i$.

We first consider the case when $f_i$ is not primitive and $n_i \geq 4$. Since $|D_1| = 2^n \cdot r$, then $f_i = 2f_2$ for some primitive indefinite quadratic form $f_2$ with $n(f_2) = n_i \geq 4$.
and $|\det f_2| = r$. Recalling that Lemma 10 and Prop. 1 were obtained without using the condition $G (f, V) = 1$, we thus have that $f_2 \sim \pm F_v + H_t + g$, for some choice of sign, $v, t > 0$, and $g$ given in Table IIX or X. Recalling that $\pm F_v \equiv H_{4v}$ and

\[ (-x_1) (x_2) = -x_1 x_2 \text{ and } (x_1 + x_2 + x_3)^2 - 2 (x_1 + x_3) (x_2 + x_3) \]

\[ = x_1^2 + x_2^2 - x_3^2 \]

we obtain that

\[ \pm x_1^2 + 2 (\pm F_v + H_t) \sim \pm x_1^2 \pm \ldots \pm x_{2i+1}^2, \text{ for some choice of sign...} (11) \]

Thus, for the case when $f_1$ is not primitive, each $h = 2g$ appears in Table XV.

Therefore, it suffices to consider the case when

\[ f \sim \pm x_1^2 \pm \ldots \pm x_{n-4}^2 + f_1, \]

where $f_1$ is primitive and $|D_1| = 16r$.

If $r = 3$, we first analyze 3-adic equivalence. From $n_1 = 4, 3 \parallel D_1$, and Lemma 3, we have $f_1 \sim H_1 + cx_3^2 + 3dx_4^2$. Since

\[ H_1 (x_1-x_2, x_1 + x_2) = x_1^2 - x_2^2 \text{ and } cx_3^2 + 3dx_4^2 \equiv \pm x_2 + 3x_4^2 \pmod{3} \]

then

\[ f \sim x_1^2 - x_2^2 \pm x_3^2 + 3x_4^2 ; \]

that is, there is exactly one 3-adic equivalence class for each of $D_1 = 48, D_1 = -48, D_1 = -48$. Hence, for $r = 3$ (and so for all $1 \leq r \leq 4$), congruential equivalence follows from 2-adic equivalence. We write $f_1 \sim g_0 + 2g_1$; Lemma 11 implies that $g_0 = cx_1^2$ or $g_0 = \pm x_1^2 + cx_2^2$.

If $r = 1$, then $\det g_1$ is odd and so $g_1 = E_2$ or $g_1 = H_1$. Since $n_1 = 4$, by Lemma 4 either $f_1 \sim \pm x_1^2 \pm 3x_2^2 + 2E_2$ or $f_1 \sim \pm x_1^2 + x_2^2 + 2H_1$, with the latter equivalent to $\pm x_1^2 \pm \ldots \pm x_4^2$ by (11). We note that $3 (x_1 + x_2)^2 - (x_1 + x_2 + x_3)^2 - (x_1 + x_3)^2 \equiv x_1^2 \pm 2E_2 \pmod{4}$ and so Lemma 4 implies that

\[ x_1^2 \pm 2E_2 \sim 3x_1^2 - x_2^2 - x_3^2 . \]

...(12)
Since $\pm 3x_1^2 \pm 3x_2^2 \equiv \mp x_1^2 \mp x_2^2 \pmod{4}$, again using Lemma 4 we have $\pm 3x_1^2 \pm 3x_2^2 \sim \pm x_1^2 \mp x_2^2$. Combining these results,

$\pm x_1^2 \pm 3x_2^2 + 2E \sim$

$\pm x_1^2 \pm x_2^2 \pm 3x_4^2 \pm 3x_5^2 \sim \pm x_1^2 \pm x_2^2 \pm x_3^2 \pm x_6^2$;

that is, $h = 0$ when $r = 1$. 

In a similar manner, if $r = 3$ then either $f_1 \sim \pm x_1^2 \pm x_2^2 + 2E_2$
or $f_1 \sim \pm x_1^2 \pm 3x_2^2 + 2H_1$. Again using (11), (12), $\pm 3 \equiv \mp 1 \pmod{4}$, and Lemma 4, an analysis of disjoint summands yields $h = 2E_2$ when $r = 3$.

Recalling that $f_1 \sim g_0 + 2g_1$ with $g_0 = cx_1^2$ or $\pm x_1^2 + cx_2$,

for $r = 2$ we have $2\| \det g_1$. From $n (g_1) \geq 2$ we thus obtain $g_1 = H_1 + bE_1$.

Hence, $f_1 \sim cx_1^2 + 2 (x_2 x_3 + bx_4^2)$. By considering $-f_1$ if necessary, we may assume that $c = 1, 3$. Also, since $\pm cb$ is a 2-adic square, we may assume that $b = \pm c$. If $c = 1$, then (11) implies that $f_1 \sim x_1^2 + x_2^2 - x_3^2 \pm 2x_4^2$; $h = 2E_1$.

For $c = 3$, we consider the cases $b = \pm 3$ separately: From

$$(x_1 + 2x_3)^2 + 2 (x_1 + x_2)^2 \equiv 3 (x_1^2 + 2x_2^2) \pmod{8}$$

and Lemma 4 we obtain $3 (x_1^2 + 2x_2^2) \sim x_1^2 + 2x_2^2$.

Hence, using (11),

$$3 (x_1^2 + 2x_4^2) + 2H_1 \sim x_1^2 + 2x_4^2 + 2H_1 \sim x_1^2 + x_2^2 - x_3^2 + 2x_4^2;$$

$h = 2E_1$. Finally, for $b = -3$, we observe that

$$3 (x_1 + x_2 + x_3)^2 + 2 (-x_1 - x_3) (x_2 + x_3)$$

$$\equiv -x_2^2 - x_3^2 - 3x_5^2 \pmod{4};$$
\[ 3E_1 + 2H_1 \sim -x_1^2 - x_2^2 - 3x_4^2, \text{ by Lemma 4.} \]

Therefore,

\[ 3x_1^2 + 2H_1 - 6x_4^2 \sim -x_1^2 - x_2^2 - 3(x_3^2 + 2x_4^2) \]

as above, and \( h = -2E_1 \).

For \( r = 4 \), we again write \( f_2 \sim g_0 + 2g_1 \), where \( g_0 = cx^2 \) or \( \pm x_1^2 + cx_2^2; 4 \parallel \det g_1 \). Considering \(-f_1\), if necessary, we may assume that \( c = 1, 3 \) and obtain the possibilities given in Table XVI, (using the determinant argument from Lemma 4). The empty entries of column 3 are discussed below.

**Table XVI**

<table>
<thead>
<tr>
<th>( g_0 )</th>
<th>( g_1 )</th>
<th>( f_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1^2 )</td>
<td>( H_1 \pm 2x_4^2 )</td>
<td>( x_1^2 \pm x_2^2 - x_3^2 \pm 4x_4^2 ), by (11)</td>
</tr>
<tr>
<td>( 3x_1^2 )</td>
<td>( H_1 \pm 6x_4^2 )</td>
<td>( \pm x_1^2 \pm x_2^2 \pm 2x_3^2 \pm 2x_4^2 )</td>
</tr>
<tr>
<td>( \pm x_1^2 \pm x_2^2 )</td>
<td>( \pm x_3^2 \pm x_4^2 )</td>
<td>( \pm x_1^2 \pm x_2^2 \pm 2x_3^2 \pm 2x_4^2 )</td>
</tr>
<tr>
<td>( \pm x_1^2 + 3x_2^2 )</td>
<td>( \pm x_3^2 \pm 3x_4^2 )</td>
<td>( 2H_1; c = 1 )</td>
</tr>
<tr>
<td>( \pm x_1^2 + cx_2^2 )</td>
<td>( 2H_1; c = 1 )</td>
<td>( \pm x_1^2 + x_2^2 + 4H_1 )</td>
</tr>
<tr>
<td>2E_1; c = 3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each entry in column 3 of Table XVI is listed in Table XV. Noting that

\[ 3x_1^2 \pm 12x_4^2 \equiv 3x_1^2 \mp 4x_1^2 \pmod{16} \]

and

\[ 3x_1^2 + 2H_1 \equiv -x_1^2 + 2H_1 \pmod{4} \]

we obtain

\[ 3x_1^2 + 2H_1 \pm 12x_4^2 \sim 3x_1^2 + 2H_1 \mp 4x_4^2 \]

\[ \sim -x_1^2 + 2H_1 \mp 4x_4^2 \]

\[ \sim x_1^2 + x_2^2 - x_3^2 \mp 4x_4^2, \]
the latter by (11). It thus suffices to consider
\[ f_1 \sim \pm x_1^2 + 3x_2^2 \pm 2x_3^2 \pm 6x_4^2 \]
and
\[ f_1 \sim \pm x_1^2 + 3x_2^2 + 4E_2. \]
We note that
\[ \pm (3x_2)^2 + 3 (3x_1 + 4x_2 + 2x_3)^2 + 6 (x_1 + 2x_2 + x_3)^2 \]
\[ \equiv x_1^2 \pm x_2^2 \pm 2x_3^2 \pmod{8}; \]
\[ 2x_1 + x_2 + 2x_3)^2 + 3 (x_1 + 2x_2)^2 - 6 (2x_1 + 3x_2 + x_3)^2 \]
\[ \equiv -x_1^2 - x_2^2 - 2x_3^2 \pmod{8}; \]
\[ - (x_1 + 2x_2 + 2x_3)^2 + 3 (x_2 + 2x_3)^2 - 6 (x_1 + x_2 + x_3)^2 \]
\[ \equiv x_1^2 + x_2^2 + 2x_3^2 \pmod{8} \]
and also that the determinant of each transformation is odd. Hence, by Lemma 4.
\[ \pm x_1^2 + 3x_2^2 \pm 6x_4^2 \sim \pm x_1^2 \pm 2x_3^2 x_4^2 \]
for some choice of signs and so the fourth row of Table XVI has \( h = 2 (E_1 \pm E_1) \).
Moreover,
\[ 3 (x_1 + 2x_2 - 2x_3)^2 + 4E_2 (2x_1 + x_2 - x_3, x_1 + x_3) \]
\[ \equiv -x_1^2 + 4x_2 x_3 \pmod{16}; \]
\[ -3 (x_1 + 2x_2 + 2x_3)^2 + 4E_2 (-x_2 + x_3, x_1 + 2x_2 + x_3) \]
\[ \equiv x_1^2 + 4x_2 x_3 \pmod{16}; \]
where again each determinant is odd. By Lemma 4, \( \pm 3x_1^2 + 4E_2 \sim \pm x_1^2 + 4H_1 \)
and the sixth row of Table XVI satisfies the conclusion with \( h = 4H_1 \).

Finally we consider the second row; namely, \( f_1 \sim 3 (x_1^2 \pm 4x_i)^2 + 2H_1 \). We note
that \( 3 ((x_1 + 4x_i)^2 + 4 (x_1 + x_4)^2) \equiv -x_1^2 - 4x_4^2 \pmod{16}; \)
3 \left((3x_1 + 4x_4)^2 - 4\left(x_1 + x_4\right)^2\right) \equiv -x_1^2 + 4x_4^2 \pmod{16}.

Since each of these determinants is odd, \(3\left(x_1 \pm 4x_4\right)^2 \equiv -x_1^2 \mp 4x_4^2\).

Hence, by (11),

\[ f_1 \sim -x_1^2 + 2H_1 + 4x_4^2 - x_1^2 - x_2^2 + x_3^2 \mp 4x_4^2; \quad h = 4E_1. \]

Summarizing, we are considering \(f\) for which \(f \sim \pm x_1^2 \pm \ldots \pm x_{n-4}^2 + f_1\) for some choice of signs, where \(f_1\) is indefinite, \(n(f_1) = 4\), and \(|\det(f_1)| = r \cdot 2^n\), for some \(r \leq 4\). We have shown that there exists a choice of signs such that \(f_1 \equiv \pm x_{n-3}^2 \pm x_{n-2}^2 \pm g\), where \(g\) is given in Table XVII.

| Table XVII |
|---|---|---|---|---|
| \(r\) | 1 | 2 | 3 | 4 |
| 0 | 2E_1 | 2E_2 | 4E_1; 2(E_1 ± E_1); 4H_1 |

Setting \(h_1 = \pm x_{n-3}^2 \pm x_{n-2}^2 \pm g\), we thus have that \(f_1 \equiv h\) and so \(s(f_1) = s(h_1) \pmod{8}\). Since \(f_1\) is indefinite and \(n(f_1) = 4\), then \(|s(f_1)| \leq 3\) and so \(s(f_1) = s(h_1)\). Lemma 2 thus implies that \(f_1 \sim h_1\), which completes the proof of Proposition 3.

For convenience, we set \(K_{t_1,t_2} = -x_3^2 - \ldots - 2x_{t_1}^2 + x_{t_1+1}^2 + \ldots + x_{t_1+t_2}^2\).

**Proposition 4**—Let \(f\) be an indefinite, primitive, integral quadratic form of rank \(n \geq 21\). If \(V \in \mathbb{R}^n\) for which \(G(f, V) = n/|D| > \frac{1}{2}\), and \(G(f, V) = 2\), then \(f = K_{t_1,t_2} \pm h\), for some \(h\) given in Table XV. Moreover, under this equivalence, for all \(1 \leq i \leq t_1 + t_2, v_i \equiv \frac{1}{2} \pmod{1}\). Also,

\[ \{K_{t_1,t_2}(X + V) : X \text{ is integral}\} = K_{t_1,t_2}(1/2) + 2 \mathbb{Z}. \]  \hfill (13)

**Proof**: From Lemma 6 we know that \(f \equiv L^2 \pmod{2}\) and so in particular, for \(i \neq j\), \(a_{ij}\) is even. Hence, we obtain the first conclusion from Proposition 3. By the disjointness of the sum it suffices to complete the proof for \(f = K_{t_1,t_2}\). For any \(i\), we take \(0 \leq v_i < 1\). Again by disjointness, \(G(f, V) = 2\) implies that \((1 \pm v_i)^2 - v_i^2\) is zero or is at least 2; that is, \(v_i = \frac{1}{2}\). The set equality (13) follows from Theorem 2 (i) in Watson\(^3\), completing the proof of Prop. 4.
<table>
<thead>
<tr>
<th>$2 - n \mid D \mid$</th>
<th>$\pm h$</th>
<th>$V_2 \text{ (mod 1)}$</th>
<th>$\pm h (V_3) \text{ (mod 2)}$</th>
<th>if $P^+ (f, V) = 2$</th>
<th>$-t_1^+ t_2 \text{ (mod 8)}$</th>
<th>$s(f) \text{ (mod 8)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>---</td>
<td>0</td>
<td>(0); (u = 0) (2)</td>
<td>0</td>
<td>(\pm 1)</td>
</tr>
<tr>
<td>2</td>
<td>(2E_1) (\frac{u}{2})</td>
<td>0; (u = 0) (2)</td>
<td>0</td>
<td>(u = 1) (2)</td>
<td>(\mp 2)</td>
<td>(\mp 1)</td>
</tr>
<tr>
<td>3</td>
<td>(2E_2) (\frac{u}{3}) (1, 1)</td>
<td>(\frac{2}{3} u = \pm 1) (3)</td>
<td>none</td>
<td>none</td>
<td>(0); (u = 0) (3)</td>
<td>(0)</td>
</tr>
<tr>
<td>4</td>
<td>(2E_3) (\frac{u}{3} ) (0, 1, 1, 1, 0, -1)</td>
<td>(-\frac{2}{3} u = \pm 1) (3)</td>
<td>none</td>
<td>none</td>
<td>(0); (u = 0) (3)</td>
<td>(0)</td>
</tr>
<tr>
<td>4</td>
<td>(2(E_1 + E_1)) (\frac{1}{2}) ((u_1, u_2))</td>
<td>(0); (u_1, u_2 = 0) (2)</td>
<td>(0)</td>
<td>(u_1, u_2 = 1) (2)</td>
<td>(\mp 4)</td>
<td>(\mp 2)</td>
</tr>
<tr>
<td>5</td>
<td>(E_1) (\frac{u}{4})</td>
<td>(0); (u = 0) (4)</td>
<td>(0)</td>
<td>(u = 1) (4)</td>
<td>(\mp 4)</td>
<td>(\mp 3)</td>
</tr>
<tr>
<td>6</td>
<td>(2E_3) (\frac{u}{4}) (1, 1, 1)</td>
<td>(0); (u = 0) (4)</td>
<td>(0)</td>
<td>(u = 1) (4)</td>
<td>(\mp 4)</td>
<td>(\mp 3)</td>
</tr>
<tr>
<td>7</td>
<td>(2E_4) (\frac{u}{4}) (u + v, u, v, 0)</td>
<td>(0); (u = 0) (4)</td>
<td>(0)</td>
<td>(u = v = 0) (2)</td>
<td>(\mp 4)</td>
<td>(\mp 4)</td>
</tr>
<tr>
<td>8</td>
<td>(2E_5) (\frac{u}{4}) (1, 1, 2, 0, 2)</td>
<td>(0); (u = 0) (4)</td>
<td>(0)</td>
<td>(u = v = 1) (2)</td>
<td>(\mp 4)</td>
<td>(\mp 4)</td>
</tr>
<tr>
<td>9</td>
<td>(2E_6) (\frac{u}{4}) (0, 0, u, v, u + v, u + v)</td>
<td>(0); (u = 0) (2)</td>
<td>(0)</td>
<td>(u = v = 0) (2)</td>
<td>(\mp 4)</td>
<td>(\mp 4)</td>
</tr>
</tbody>
</table>
### Table XIX

| $s(f)$ (mod 8) | $h$ | $V_2$ (mod 1) | $P^+(f, V)n/|D|$ |
|----------------|-----|---------------|-----------------|
| 0              | 0   | $-$           | 1               |
| $\pm 2 (E_1 + E_1)$ | 0   | $\frac{1}{2} (u_1, u_2); u_1 \neq u_2$ (2) | $\frac{1}{4}$ |
| $\pm 2 (E_1 - E_1)$ | 0   | $\frac{1}{2} (u_1, u_2); u_1 = u_2$ (2) | $\frac{1}{4}$ |
| $\pm 4E_1$     |     | $\pm \frac{1}{2}$ |                 |
| $\pm 2E_3$     |     | $\pm \frac{1}{2} (1, 1, 1)$ |                 |
| $4H_1$         |     | $\frac{1}{2} (u_1, u_2); u_1, u_2 = 0$ (2) |                 |
| $2E_4$         |     | $\frac{1}{2} (1, u, v, 0); u \neq v$ (2) |                 |
|                |     | $\frac{1}{2} (0, 1, 1, 0)$ |                 |
| $\pm 2E_5$     |     | $\pm \frac{1}{2} (1, 1, 2, 0, 2)$ | $\frac{1}{2}$ |
| $\pm 2E_7$     |     | 0             | $\frac{1}{3}$ |
| $\mp 2E_1$     |     | $\frac{1}{2}$ |                 |
| $\pm 4E_1$     |     | 0             | $\frac{1}{4}$ |
| $\mp 2E_3$     |     | $\frac{1}{2} (1, 1, 1)$ |                 |
| $\pm 2E_5$     |     | $\frac{1}{2} (1, 1, 0, 0, 0)$ | $\frac{1}{4}$ |
| $\pm 2$        | $2E_4$ | 0             | $\frac{1}{3}$ |
| $\pm 2 (E_1 \pm E_1)$ | 0   | 0             |                 |
| $\pm 2 (E_1 + E_1)$ | 0   | $\frac{1}{2} (1, 1)$ | $\frac{1}{4}$ |
| $\pm 2 (E_1 - E_1)$ | 0   | $\frac{1}{2} (0, 1)$ | $\frac{1}{4}$ |
| $\mp 2E_8$     | $\hat{E}_8$ | 0             |                 |
| $\pm 2E_8$     | $\hat{E}_8$ | $\frac{1}{2} (0, 0, 1, 1, 0, 0)$ |                 |
| $\pm 3$        | $4E_1$ | $\frac{1}{2}$ | $\frac{1}{4}$ |
| $\pm 2E_3$     |     | 0             |                 |
| $\mp 2E_5$     |     | 0             |                 |
| $2E_4$         |     | 0             | $\frac{1}{4}$ |
| $2E_6$         | $\hat{E}_6$ | 1 (0, 0, u, v, 1, 1); u \neq v (2) | $\frac{1}{4}$ |

### Table XX

Consider the case when $P^+(f, V) < 2$

| $\pm h$ | $P^+(f, V) \leq$ | $P^+(f, V)n/|D| \leq$ |
|----------|------------------|--------------------------|
| $2E_4; 2E_6$ | $\frac{23}{12}$ | $\frac{1}{3} \left( \frac{23}{24} \right)^n$ |
| Otherwise | $\frac{7}{4}$   | $\frac{1}{2} \left( \frac{7}{8} \right)^n$ |
Proof of the theorems for the case $G(f, V) = 2$

If $P^* (f, V)^n/ | D | > \frac{1}{6}$ holds, then $G(f, V)^n/ | D | > \frac{1}{6}$. From $G(f, V) = 2$, we obtain $f \sim K_{t_1, t_2} + h$, for some choice of $t_1, t_2$ and for some $h$ listed in Table XV. Moreover, separating $V = (V_1, V_2)$ as in the sum $K_{t_1, t_2} + h$, by Prop. 4 we have $V_1 = (1/2) (mod 1)$. Also, from the disjointness of $K_{t_1, t_2} + h$, Lemma 6 implies that if $M(X)$ is the linear part of $h(X + V_2)$ then $M$ is integral and $M^2(X) \equiv h(X) (mod 2)$. Each $h$ given in Table XV has even coefficients. Hence $M^2(X) \equiv h(X) (mod 2)$ is equivalent to requiring that each coefficient in $M$ is even. Since $h = 2g$, for some $g$ in Table XII, we thus obtain the first three columns of Table XVIII from Table XII. Column 4 is computed directly from the preceding columns. For the remainder of Table XVIII, we recall (13), obtaining

$$f(V + X) \equiv K_{t_1, t_2} (1/2) + h(V_2) (mod 2) \equiv \frac{1}{2} (+ t_1 + t_2) + h(V_2)$$

Hence, $P^* (f, V) = 2$ exactly when $- t_1 + t_2 \equiv -4 h(V_2) (mod 8)$. For the last column we use $s(f^2) \equiv - t_1 + t_2 + s(\pm h) (mod 8)$.

Summarizing, if $P^* (f, V)^n/ | D | > \frac{1}{6}$, and $P^* (f, V) = 2$, then we have $f \sim K_{t_1, t_2} + h$ where $h$ and $V = (0, V_2)$ are given below (Tables XIX and XX).

Since $n \geq 21$, each of the entries in the third column is less then $\frac{1}{6}$. This completes the proofs of Theorems 1 and 2.

For Theorem 3 we note that $P((f, V)^n/ | D | > 1/5$ implies that $P^* (f^2)^n/ | D | > 1/5$. Therefore, when $P((f, V)^n/ | D | > 1/5$ and $G(f, V) = 2$, the above argument applies and so $P^* (f, V) = 2$. Hence, $f$ and $V$ are given in Table XIX. But, for any of these we have $P(f, V) = 0$ and so $P(f, V)^n/ | D | \leq 1/5$ for all $f, V$.

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References