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ON COMMON FIXED POINTS IN METRIC SPACES

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Some fixed point theorems for certain contractive type mapping are presented in this note.

Throughout this paper (X, d) will denote a complete metric space unless otherwise stated and R^+, the set of non-negative reals. Recently Kiventidis[1] proved the following :

*Theorem TK*1—Let T be a self-mapping of X such that

$$d(Tx, Ty) \leqslant d(x, y) - W(d(x, y)) \;\forall\; x, y \text{ in } X \qquad \ldots(1)$$

where $W : R^+ \to R^+$ is a continuous function such that $0 < W(r) < r$ for all $r \in R^+ - \{0\}$

Then T has a unique fixed point :

In what follows first we prove a theorem which gives Theorem TK 1 as a special case.

Theorem 1—Let T be a continus mapping and T_1, T_2 be any other two mappings of X into itself such that

$$TT_i = T_i T \; (i = 1, 2) \qquad \ldots(2)$$

$$\bigcup_{i=1}^{2} T_i(X) \subseteq T(X)$$

and

$$\text{for all } x, y \text{ in } X \qquad \ldots(3)$$

$$d(T_1 x, T_2 y) \leqslant d(Tx, Ty) - W(d(Tx, Ty))$$

where $W : R^+ \to R^+$ is a continuous function, with

$$0 < W(r) < r \text{ for all } r \in R^+ - \{0\}.$$

Then $F_{T,T_1,T_2} = \{x \in X : x = Tx = T_1 x = T_2 x\}$ is non-empty. Furthermore $F_{T_1} = F_{T_2} = F_{T,T_1,T_2} = \{u\}$, for some u in X.

PROOF : Let x_0 be an arbitrary point in X.

Since $T_1(X)$ and $T_2(X)$ are subsets of $T(X)$, we let $T_1 x_{2n} = Tx_{2n+1}$ and $T_2 x_{2n+1} = Tx_{2n+2}$, $n = 0, 1, 2, \ldots$

Then from (3) we have for all $n \geqslant 1$, $x \in X$,

$$\sum_{r=0}^{n} w(d(Tx_r, Tx_{r+1})) \leqslant d(Tx_0, Tx_1).$$

So the series of non-negative terms

$$\sum_{r=0}^{n} W(d(Tx_r, Tx_{r+1})) \text{ is convergent.}$$

From this it follows that $\lim\limits_{r \to \infty} W(d(Tx_r, Tx_{r+1})) = \theta$.

Since $W(0) = 0$, so from the continuity of W we get

$$\lim_{r \to \infty} W(d(Tx_r, Tx_{r+1})) = 0$$

$$\Rightarrow W\left(\lim_{r \to \infty} d(Tx_r, Tx_{r+1})\right) = 0$$

$$\Rightarrow \lim_{r \to \infty} d(Tx_r, Tx_{r+1}) = 0$$

which implies that $\{Tx_n\}$ is Cauchy and so it converges to a point u in X, since X is complete.

Therefore $\{Tx_{2n+1} = T_1 x_{2n}\}$, $\{Tx_{2n+2} = T_2 x_{2n+1}\}$ and $\{Tx_{2n} = T_2 x_{2n-1}\}$ being subsequences of $\{Tx_n\}$ converge to u also. But $TT_i = T_i T$, $i = 1, 2$ and the continuity of T implies that $\lim\limits_{n \to \infty} T(Tx_{2n}) = Tu$, $\lim\limits_{n \to \infty} T(Tx_{2n+1}) = Tu$, $\lim\limits_{n \to \infty} T_1(Tx_{2n}) = \lim\limits_{n \to \infty} T(T_1 x_{2n}) = Tu$ and $\lim\limits_{n \to \infty} T_2(Tx_{2n+1}) = \lim\limits_{n \to \infty} T(T_2 x_{2n+1}) = Tu$.

Now from (3)

$$d(T_1(Tx_{2n}), T_2 u) \leqslant d(T(Tx_{2n}), Tu) - W(d(T(Tx_{2n}), Tu)$$

Proceeding to the limit $n \to \infty$, we obtain $Tu = T_2 u$.

In a similar manner we can show that $Tu = T_1 u$. Suppose $u \neq Tu$

Now

$$d(T_1(Tx_{2n}), T_2 x_{2n+1})$$
$$\leqslant d(T(Tx_{2n}), Tx_{2n+1}) - W(d(T(Tx_{2n})), Tx_{2n+1}).$$

Proceeding to the limit $n \to \infty$, we obtain

$$d(Tu, u) \leqslant d(Tu, u) - W(d(Tu, u)),$$

which is a contradiction. Thus $u = Tu$.

So $u = Tu = T_1 u = T_2 u$. So F_{T,T_1,T_2} is nonempty. It follows easily from (3)

$$F_{T_1} = F_{T_2} = F_{T,T_1,T_2} = \{u\}.$$

Remarks : Theorem *TK* 1 follows from Theorem 1 is one takes $T_1 = T_2$ and $T = I_X$ where I_X is the identity mapping on X.

In what follows we don't take X as a complete metric space.

Theorem 2—Let T be continuous mapping of a metric space X into itself satisfying (1). If there exists a subset M of X and a point x_0 in M such that

$$d(x, x_0) - d(Tx, Tx_0) \geqslant 2d(x_0, Tx_0) \text{ for every } x \text{ in } x - M \qquad \ldots(4)$$

and if T maps M into a compact subset of X then there exists a unique fixed point of T.

PROOF : Since T maps M into a compact set, Theorem 2 will follow from Theorem *TK* 1 if it is shown that $x_n \in M$ for every n, where $x_n = T^n x_0$, $n = 1,2,3,\ldots$ Let us suppose that $x_0 \neq Tx_0$. Then it follows easily that the sequence $\{C_n\}$, where $C_n = d(x_n, x_{n+1})$, is non-increasing and since $x_0 \neq Tx_0$, we get $d(x_n, x_{n+1}) < d(x_0, Tx_0)$.

But

$$d(x_n, x_0) \leqslant d(x_n, x_{n+1}) + d(Tx_n, Tx_0).$$

So

$$d(x_n, x_0) - d(Tx_n, Tx_0) \leqslant d(x_n, x_{n+1}) + d(x_0, Tx_0) < 2d(x_0, Tx_0).$$

Hence it follows from (4) that $x_n \in M$ for every n.

This completes the proof of Theorem 2.

REFERENCES

1. Thomas Kiventidis, *Indian J. pure appl. Math.* 16 (1985), 1420-24.
2. E. Rakotch, *Proc. Am. Math. Soc.* 13 (1962), 459-65.