ON COMMON FIXED POINTS IN METRIC SPACES

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Some fixed point theorems for certain contractive type mapping are presented in this note.

Throughout this paper \((X, d)\) will denote a complete metric space unless otherwise stated and \(R^+\), the set of non-negative reals. Recently Kiventidis\(^1\) proved the following:

**Theorem TK1**—Let \(T\) be a self-mapping of \(X\) such that

\[ d(Tx, Ty) \leq d(x, y) - W(d(x, y)) \forall x, y \in X \]  \(\text{...(1)}\)

where \(W : R^+ \to R^+\) is a continuous function such that \(0 < W(r) < r\) for all \(r \in R^+ - \{0\}\).

Then \(T\) has a unique fixed point:

In what follows first we prove a theorem which gives Theorem TK 1 as a special case.

**Theorem 1**—Let \(T\) be a continuos mapping and \(T_1, T_2\) be any other two mappings of \(X\) into itself such that

\[ TT_i = T_i T (i = 1, 2) \]  \(\text{...(2)}\)

\[ \bigcup_{i=1}^{\infty} T_i (X) \subseteq T(X) \]

and

\[ d(T_1 x, T_2 y) \leq d(Tx, Ty) - W(d(Tx, Ty)) \]  \(\text{...(3)}\)

where \(W : R^+ \to R^+\) is a continuous function, with

\[ 0 < W(r) < r \text{ for all } r \in R^+ - \{0\}. \]

Then \(F_{T_1, T_2} = \{x \in X : x = Tx = T_1 x = T_2 x\}\) is non-empty. Furthermore \(F_{T_1} = F_{T_2} = F_{T_1, T_2} = \{u\}, \) for some \(u\) in \(X\).

**Proof** : Let \(x_0\) be an arbitrary point in \(X\).
Since $T_1(X)$ and $T_2(X)$ are subsets of $T(X)$, we let $T_1 x_{2n} = T x_{2n+1}$ and $T_2 x_{2n+1}$
$= T x_{2n+2}$, $n = 0, 1, 2, ...$

Then from (3) we have for all $n \geq 1$, $x \in X$,

$$\sum_{r=0}^{n} w(d(T x_r, T x_{r+1})) \leq d(T x_0, T x_1).$$

So the series of non-negative terms

$$\sum_{r=0}^{n} W(d(T x_r, T x_{r+1}))$$

is convergent.

From this it follows that $\lim_{r \to \infty} W(d(T x_r, T x_{r+1})) = 0$.

Since $W(0) = 0$, so from the continuity of $W$ we get

$$\lim_{r \to \infty} W(d(T x_r, T x_{r+1})) = 0$$

$$\Rightarrow W(\lim_{r \to \infty} d(T x_r, T x_{r+1})) = 0$$

$$\Rightarrow \lim_{r \to \infty} d(T x_r, T x_{r+1}) = 0$$

which implies that $\{T x_n\}$ is Cauchy and so it converges to a point $u$ in $X$, since $X$ is complete.

Therefore $\{T x_{2n+1} = T_1 x_{2n}\}$, $\{T x_{2n+2} = T_2 x_{2n+1}\}$ and $\{T x_{2n} = T_2 x_{2n-1}\}$ being subsequences of $\{T x_n\}$ converge to $u$ also. But $T T_i = T_i T$, $i = 1, 2$ and the continuity of $T$ implies that $\lim_{n \to \infty} T(T x_{2n}) = T u$, $\lim_{n \to \infty} T(T x_{2n+1}) = T u$, $\lim_{n \to \infty} T(T x_{2n})$ = $\lim_{n \to \infty} T_1 x_{2n} = T u$ and $\lim_{n \to \infty} T_2 (T x_{2n+1}) = \lim_{n \to \infty} T_2 x_{2n+1} = T u$.

Now from (3)

$$d(T_1(T x_{2n}), T_2 u) \leq d(T(T x_{2n}), T u) - W(d(T(T x_{2n}), T u))$$

Proceeding to the limit $n \to \infty$, we obtain $Tu = T_2 u$.

In a similar manner we can show that $Tu = T_1 u$. Suppose $u \neq Tu$

Now

$$d(T_1(T x_{2n}), T_2 x_{2n+1})$$

$$\leq d(T(T x_{2n}), T x_{2n+1}) - W(d(T(T x_{2n}), T x_{2n+1})).$$

Proceeding to the limit $n \to \infty$, we obtain

$$d(T u, u) \leq d(T u, u) - W(d(T u, u)),$$

which is a contradiction. Thus $u = Tu$. 
So \( u = Tu = T_1 u = T_2 u \). So \( F_{T,T_1,T_2} \) is nonempty. It follows easily from (3)

\[ F_{T_1} = F_{T_2} = F_{T,T_1,T_2} = \{u\} \]

**Remarks:** Theorem TK 1 follows from Theorem 1 is one takes \( T_1 = T_2 \) and \( T = I_X \) where \( I_X \) is the identity mapping on \( X \).

In what follows we don’t take \( X \) as a complete metric space.

**Theorem 2**—Let \( T \) be continuous mapping of a metric space \( X \) into itself satisfying (1). If there exists a subset \( M \) of \( X \) and a point \( x_0 \) in \( M \) such that

\[ d(x, x_0) - d(Tx, Tx_0) \geq 2d(x_0, Tx_0) \text{ for every } x \in x - M \]  \( \ldots (4) \)

and if \( T \) maps \( M \) into a compact subset of \( X \) then there exists a unique fixed point of \( T \).

**Proof:** Since \( T \) maps \( M \) into a compact set, Theorem 2 will follow from Theorem TK 1 if it is shown that \( x_n \in M \) for every \( n \), where \( x_n = T^n x_0, n = 1,2,3,... \), Let us suppose that \( x_0 \neq Tx_0 \). Then it follows easily that the sequence \( \{C_n\} \), where \( C_n = d(x_n, x_{n+1}) \), is non-increasing and since \( x_0 \neq Tx_0 \), we get \( d(x_n, x_{n+1}) < d(x_0, Tx_0) \).

But

\[ d(x_n, x_0) \leq d(x_n, x_{n+1}) + d(Tx_n, Tx_0). \]

So

\[ d(x_n, x_0) - d(Tx_n, Tx_0) \leq d(x_n, x_{n+1}) + d(x_0, Tx_0) < 2d(x_0, Tx_0). \]

Hence it follows from (4) that \( x_n \in M \) for every \( n \).

This completes the proof of Theorem 2.

**References**